PHOTOCENTER SHIFT FOR A LOMMEL-SEELIGER ELLIPSOID

Disk-integrated brightness:

$$L = \int_{A_+} dA \; S(\mu_\odot, \mu_\oplus, \alpha)$$

Photocenter shift: $\qquad P_\perp \equiv 1 - \hat{e}_\oplus \hat{e}_\oplus$

$$\bar{\delta} = \langle P_\perp^T \bar{x} \rangle = \frac{1}{L} \, \hat{e}_\perp^T \int_{A_+} dA \; \bar{x} \, S(\mu_\odot, \mu_\oplus, \alpha)$$

$$= \frac{1}{L} \, P_\perp^T \, \frac{1}{3} \tilde{\omega} \, P_{11}(\alpha) \, \frac{Fabc}{A} \, |\bar{N}'_\odot| \, |\bar{N}'_\oplus| \cdot D^{-1} \sqrt{D} \, R^T \int_{\alpha' - \frac{\pi}{2} - \lambda}^{\frac{\pi}{2} - \lambda} d\varphi \begin{pmatrix} \cos(\varphi+\lambda) \\ \sin(\varphi+\lambda) \\ 0 \end{pmatrix} \frac{\cos\alpha' \cos^2(\varphi+\lambda) + \sin\alpha' \cos(\varphi+\lambda) \sin(\varphi+\lambda)}{\cos\varphi}$$

$$= \frac{1}{L} \, \frac{1}{3} \, \tilde{\omega} \, P_{11}(\alpha) \, \frac{Fabc}{A} \, |\bar{N}'_\odot| \, |\bar{N}'_\oplus| \cdot P_\perp^T D^{-\frac{1}{2}} R^T \cdot \begin{pmatrix} I_x \\ I_y \\ I_z \end{pmatrix}.$$

where $I_z = 0$ and

$$\begin{cases} I_x = \displaystyle\int_{\alpha' - \frac{\pi}{2} - \lambda}^{\frac{\pi}{2} - \lambda} d\varphi \; \cos(\varphi+\lambda) \, \frac{\cos\alpha' \cos^2(\varphi+\lambda) + \sin\alpha' \cos(\varphi+\lambda) \sin(\varphi+\lambda)}{\cos\varphi} \\[2ex] I_y = \displaystyle\int_{\alpha' - \frac{\pi}{2} - \lambda}^{\frac{\pi}{2} - \lambda} d\varphi \; \sin(\varphi+\lambda) \, \frac{\cos\alpha' \cos^2(\varphi+\lambda) + \sin\alpha' \cos(\varphi+\lambda) \sin(\varphi+\lambda)}{\cos\varphi} \end{cases}$$

$Q_0 \equiv$

$$\cos\alpha' \cos^2(\varphi+\lambda) + \sin\alpha' \cos(\varphi+\lambda)\sin(\varphi+\lambda) =$$

(p. 5, LD1)

$$\cos^2\varphi \cos\lambda \cos(\lambda-\alpha') - \cos\varphi \sin\varphi \cos\lambda \sin(\lambda-\alpha')$$
$$- \sin\varphi \cos\varphi \sin\lambda \cos(\lambda-\alpha') + \sin^2\varphi \sin\lambda \sin(\lambda-\alpha')$$
$$= \cos^2\varphi \cos(\lambda+(\lambda-\alpha')) - \cos\varphi \sin\varphi \sin(\lambda+(\lambda-\alpha'))$$
$$+ \sin\lambda \sin(\lambda-\alpha')$$
$$= \cos^2\varphi \cos(2\lambda-\alpha') - \cos\varphi \sin\varphi \sin(2\lambda-\alpha') + \sin\lambda \sin(\lambda-\alpha')$$

$$\cos(\varphi+\lambda) = \cos\varphi \cos\lambda - \sin\varphi \sin\lambda \equiv Q_x$$
$$\sin(\varphi+\lambda) = \sin\varphi \cos\lambda + \cos\varphi \sin\lambda \equiv Q_y$$

$$Q_x Q_0 = \cos^3\varphi \cos\lambda \cos(2\lambda-\alpha') - \cos^2\varphi \sin\varphi \cos\lambda \sin(2\lambda-\alpha')$$
$$+ \cos\varphi \cos\lambda \sin\lambda \sin(\lambda-\alpha')$$
$$- \cos^2\varphi \sin\varphi \sin\lambda \cos(2\lambda-\alpha') + \cos\varphi \overbrace{\sin^2\varphi}^{=1-\cos^2\varphi} \sin\lambda \sin(2\lambda-\alpha')$$
$$- \sin\varphi \sin^2\lambda \sin(\lambda-\alpha')$$
$$= \cos^3\varphi \cos(3\lambda-\alpha') - \cos^2\varphi \sin\varphi \sin(3\lambda-\alpha')$$
$$+ \cos\varphi \left[\cos\lambda \sin\lambda \sin(\lambda-\alpha') + \sin\lambda \sin(2\lambda-\alpha')\right]$$
$$- \sin\varphi \sin^2\lambda \sin(\lambda-\alpha')$$

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$$Q_y Q_0 = \cos^3\varphi \sin\lambda \cos(2\lambda-\alpha') - \cos^2\varphi \sin\varphi \sin\lambda \sin(2\lambda-\alpha')$$

$$+ \cos\varphi \sin^2\lambda \sin(\lambda-\alpha')$$

$$+ \cos^2\varphi \sin\varphi \cos\lambda \cos(2\lambda-\alpha') - \cos\varphi \overbrace{\sin^2\varphi}^{=1-\cos^2\varphi} \cos\lambda \sin(2\lambda-\alpha')$$

$$+ \sin\varphi \cos\lambda \sin\lambda \sin(\lambda-\alpha')$$

$$= \cos^3\varphi \sin(3\lambda-\alpha') + \cos^2\varphi \sin\varphi \cos(3\lambda-\alpha')$$

$$+ \cos\varphi \left[\sin^2\lambda \sin(\lambda-\alpha') - \cos\lambda \sin(2\lambda-\alpha') \right]$$

$$+ \sin\varphi \cos\lambda \sin\lambda \sin(\lambda-\alpha')$$

%.

The remaining integrals are as follows:

$$I_1 = \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \cos^2\varphi = \frac{1}{2} \Big/_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} \left(\frac{1}{2}\sin 2\varphi + \varphi\right) = \frac{1}{2}\left[\pi - \alpha + \frac{1}{2}\sin 2\lambda - \frac{1}{2}\sin 2(\lambda-\alpha')\right]$$

$$I_2 = \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \underbrace{\cos\varphi \sin\varphi}_{=\frac{1}{2}\sin 2\varphi} = \frac{1}{2}\left[\sin^2\left(\frac{\pi}{2}-\lambda\right) - \sin^2\left(\alpha'-\frac{\pi}{2}-\lambda\right)\right]$$

$$= \frac{1}{2}\left(\cos^2\lambda - \cos^2(\lambda-\alpha')\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2\lambda + \cancel{\frac{1}{2}} - \frac{1}{2}\cos 2(\lambda-\alpha') - \cancel{\frac{1}{2}}\right)$$

$$I_3 = \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi = \frac{\pi}{2} - \lambda - \alpha' + \frac{\pi}{2} + \lambda = \pi - \alpha'$$

$$I_4 = \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \frac{\sin\varphi}{\cos\varphi} = \Big/_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} -\ln(\cos\varphi) = -\ln\left[\cos\left(\frac{\pi}{2}-\lambda\right)\right] + \ln\left[\cos\left(\alpha'-\frac{\pi}{2}-\lambda\right)\right]$$

$$= \ln\left(\frac{-\sin(\lambda-\alpha')}{\sin\lambda}\right)$$

%.

Therefore,

$$I_x = I_1 \cos(3\lambda-\alpha') - I_2 \sin(3\lambda-\alpha')$$
$$+ I_3 \left[\cos\lambda \sin\lambda \sin(\lambda-\alpha') + \sin\lambda \sin(2\lambda-\alpha')\right]$$
$$- I_4 \sin^2\lambda \sin(\lambda-\alpha')$$

$$I_y = I_1 \sin(3\lambda-\alpha') + I_2 \cos(3\lambda-\alpha')$$
$$+ I_3 \left[\sin^2\lambda \sin(\lambda-\alpha') - \cos\lambda \sin(2\lambda-\alpha')\right]$$
$$+ I_4 \cos\lambda \sin\lambda \sin(\lambda-\alpha')$$

Cleaning up:

$$I_x = (\pi-\alpha')\left[\overset{= \frac{1}{2}[\cos 2\lambda \cos(\lambda-\alpha') - \sin 2\lambda \sin(\lambda-\alpha')]}{\frac{1}{2}\cos(3\lambda-\alpha')} + \overset{= \frac{1}{2}\sin 2\lambda}{\cos\lambda \sin\lambda} \sin(\lambda-\alpha') + \sin\lambda \sin(2\lambda-\alpha')\right]$$
$$+ \frac{1}{4}\left[\cos(3\lambda-\alpha')\left(\sin 2\lambda - \sin 2(\lambda-\alpha')\right) - \sin(3\lambda-\alpha')\cdot\left(\cos 2\lambda - \cos 2(\lambda-\alpha')\right)\right]$$
$$- \ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right) \cdot \sin^2\lambda \sin(\lambda-\alpha')$$

$$= (\pi-\alpha')\left[\frac{1}{2}\cos 2\lambda \cos(\lambda-\alpha') + \sin\lambda \sin(2\lambda-\alpha')\right]$$
$$+ \frac{1}{4}\left[\sin(2\lambda-(3\lambda-\alpha')) + \sin(3\lambda-\alpha'-2(\lambda-\alpha'))\right]$$
$$- \ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right) \sin^2\lambda \sin(\lambda-\alpha')$$

$$= (\pi-\alpha')\left[\frac{1}{2}\cos^2\lambda \cos(\lambda-\alpha') - \frac{1}{2}\sin^2\lambda \cos(\lambda-\alpha')\right.$$
$$\left. + \sin^2\lambda \cos(\lambda-\alpha') + \sin\lambda \cos\lambda \sin(\lambda-\alpha')\right]$$
$$+ \frac{1}{4}\left[-\sin(\lambda-\alpha') + \sin(\lambda+\alpha')\right] - \ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right) \sin^2\lambda \sin(\lambda-\alpha')$$

$$\Rightarrow I_x = \frac{1}{2}(\pi-\alpha')\left[\cos(\lambda-\alpha') + \sin 2\lambda \sin(\lambda-\alpha')\right]$$

$$-\frac{1}{4}\sin(\lambda-\alpha') + \frac{1}{4}\sin(\lambda+\alpha') - \ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right)\sin^2\lambda \sin(\lambda-\alpha')$$

$$I_y = (\pi-\alpha')\Big[\underbrace{\frac{1}{2}\sin(3\lambda-\alpha')}_{=\frac{1}{2}[\sin 2\lambda\cos(\lambda-\alpha')+\cos 2\lambda \sin(\lambda-\alpha')]} + \underbrace{\sin^2\lambda}_{=\frac{1}{2}(1-\cos 2\lambda)} \sin(\lambda-\alpha') - \cos\lambda \sin(2\lambda-\alpha')\Big]$$

$$+\frac{1}{4}\left[\sin(3\lambda-\alpha')\left(\sin 2\lambda - \sin 2(\lambda-\alpha')\right) + \cos(3\lambda-\alpha')\left(\cos 2\lambda - \cos 2(\lambda-\alpha')\right)\right]$$

$$+\ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right)\cdot\cos\lambda\sin\lambda\sin(\lambda-\alpha')$$

$$= (\pi-\alpha')\left[\frac{1}{2}\sin 2\lambda\cos(\lambda-\alpha') - \cos\lambda\sin(2\lambda-\alpha') + \frac{1}{2}\sin(\lambda-\alpha')\right]$$

$$+\frac{1}{4}\left[\cos(3\lambda-\alpha'-2\lambda) - \cos(3\lambda-\alpha'-2(\lambda-\alpha'))\right]$$

$$+\ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right)\cos\lambda\sin\lambda\sin(\lambda-\alpha')$$

$$= (\pi-\alpha')\Big[\sin\lambda\cos\lambda\cos(\lambda-\alpha') - \cos\lambda\left(\sin\lambda\cos(\lambda-\alpha') + \cos\lambda\sin(\lambda-\alpha')\right) + \frac{1}{2}\sin(\lambda-\alpha')\Big]$$

$$+\frac{1}{4}\left[\cos(\lambda-\alpha') - \cos(\lambda+\alpha')\right]$$

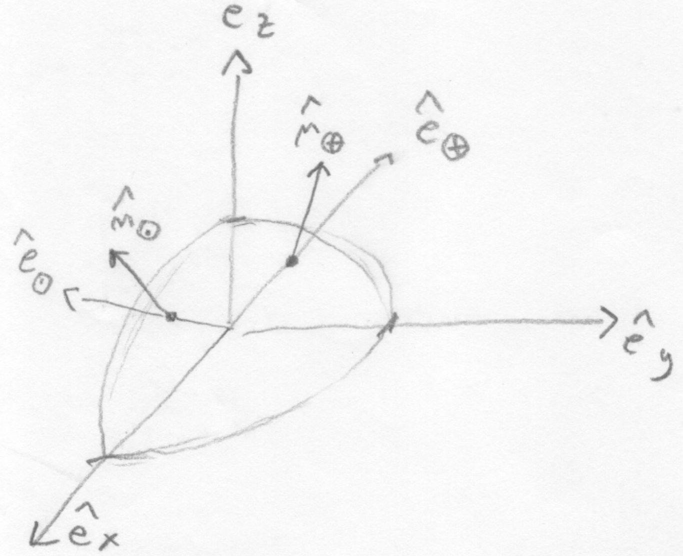
$$+\ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right)\cos\lambda\sin\lambda\sin(\lambda-\alpha')$$

$$= \frac{1}{2}(\pi-\alpha')\Big[\overbrace{+\sin(\lambda-\alpha') - (1+\cos 2\lambda)\sin(\lambda-\alpha')}^{=-\cos 2\lambda\sin(\lambda-\alpha')}\Big]$$

$$+\frac{1}{4}\cos(\lambda-\alpha') - \frac{1}{4}\cos(\lambda+\alpha')$$

$$+\ln\left(-\frac{\sin(\lambda-\alpha')}{\sin\lambda}\right)\cos\lambda\sin\lambda\sin(\lambda-\alpha')$$

Rotation matrix R



$$\hat{e}_\odot \cdot \hat{e}_\oplus = \cos\alpha$$
$$\hat{m}_\odot \cdot \hat{m}_\oplus = \cos\alpha'$$

$$\begin{cases} \hat{e}_{x'} = \hat{m}_\odot \\ \hat{e}_{z'} = \dfrac{\hat{m}_\odot \times \hat{m}_\oplus}{|\hat{m}_\odot \times \hat{m}_\oplus|} \end{cases}$$

<u>Euler</u>: $\cos\beta_E = \hat{e}_z \cdot \hat{e}_{z'} = \hat{e}_z \cdot \dfrac{\hat{m}_\odot \times \hat{m}_\oplus}{|\hat{m}_\odot \times \hat{m}_\oplus|}$

$$\begin{cases} \hat{e}_{z'} \cdot \hat{e}_x = \sin\beta_E \cos\gamma_E \\ \hat{e}_{z'} \cdot \hat{e}_y = \sin\beta_E \sin\gamma_E \end{cases}$$

$$\begin{cases} \hat{e}_{x''} \cdot \hat{m}_\odot = \cos\alpha_E \\ \hat{e}_{y''} \cdot \hat{m}_\odot = \sin\alpha_E \end{cases}$$

$$\begin{cases} \hat{e}_{y''} = \cos(\gamma_E + \frac{\pi}{2})\,\hat{e}_x + \sin(\gamma_E + \frac{\pi}{2})\,\hat{e}_y \\ \hat{e}_{x''} = \hat{e}_{y''} \times \hat{e}_{z'} \end{cases}$$

$$\Rightarrow \begin{cases} \hat{e}_{y''} = -\sin\gamma_E\,\hat{e}_x + \cos\gamma_E\,\hat{e}_y \\ \hat{e}_{x''} = \hat{e}_{y''} \times \hat{e}_{z'} \end{cases}$$

Appendix:

$$D\bar{x} = \sqrt{D}\,\hat{r}$$

$$\Rightarrow \quad \bar{x} = D^{-1}\sqrt{D}\,\hat{r}$$

$$\hat{r} = R^T \hat{r}'$$

$$\Rightarrow \quad \bar{x} = D^{-1}\sqrt{D}\,R^T \hat{r}'$$

$$\int_0^\pi d\gamma' \sin\gamma' \cos\gamma' = \frac{1}{2}\int_0^\pi d\gamma' \sin 2\gamma'$$

$$= \frac{1}{2} \Big/_0^\pi -\frac{1}{2}\cos 2\gamma'$$

$$= 0$$

$$\int_0^\pi d\gamma' \sin^3\gamma'$$

$$= \int_0^\pi d\gamma' \sin\gamma' (1-\cos^2\gamma')$$

$$= \Big/_0^\pi -\cos\gamma' + \cos^3\gamma' \cdot \frac{1}{3}$$

$$= +1+1-\frac{1}{3}-\frac{1}{3} = \frac{4}{3}$$

$$\int_0^\pi d\gamma' \sin^2\gamma' \cos\gamma'$$

$$= \Big/_0^\pi \frac{1}{3}\sin^3\gamma' = 0$$

$$D^{-1} = \mathrm{diag}(a^2, b^2, c^2)$$

$$\sqrt{D} = \mathrm{diag}\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

$$D^{-1}\sqrt{D} = \mathrm{diag}(a, b, c) \equiv D^{-\frac{1}{2}}$$

$$\frac{2}{\pi} \cdot \frac{4}{3} = \frac{8}{3\pi}$$

$$\frac{8}{3\pi} \cdot \frac{\pi}{8} = \frac{1}{3}$$

$$-\frac{d}{d\varphi} \ln(\cos\varphi) = -\frac{1}{\cos\varphi} \cdot -\sin\varphi$$

$$-\frac{1}{2}\sin 2\left(\alpha' - \frac{\pi}{2} - \lambda\right) = -\frac{1}{2}\sin(2\alpha' - 2\lambda - \pi) = +\frac{1}{2}\sin(2\alpha' - 2\lambda)$$

$$= -\frac{1}{2}\sin 2(\lambda - \alpha')$$

$$\bar{N}'^{T}_{\odot}\bar{N}'_{\odot} = \hat{e}'^{T}_{\odot} R\sqrt{D}R^{T} R\sqrt{D}R^{T}\hat{e}'_{\odot}$$

$$= \hat{e}'^{T}_{\odot} R D R^{T} \hat{e}'_{\odot} = \hat{e}^{T}_{\odot} D\, \hat{e}_{\odot}$$

$$\cos(\lambda-\alpha') + \cos(\lambda-\alpha'+2\lambda-2\alpha')$$

$$= \cos(\lambda-\alpha') + \cos(\lambda-\alpha')\cos(2\lambda-\alpha') - \sin(\lambda-\alpha')\sin(2\lambda-\alpha')$$