Applied Quantitative Finance for Equity Derivatives

Third Edition

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Contents

Preface

This book presents the most significant equity derivatives models used these days. It is not a book around esoteric or cutting-edge models, but rather a book on relatively simple and standard models, viewed from the angle of a practitioner. Most books present models in an abstract manner, often disconnected from how to apply them in the real world. This book intends to fill that gap, with the ambitious goal of transforming a reader unfamiliar with equity derivatives models into a specialist of such models.

There is no introductory mathematical chapter. To learn stochastic calculus, the very concise book of Mikosch [277] is highly recommended. Shreve's book [333] is a nice complement with a more detailed, and very accessible mathematical presentation of theorems relevant to finance. John Hull offers a good even if slightly austere introduction to financial derivatives and various rate conventions in his book [176].

The first chapter of this book introduces the specificities of the equity derivatives market in terms of modeling, with a close look at the dividend curves and the forward price. We then move on to the vanilla options, with the famous Black-Scholes model, paying attention to the various adjustments used in practice. After giving the most standard practices for European vanilla options, we follow with the issues raised by discrete cash dividends on the option price and study recent analytical approximations. Regarding American vanilla options, we detail fast and stable finite difference schemes, and proceed to analyze the inclusion of cash or proportional dividends, paying particular attention to the effect of the dividend model on the exercise boundary.

Chapter 4 introduces the Monte-Carlo method to price financial derivatives on a basket of equities. The parallelization of random numbers generation, the randomization of quasi-random numbers and the various ways of generating of correlated normal variates as well as the use of control variates and their caveats are carefully explained. We then present adjoint algorithmic differentiation techniques to compute sensitivities and finish the chapter with various techniques to include the American or more precisely, Bermudan exercise, in particular non-parametric regressions.

In Chapter 5, 6 and 7, we look at how to imply volatilities in practice, and common volatility representations, be it parametric, Dupire local volatility, or stochastic volatility. We describe precisely how to accurately simulate the different models with the Monte-Carlo method or through finite difference methods. In doing so, we expose the many issues that arise with the classical approach to the Dupire local volatility along with solutions and explain how to handle discrete cash dividends in the Dupire framework. We conclude the chapter with an analysis of the particle method and its close relatives for Heston and Schobel-Zhu stochastic-local volatility models, detailing the use of quasi-random sequences with the method.

Progressively we consider other commonly traded options: forward start, digital, barrier, Asian, quanto, compo, etc. We will however not present the rarely traded options such as compound or chooser, even when they have apparently nice analytical formulas. On each subject, pricing techniques are presented in great detail, be it through the simplest analytical formula, a Monte-Carlo simulation, or the finite difference method.

In Chapter 10, we have a look at common volatility derivatives, that is, variance swaps, volatility swaps and options. Discrete, continuous or model-free replication of variance swaps is analyzed. Newer listed derivatives such as VIX options and dividend derivatives are subsequently covered.

We finally present common exotics and how to evaluate those in a forward Monte-Carlo manner or in a backward PDE manner. Even if those tend to be less traded nowadays, they are still in many traders books, and remain popular in Asia, especially the autocallables.

Second edition

In the second edition, various typos have been corrected, and the text has been slightly updated. New arbitrage-free implied volatility interpolations were added to Chapter 5, and different types of warrants, including callable bull/bear contracts (CBBCs) are covered in Chapter 8. The book layout has also been significantly updated to allow for a hardback book publication.

Third edition

In this third edition, the physical exercise feature is briefly discussed in the first two chapters. The section on how to imply the Black-Scholes volatility has been updated. More details around the projected successive over-relaxation method for the pricing of American options have been added, as well as the relatively recent policy iteration method, particularly relevant in the case of negative interest rates, in Chapter 3. In addition, we present the Andersen-Lake algorithm instead of the Ju-Zhong formula as fast pricing routine for the case of vanilla American options under the Black-Scholes model with constant parameters. Chapter 4 has been updated around random number generation, antithetic variates, and the vectorization of the Monte-Carlo simulation. Important corrections have been made around the pricing of forward starting options with finite difference methods in Chapter 8, as well as to the pricing of knock-in options.

Additional changes include paragraphs on the radial basis function interpolation of implied volatilities, the Cos and the Andersen-Lake methods for European options under stochastic volatility models, the Vega in stochastic volatility models, the interpolation of prices on the finite difference grid. Example code for a set of important algorithms detailed in this book will appear progressively on this book's website at `https://jherekhealy.github.io`.

Acknowledgments

I would like to thank Liam Henry for kindly providing feedback on various typos in the first edition of this book. I am grateful to the many quants in the industry and in academia who have always answered my questions, in reverse alphabetical order: Mike Staunton, Alexander Sokol, Christoph Reisinger, Cornelis (Kees) Oosterlee, Matthew Lorig, Timothy Klassen, Gary Kennedy, Peter Jäckel, Brian Huge, Marc Henrard, Julien Guyon, Lech Grzelak, Daniel Duffy, Peter Carr, Alexander Antonov, Jesper Andreasen, and all the others. I also thank Yu-Ting Lin for instilling the idea of writing a book in my mind. Finally, I am grateful to my family, for its understanding and endless love, through the duration of the writing of this book.

Chapter One

The Forward

The equity forward, the expected price of a stock at a future date, is key to understanding and explaining the valuation of many equity derivatives. It is directly linked to one of the simplest equity derivatives contract, the forward contract, where an exchange of cash against stock occurs at a specific future date, the maturity date.

Compared to other asset classes, for example, foreign exchange (FX) derivatives or interest rate derivatives, there are two specificities of equities that are going to play a major role in the valuation of the forward price and in the pricing of equity derivatives: the dividend and the borrow cost.

1.1 THE BORROW COST

Through a repurchase agreement contract (in common language, a repo), one can borrow money secured by a stock at a specific rate, usually lower than the rate that would be obtained by borrowing money unsecured. Therefore the stock price will grow at its repo rate r_R, while the option price will grow either at the so called risk-free rate (really the unsecured funding rate) r_F, or at the collateral rate r_C in the case of a collateralized trade in the risk-neutral measure [300].

In order to look at the evolution of repo costs, it can be more meaningful to represent the repo in terms of annualized spread s_R against the risk-free rate: $s_R = r_F - r_R$. This spread corresponds to the rate charged over the risk-free rate to go short. The spread is positive when there is demand for a security and negative when there is demand for cash. The repo spread for equity indices was traditionally very close to zero. Since the 2008 financial crisis, it is not uncommon for it to be negative, for example the 7 years repo spread on Eurostoxx offered by BNP was around -0.3% in December 2012.

In practice, a borrow curve defines the borrow rate[1] $\bar{r}_R(t,T)$ between dates t and

[1] This is a cumulative rate, we have $\bar{r}_R(t,T) = \int_t^T r_R(u)\,\mathrm{d}u$ if we model $r_R(t)$ as a short rate.

T through a simple interpolation of brokers "borrow rates" (or spreads) against the unsecured funding rates. Common interpolations are linear or spline on spreads or on the logarithm of discount factors. The term structure of repo spreads is typically downward sloping.

1.2 THE DIVIDENDS

It is common for stocks to pay a fixed amount in cash regularly, typically quarterly for U.S. stocks. For the owner of a stock, this amount, the dividend, is guaranteed to be paid at the so called dividend ex-date, and usually actually paid a few days later at the dividend payment date (see Table 1.1 for an example of dividend payment information).

Table 1.1. AAPL Dividend history since 2014. The jump between May 2014 and August 2014 is due to a stock split.

Ex-Div. Date	Amount	Declaration Date	Record Date	Payment Date
11/5/2015	0.52	10/27/2015	11/9/2015	11/12/2015
8/6/2015	0.52	7/21/2015	8/10/2015	8/13/2015
5/7/2015	0.52	4/27/2015	5/11/2015	5/14/2015
2/5/2015	0.47	1/27/2015	2/9/2015	2/12/2015
11/6/2014	0.47	10/20/2014	11/10/2014	11/13/2014
8/7/2014	0.47	7/22/2014	8/11/2014	8/14/2014
5/8/2014	3.29	4/23/2014	5/12/2014	5/15/2014
2/6/2014	3.05	1/27/2014	2/10/2014	2/13/2014

Under the assumption of deterministic dividends, the absence of arbitrage then requires that the stock value S drops from the dividend amount d_i at the dividend ex-date t_i [160]:

$$S(t_i) = S(t_i^-) - d_i. \tag{1.1}$$

In reality, as well as under stochastic dividend models [292], even though the stock will of course not drop from the exact dividend amount, the drop will not be far off and will be influenced by tax rules and tick sizes [190].

1.3 THE FORWARD CONTRACT

A forward contract is an agreement between two counter-parties to exchange n stocks at specific price K, the strike price, on a specific date T, the maturity date. Usually forward contracts are settled in cash, a few days (typically two days) after the maturity date. The value of such a contract at time t is therefore:

$$V(t) = n\mathbb{E}_Q\left[e^{-\int_t^{T_p} r(u)\,\mathrm{d}u}(S(T) - K) \mid S(t)\right], \tag{1.2}$$

where Q is the risk-neutral measure, T_p is the settlement date and r the relevant instantaneous interest rate (eventually stochastic). According to [300], in the case of a collateralized forward contract, we have $r = r_C$, otherwise, $r = r_F$. If the forward contract is settled physically, the maturity date is the same as the settlement date, when the shares are exchanged against cash.

We may change measure and express Equation 1.2 under the T-forward measure Q_T defined by the numeraire $B(t,T) = \mathbb{E}_{Q_T}\left[e^{-\int_t^T r(u))du}\right]$:

$$V(t) = nB(t, T_p)\mathbb{E}_{Q_T}\left[S(T) - K \mid S(t)\right], \tag{1.3}$$

where B is the price of a zero coupon bond paying 1 unit of currency. In the case of a non-collateralized forward contract,

$$B(t,T) = e^{-\int_t^T r_F(t)\,\mathrm{d}t}, \tag{1.4}$$

while in the case of a collateralized forward contract,

$$B(t,T) = e^{-\int_t^T r_C(t)\,\mathrm{d}t}. \tag{1.5}$$

The forward price $F(t,T)$ is defined as the strike that makes the value of the contract zero at time t, and we have thus

$$F(t,T) = \mathbb{E}_{Q_T}\left[S(T)|S(t)\right]. \tag{1.6}$$

The forward price thus corresponds to the first moment of S, and will be key for the pricing of equity derivatives.

1.4 THE DIVIDEND CURVE

In order to stay in a continuous world, and therefore to translate the discrete cash events into a continuous rate, it is common to build a dividend curve, defining a dividend yield[2] $\bar{q}(t,T)$ from the cash dividends between t and T in terms of the current stock price $S(t)$. It is defined so as to make the forward price $F(t,T)$ equal in the two worlds. Usual daycount conventions for the yield are ACT/365, ACT/360 or BU/252 [176].

In practice, longer term dividends don't have the same dynamic as the short term dividends: while the short term dividends are well represented by a fixed cash

[2] A cumulative continuously compounded rate.

amount, long term dividends are better represented by a cash amount defined as proportion of the stock price: if the stock price drops by fifty percent in two years, we expect the dividends to drop as well by fifty percent.

In terms of discrete dividends of cash amount α_i and proportional amount β_i at dates t_i, and in absence of default event, we have $d_i = \alpha_i + \beta_i S(t_i^-)$ and the forward price F is[3] [53]:

$$F(t,T) = S(t)C(t,T) - \sum_{i:t<t_i\leq T} \alpha_i C(t_i^p, T)\,, \tag{1.7}$$

where $C(t,T)$ is the capitalization factor $R(t,T)$ adjusted by the proportional dividends. The inclusion of dividends depends on the dividend ex-date t_i and the capitalisation depends on the dividend payment date t_i^p. Assuming a continuously compounded repo rate $\bar{r}_R(t,T)$ and T measured according to the relevant daycount convention, we have

$$R(t,T) = e^{\bar{r}_R(t,T)(T-t)}, \tag{1.8}$$

$$C(t,T) = R(t,T) \prod_{i:t<t_i\leq T} (1-\beta_i)\,. \tag{1.9}$$

The advantage of reasoning through capitalisation and discount factors is to avoid thinking about the daycount convention, and to be able to mix different daycount conventions while keeping the same measure of time T. Note that each dividend cash value grows at the repo rate r_R because the repurchase agreement contract is written on the cash equivalent value of the stock.

In terms of continuous dividend yield $\bar{q}(t,T)$, the forward price is:

$$F(t,T) = S(t)R(t,T)e^{-\bar{q}(t,T)(T-t)}. \tag{1.10}$$

Using Equations 1.7 and 1.10, we can express the continuous dividend yield in terms of cash dividends:

$$\bar{q}(t,T) = -\frac{1}{T-t}\ln\left(\prod_{i:t<t_i\leq T}(1-\beta_i) - \frac{\sum_{i:t<t_i\leq T}\alpha_i C(t_i^p,T)}{S(t)R(t,T)}\right). \tag{1.11}$$

Let us consider a concrete example, taken from [305]: a stock trades at EUR 100, the 3-months interest rate is 5% Actual/360, the company pays a EUR 2 dividend in 31 days, the borrowing fee is 0.20% annual Actual/365, and the forward interest rate with a 2-months maturity is 4% annual Actual/360. This example is interesting since the author in [305] does not take properly into account the borrowing fees. The theoretical 3-month forward, assuming 92 calendar days in the period, is calculated

[3] We will see later in Section 2.7.3 and through equation 2.59 that the forward price can actually depend on the dividend policy, that is, on what happens when the stock price is below the theoretical dividend amount. In practice however, this issue is often ignored.

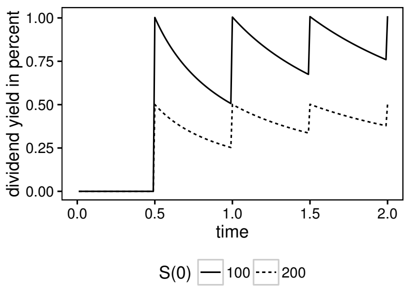
as follows:

$$F(t,T) = 100 * (1 + 5/100 * 92/360 - 0.20/100 * 92/365) \\ - 2 * (1 + 4/100 * 61/360 - 0.20/100 * 61/365) = 91.2145\,.$$

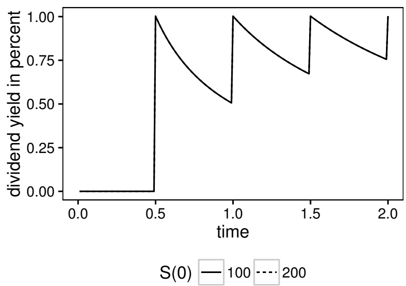
The equivalent dividend yield in continuous Actual/365 is calculated as

$$\bar{q}(t,T) = -\frac{365}{92} \ln \frac{91.2145}{100 * (1 + 5/100 * 92/360 - 0.20/100 * 92/365)} = 3.4607\%\,.$$

The dividend yield resulting from discrete cash dividends exhibits a sawtooth shape. One can build exactly the same dividend yield term structure from proportional dividends (Figure 1.1). The big difference is that the dividend yield term structure is going to stay the same when the underlying spot price moves in the case of proportional dividends, but not so in the case of cash dividends. The sawtooth shape stems from looking at the dividend yield in the units of a rate. If we were to look at the dividend yields in terms of logarithm of yield discount factor $-\bar{q}(t,T)(T-t)$, we would clearly see the staircase implied by the jumps at the dividend ex-dates (Figure 1.2).



(a) Cash with $\alpha_i = 1, t_i = \frac{1}{2}i$.



(b) Proportional with $\beta_j = 0.01, t_j = \frac{1}{2}j$.

Figure 1.1. Dividend yield from cash or proportional dividends when the asset spot $S(0)$ moves from $S(0) = 100$ to $S(0) = 200$ with $r_R = 0, r_C = 0$.

Instead of being defined in terms of cash dividends, the dividend curve can also be directly defined in terms of yields. Those yield are implied from specific equity derivatives market prices, for example equity forwards prices, equity index futures prices or option prices. In order to build a dividend yield term structure from futures prices, we can just invert equation (1.10). $F(t,T)$ is also the price of a future with maturity T if we neglect the forward-future spread that involves the covariance of the discount process and the stock process in the presence of stochastic rates [333, p. 240-241, Section 5.6.1]. In order to imply the yield from option prices, the put-call

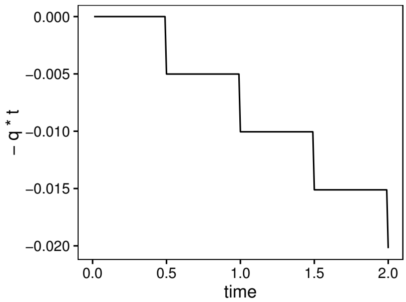


Figure 1.2. Logarithm of dividend yield discount factors $q(t,T)(T-t)$ with the same parameters as in Figure 1.1.

parity relationship is used as we shall see later in Section 2.3 on page 17.

When the dividend curve is directly defined through a yield term structure, a linear interpolation on yields or on the logarithm of discount factors[4] is commonly used to obtain a yield in between futures or options maturities.

1.5 BLENDING DIVIDENDS

Cash dividends are a better representation for short term dividends, and proportional dividends a better representation for long term dividends. There is however the need to move smoothly from one to the other, this is where blending comes in.

Let $(t_i)_{i=0,\dots,n}$ denote the dividend dates. k the index of the last full cash dividend, l the index of the first full proportional dividend, the linear blending proposed by Klassen [208] leads to the following term structure of dividends:

$$\alpha_i \text{ for } i \in \{0,\dots,k\} \tag{1.12}$$

$$\frac{t_l - t_i}{t_l - t_k}\alpha_i + \frac{t_i - t_k}{t_l - t_k}\beta_i S(t_i^-) \text{ for } i \in \{k+1,\dots,l-1\} \tag{1.13}$$

$$\beta_i S(t_i^-) \text{ for } i \in \{l,\dots,n\} \tag{1.14}$$

It can be useful to express the blending in a more trader-friendly manner. Borrowing from Reghai [306], we can define the cash and proportional parts from a mixing

[4] Also called flat forward interpolation.

weight w_i and a total dividend amount d_i expected at t_i by:

$$\beta_i = w_i \frac{d_i}{F(t, t_i^-)}, \tag{1.15}$$

$$\alpha_i = (1 - w_i) d_i. \tag{1.16}$$

A mixing weight $w_i = 0$ corresponds to a pure cash dividend of amount d_i while a mixing weight $w_i = 1$ corresponds to a pure proportional dividend of absolute expected amount d_i. A trader can then enter a schedule of discrete dividend amounts, typically forecasted by applying a growth yield on the current or past year dividends, along the corresponding mixing weights.

1.6 TRADING REPO VIA A TOTAL RETURN SWAP

A total return swap (TRS) is an over-the-counter derivative product where the equity amount payer makes floating payments equal to the total return of an equity (or more often of an equity index) and receives amounts either corresponding to a fixed or a floating rate in exchange. The total return includes the equity index return as well as the dividends.

In case of physical delivery, the equity amount payer receives the underlying asset at the trade start date, to return it at maturity. During the life of the swap, the equity amount payer then pays only the dividends to the receiver, either at each dividend payment dates, or at the end of each period of the swap, or by adjusting the settlement price at each period if the dividends are reinvested.

For a cash-settled multi-period TRS with m interest periods, the payer receives the standard Libor swap floating leg with spread s

$$\begin{aligned} V_{\text{payer}}(t) &= N \sum_{i=1}^{m} B(t, T_i^p) \left(\mathbb{E}_{Q_{T_i}} \left[L(T_{i-1}, T_i) + s \right] \delta_i \right) \\ &= N \sum_{i=1}^{m} B(t, T_i^p) \left(L(t, T_{i-1}, T_i) \delta_i + s \delta_i \right) \\ &= N \sum_{i=1}^{m} B(t, T_i^p) \left(\frac{B_L(t, T_{i-1})}{B_L(t, T_i)} - 1 + s \delta_i \right), \end{aligned}$$

where δ_i denotes the accrual period starting at T_{i-1} and ending at T_i in the relevant daycount convention, B_L is the pseudo-discount factor associated to the relevant Libor rate. The Libor rate is typically fixed two days before the start of each period. The receiver receives the performance for the n performance periods

$$V_{\text{receiver}}(t) = N \sum_{i=1}^{n} B(t, T_i^p) \left(\mathbb{E}_{Q_{T_i}} \left[\frac{S(T_i) - S(T_{i-1}) + \sum_{T_{i-1} < t_k \leq T_i} \alpha_k 1(t_k^p)}{S(T_{i-1})} \right] \right), \tag{1.17}$$

when the TRS contract specifies that the dividends are paid at the dividend payment dates. Note that the dividends may be paid at the performance leg payment dates instead of the dividend payment dates, in which case the discounting needs to be adjusted accordingly. In terms of the equity forward price and the capitalization factor, Equation (1.17) becomes

$$V_{\text{receiver}}(t) = N\left\{\sum_{i=1}^{n} B(t,T_i^p)\left(C(T_{i-1},T_i)-1\right) + \sum_{T_i<t_k\leq T_{i+1}} \frac{\alpha_k B(t,T_k^p)}{F(t,T_{i-1})}\right\}. \tag{1.18}$$

When the interest leg involves a floating Libor rate plus a spread, the spread is then close to the repo spread. Indeed if we consider a single period[5] of 3 months, and neglect the dividend payment lag, the value of the swap of notional N is:

$$\begin{aligned} V(t) &= NB(t,T_p)\left(\mathbb{E}_{Q_T}\left[L(t,T)+s\right]\delta_{T-t} - \frac{1}{S(0)}\mathbb{E}_{Q_T}\left[S(T)-S(0)+\sum_{t<t_i\leq T}\alpha_i\right]\right) \\ &\approx NB(t,T_p)\left(e^{(\bar{r}_F(t,T)+s)(T-t)} - e^{(\bar{r}_F(t,T)-\bar{s}_R(t,T))(T-t)}\right) \\ &= NB(t,T_p)e^{\bar{r}_F(t,T)(T-t)}\left(e^{s(T-t)} - e^{-\bar{s}_R(t,T)(T-t)}\right), \end{aligned} \tag{1.19}$$

where T is the three months maturity, Q_T is the T-forward risk-neutral measure, T_p is the settlement date, and B is the price of a zero coupon bond paying 1 unit of currency, $L(t,T)$ is the three months Libor rate, $\bar{r}_F(t,T)$ is the continuously compounded risk free rate (corresponding here to the 3 months Libor rate) and $\bar{r}_R(t,T)$ is the continuously compounded borrow spread. The swap value is approximately zero when $s=-s_R$.

We discussed here the most simple total return swap. In reality, there is a wide variety of equity swaps: some don't include dividends, some have a varying notional (the amortising swaps), some are on a foreign stock (quanto swaps), some are exotic derivatives with knock-out or best-of assets features, etc..

1.7 FURTHER READING

For the interested reader, we recommend to look at some of the referenced material, especially

- John Hull *Option, Futures and Other Derivatives* [176] for everything about rates and daycount conventions.
- Vladimir Piterbarg *Funding beyond discounting* [300] for the subtleties of the multi-curve world, and how it impacts equity derivatives.
- Marc Henrard *Interest Rate Modelling in the Multi-curve Framework* [164] for the

[5] Single period TRS are relatively common.

construction of the interest rate curves, which is key to define the continuously compounded rates $\bar{r}_F$, $\bar{r}_C$ and $\bar{r}_R$.

- Juan Ramirez *Handbook of corporate equity derivatives and equity capital markets* [305] for the concrete examples of forward contracts and equity linked swaps, along with flows and settlement details, even if the pricing examples are not always accurate.
- Hans Bühler *Volatility and dividends* [53] for a deeper overview of discrete dividends.