
Applied Quantitative Finance for Equity Derivatives

Second Edition

Jherek Healy

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Contents

Preface	xi
Acknowledgments	xiii
1 The Forward	1
1.1 The borrow cost	1
1.2 The dividends	2
1.3 Blending dividends	5
1.4 The forward contract	6
1.5 Trading repo via a total return swap	6
1.6 Further reading	7
2 Vanilla Options	8
2.1 Black	8
2.2 Greeks	9
2.2.1 Delta	9
2.2.2 Gamma	11
2.2.3 Vega	11
2.2.4 Vanna	12
2.2.5 Volga	12
2.2.6 Theta	13
2.2.7 Rho	13
2.2.8 Dividend sensitivity	13
2.2.9 Numerical greeks	14
2.2.10 Example values	14
2.3 Put-Call Parity	15
2.4 Implied volatility from the option price	16
2.5 Options on Index Futures	19
2.6 Fixed notional	19
2.7 Forward model vs. spot model	20
2.7.1 Forward model	20
2.7.2 Spot model	21
2.7.3 The option price continuity relationship	21
2.7.4 The quadrature method	23

2.7.5	Forward vs. spot model greeks	24
2.8	Hybrid models for cash dividends	26
2.8.1	Lehman's model	26
2.8.2	Bühler's model	28
2.9	Approximations for European option prices under the Spot model	29
2.9.1	Etoe and Gobet approximation	29
2.9.2	Zhang approximation	33
2.9.3	Comparison	34
2.10	Conclusion	38
2.11	Further reading	40
3	American options	41
3.1	Ju-Zhong approximation	41
3.2	Finite difference methods	45
3.3	TR-BDF2	49
3.4	Richardson extrapolation on implicit Euler	52
3.5	American option specifics	52
3.6	Choice of grid	56
3.7	Exact calibration to discount bonds and forward prices	57
3.7.1	Exact Bond Price	58
3.7.2	Exact Forward Price	59
3.7.3	Put-Call Parity	61
3.8	Exponential fitting	62
3.9	Issues with varying yield and cash dividends	64
3.10	Finite difference method for the spot model	66
3.10.1	Proportional dividends	66
3.10.2	Cash dividends	68
3.10.3	Numerical integration of American Calls under the Spot model	70
3.11	Hybrid models for cash dividends	71
3.12	Exercise boundary	72
3.13	Cash vs. Proportional	74
3.14	Conclusion	76
3.15	Further reading	76
4	Monte-Carlo and basket options	77
4.1	Monte-Carlo simulation	77
4.1.1	Single asset	78
4.1.2	Incorporating discrete dividends	81
4.1.3	Measuring convergence	82
4.1.4	Random numbers	83
4.1.5	Multiple assets	96
4.1.6	Computing sensitivities with algorithmic differentiation	101
4.1.7	Variance reduction of ordinary Monte-Carlo estimates with the Brownian bridge path construction	105

4.1.8	Control variate	111
4.1.9	American options	114
4.2	A good approximation for Vanilla basket options under the Black model	122
4.2.1	Two assets	122
4.2.2	General case	124
4.2.3	Beyond Black	128
4.3	Conclusion	129
4.4	Further reading	129
5	The volatility smile	130
5.1	Implied volatilities	130
5.2	No-arbitrage conditions	135
5.3	Basic models	137
5.3.1	Spline	137
5.3.2	Sticky strike or sticky Delta	143
5.3.3	A simple trader's model	143
5.3.4	Least-squares natural quintic	146
5.3.5	Least-squares spline	147
5.3.6	Controlling the extrapolation	148
5.4	Almost arbitrage-free models	153
5.4.1	SVI	153
5.4.2	SABR	162
5.4.3	Summary of parameterizations on AAPL	169
5.4.4	When SVI breaks down	170
5.5	Arbitrage-free models	173
5.5.1	Marking market call prices arbitrage-free	173
5.5.2	The measure	175
5.5.3	Mixture of lognormal distributions	176
5.5.4	Polynomial stochastic collocation	179
5.5.5	Stochastic collocation with splines	195
5.5.6	Arbitrage-free C^2 rational spline interpolation	210
6	Local volatility	220
6.1	Dupire Local volatility	220
6.1.1	Taking dividends into account	220
6.1.2	Calibration effects on the local volatility	226
6.1.3	Repairing a broken local volatility	227
6.1.4	Dupire local volatility in Finite Difference Methods	234
6.1.5	Dupire local volatility in Monte-Carlo	237
6.1.6	Extrapolation effects	242
6.2	Local volatility parameterizations	249
6.2.1	Quadratic	249
6.2.2	Andeasen-Huge single-step local volatility	262
6.3	Black-Scholes with a term-structure of volatilities	275

7	Stochastic volatility and beyond	277
7.1	Stochastic volatility	277
7.1.1	Models and characteristic functions	277
7.1.2	Overview of pricing formulae	281
7.1.3	Black-Scholes control variate	282
7.1.4	A robust algorithm to find the optimal α for Heston	284
7.1.5	Truncation	286
7.1.6	Accuracy of the pricing formulae with control variate	289
7.1.7	Adaptive Filon integration	291
7.1.8	Generic calibration via differential evolution	300
7.1.9	Calibration via Simulated Annealing	303
7.1.10	Calibration via Particle Swarm Optimization	304
7.1.11	Heston calibration	305
7.1.12	Schobel-Zhu calibration	318
7.1.13	Bates calibration	320
7.1.14	Double-Heston calibration	321
7.1.15	Heston simulation	323
7.1.16	Schobel-Zhu simulation	337
7.1.17	SVJ simulation	339
7.1.18	Double-Heston simulation	341
7.2	Stochastic-local volatility	341
7.2.1	QE scheme for the Heston-local volatility model	342
7.2.2	EAE scheme for the Schobel-Zhu-local volatility model	342
7.2.3	Bins	342
7.2.4	Non-conform linear regression	344
7.2.5	The particle method	346
7.2.6	Vectorization	348
7.2.7	Particle Quasi Monte-Carlo	349
7.3	Further reading	352
8	Almost Vanilla Options	354
8.1	Forward start	354
8.1.1	Fixed quantity	354
8.1.2	Fixed notional	355
8.1.3	Which volatility?	356
8.1.4	Local volatility	356
8.2	Asian	357
8.2.1	Fixed strike	358
8.2.2	Asian warrants	361
8.2.3	Floating strike	362
8.2.4	Local volatility	364
8.3	Digital	365
8.3.1	Cash-or-nothing under the Black model	365
8.3.2	Local volatility	368

8.3.3	Stochastic volatility	368
8.4	Barrier	369
8.4.1	European barriers	370
8.4.2	Continuous barriers under the Black model	371
8.4.3	CBBC warrants	380
8.4.4	Partial barriers under the Black model	381
8.4.5	Double barrier	382
8.4.6	Discrete observations adjustment	385
8.4.7	TR-BDF2 with barriers	385
8.4.8	Monte-Carlo with barriers	395
8.5	Further reading	397
9	Options on a Foreign Stock	398
9.1	Local currency	398
9.2	Quanto	399
9.2.1	The quanto process under local volatility	399
9.2.2	The quanto process under the Schobel-Zhu stochastic volatility model	400
9.2.3	Vanilla quanto under the Black model	401
9.2.4	Quanto in Monte-Carlo	403
9.3	Compo	404
9.4	A remark on the models	405
9.5	Further reading	405
10	Volatility derivatives	406
10.1	Variance Swap	406
10.1.1	Definition	406
10.1.2	Continuous replication in practice	407
10.1.3	Discrete replication	408
10.1.4	Replication examples	410
10.1.5	Model-free replication	413
10.1.6	Seasoned swap	417
10.1.7	Initial value of a discretely sampled variance swap	418
10.1.8	Forward Starting Variance Swap	418
10.1.9	Variance swap greeks	418
10.1.10	Quanto variance swap	419
10.1.11	Variance Swap and Discrete Dividends	422
10.2	Volatility Swap	423
10.2.1	Payoff	424
10.2.2	Replication	426
10.2.3	A Fast Fourier Transformation	427
10.2.4	Adaptive Filon	428
10.2.5	Integration Boundaries	431
10.3	Options, Caps and Floors	433

10.3.1	Option on Variance	433
10.3.2	Option on Volatility	434
10.3.3	Variance Cap	434
10.3.4	Volatility Cap	434
10.4	Validation	435
10.5	Finite difference method for volatility derivatives under local volatility	437
10.5.1	Variance Swap	437
10.5.2	Quanto variance swap	438
10.6	Further reading	438
11	VIX derivatives	439
11.1	The VIX index	439
11.2	VIX Futures	440
11.3	VIX options	442
12	Dividend derivatives	445
12.1	Dividend swap	445
12.1.1	Price from discrete dividends	447
12.1.2	Price from a dividend yield term-structure	447
12.2	Dividend future	448
12.3	Dividend future option	449
13	Exotics	450
13.1	Autocall	450
13.1.1	Forward evaluation	451
13.1.2	Backward evaluation	452
13.1.3	Stochastic interest rates and the equity-interest rate correlation	453
13.2	Accumulator	456
13.2.1	Forward evaluation	457
13.2.2	Backward evaluation	458
13.3	Altiplano	459
A.1	Minimizers	464
A.2	Solvers	465
A.3	Special functions	466
A.4	Quadratures	467
A.5	Random Numbers	467
A.6	Linear algebra	468
A.7	Miscellaneous	468
	Bibliography	475
	Index	499

Preface

This book presents the most significant equity derivatives models used these days. It is not a book around esoteric or cutting-edge models, but rather a book on relatively simple and standard models, viewed from the angle of a practitioner. Most books present models in an abstract manner, often disconnected from how to apply them in the real world. This book intends to fill that gap, with the ambitious goal of transforming a reader unfamiliar with equity derivatives models into a specialist of such models.

There is no introductory mathematical chapter. If the reader is interested in stochastic calculus, the very concise book of Mikosch [248] is highly recommended. Shreve's book [295] is a nice complement with a more detailed, and very accessible mathematical presentation of theorems relevant to finance. John Hull offers a good even if slightly austere introduction to financial derivatives and various rate conventions in his book [155].

The first chapter of this book introduces the specificities of the equity derivatives market in terms of modeling, with a close look at the dividend curves and the forward price. We then move on to the vanilla options, with the famous Black-Scholes model, paying attention to the various adjustments used in practice. After giving the most standard practices for European vanilla options, we follow with the issues raised by discrete cash dividends on the option price and study recent analytical approximations. Regarding American vanilla options, we detail fast and stable finite difference schemes, and proceed to analyze the inclusion of cash or proportional dividends, paying particular attention to the effect of the dividend model on the exercise boundary.

Chapter four introduces the Monte-Carlo method to price financial derivatives on a basket of equities. The parallelization of random numbers generation, the randomization of quasi-random numbers and the various ways of generating of correlated normal variates as well as the use of control variates and their caveats are carefully explained. We then present adjoint algorithmic differentiation techniques to compute sensitivities and finish the chapter with various techniques to include the American or more precisely, Bermudan exercise, in particular non-parametric regressions.

In chapter five, six and seven, we look at how to imply volatilities in practice, and common volatility representations, be it parametric, Dupire local volatility, or stochastic volatility. We describe precisely how to accurately simulate the different models with the Monte-Carlo method or through finite difference methods. In doing so, we expose the many issues that arise with the classical approach to the Dupire local volatility along with solutions and explain how to handle discrete cash dividends in the

Dupire framework. We conclude the chapter with an analysis of the particle method and its close relatives for Heston and Schobel-Zhu stochastic-local volatility models, detailing the use of quasi-random sequences with the method.

Progressively we consider other commonly traded options: forward start, digital, barrier, Asian, quanto, compo, etc. We will however not present the rarely traded options such as compound or chooser, even when they have apparently nice analytical formulas. On each subject, pricing techniques are presented in great detail, be it through the simplest analytical formula, a Monte-Carlo simulation, or the finite difference method.

In chapter ten, we have a look at common volatility derivatives, that is, variance swaps, volatility swaps and options. Discrete, continuous or model-free replication of variance swaps is analyzed. Newer listed derivatives such as VIX options and dividend derivatives are subsequently covered.

We finally present common exotics and how to evaluate those in a forward Monte-Carlo manner or in a backward PDE manner. Even if those tend to be less traded nowadays, they are still in many traders books, and remain popular in Asia, especially the autocallables.

SECOND EDITION

In this second edition, various typos have been corrected, and the text has been slightly updated. New arbitrage-free implied volatility interpolations were added to chapter five, and different types of warrants, including callable bull/bear contracts (CBBs) are covered in chapter eight. The book layout has also been significantly updated to allow for a hardback book publication.

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Chapter One

The Forward

Second Edition

The equity forward, the expected price of a stock at a future date, is key to understanding and explaining the valuation of many equity derivatives. It is directly linked to one of the simplest equity derivatives contract, the forward contract, where an exchange of cash against stock occurs at a specific future date, the maturity date.

Compared to other asset classes, for example, foreign exchange (FX) derivatives or interest rate derivatives, there are two specificities of equities that are going to play a major role in the valuation of the forward price and in the pricing of equity derivatives: the dividend and the borrow cost.

1.1 THE BORROW COST

Through a repurchase agreement contract (in common language, a repo), one can borrow money secured by a stock at a specific rate, usually lower than the rate that would be obtained by borrowing money unsecured. Therefore the stock price will grow at its repo rate r_R , while the option price will grow either at the so called risk-free rate (really the unsecured funding rate) r_F , or at the collateral rate r_C in the case of a collateralized trade in the risk-neutral measure [268].

In order to look at the evolution of repo costs, it can be more meaningful to represent the repo in terms of annualized spread s_R against the risk-free rate: $s_R = r_F - r_R$. This spread corresponds to the rate charged over the risk-free rate to go short. The spread is positive when there is demand for a security and negative when there is demand for cash. The repo spread for equity indices was traditionally very close to zero. Since the 2008 financial crisis, it is not uncommon for it to be negative, for example the 7 years repo spread on Eurostoxx offered by BNP was around -0.3% in December 2012.