DISK-INTEGRATED BRIGHTNESS OF A LOMMEL-SEELIGER ELLIPSOID

Disk-integrated brightness:

$$L = \int_{A_+} dA \, S(\mu_\odot, \mu_\oplus, \alpha),$$

where S is the scattering law, $\mu_\odot = \cos\varepsilon$ and $\mu_\oplus = \cos\imath$ ($\varepsilon, \imath$ are the local angles of emergence and incidence), α is the phase angle, and A_+ is the surface area both illuminated by the Sun and observable at the Earth.

Triaxial ellipsoid (reference frame K):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \iff \bar{x}^T D \bar{x} = 1, \quad D = \mathrm{diag}\left(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}\right)$$

Transformation to coordinates ϑ, φ:

$$\begin{cases} x = a \sin\vartheta \cos\varphi \\ y = b \sin\vartheta \sin\varphi \\ z = c \cos\vartheta \end{cases}$$

$$\begin{cases} dA = \Big| \underbrace{\dfrac{\partial \bar{x}}{\partial \vartheta} \times \dfrac{\partial \bar{x}}{\partial \varphi}}_{\equiv \bar{J}(\vartheta,\varphi)} \Big| \, d\vartheta \, d\varphi = J(\vartheta,\varphi) \, d\vartheta \, d\varphi, \\ J(\vartheta,\varphi) = |\bar{J}(\vartheta,\varphi)| \end{cases}$$

Partial derivatives:

$$\frac{\partial \bar{x}}{\partial \vartheta} = \begin{pmatrix} a \cos\vartheta \cos\varphi \\ b \cos\vartheta \sin\varphi \\ -c \sin\vartheta \end{pmatrix}, \quad \frac{\partial \bar{x}}{\partial \varphi} = \begin{pmatrix} -a \sin\vartheta \sin\varphi \\ b \sin\vartheta \cos\varphi \\ 0 \end{pmatrix}$$

Jacobian:

$$\bar{J}(\vartheta,\varphi) = \begin{pmatrix} bc\sin^2\vartheta\cos\varphi \\ ac\sin^2\vartheta\sin\varphi \\ ab\sin\vartheta\cos\vartheta \end{pmatrix} = abc\sin\vartheta \begin{pmatrix} \frac{1}{a}\sin\vartheta\cos\varphi \\ \frac{1}{b}\sin\vartheta\sin\varphi \\ \frac{1}{c}\cos\vartheta \end{pmatrix}$$

$$\Rightarrow J(\vartheta,\varphi) = abc\sin\vartheta\sqrt{\frac{1}{a^2}\sin^2\vartheta\cos^2\varphi + \frac{1}{b^2}\sin^2\vartheta\sin^2\varphi + \frac{1}{c^2}\cos^2\vartheta}$$

Disk-integrated brightness using ϑ,φ:

$$L = \int_{\Omega_+} d\vartheta\, d\varphi\, abc\sin\vartheta\sqrt{\frac{1}{a^2}\sin^2\vartheta\cos^2\varphi + \frac{1}{b^2}\sin^2\vartheta\sin^2\varphi + \frac{1}{c^2}\cos^2\vartheta}\, S(\mu_\odot,\mu_\oplus,\alpha),$$

where Ω_+ denotes the angular regime both illuminated and observable.

The scattering law $S(\mu_\odot,\mu_\oplus,\alpha)$ is related to the reflection coefficient $R(\mu_\odot,\mu_\oplus,\alpha)$ through

$$S(\mu_\odot,\mu_\oplus,\alpha) = \mu_\odot\mu_\oplus R(\mu_\odot,\mu_\oplus,\alpha)\, F,$$

where πF is the incident flux density. For the Lommel-Seeliger scattering law,

$$\begin{cases} R(\mu_\odot,\mu_\oplus,\alpha) = \frac{1}{4}\tilde{\omega} P_{11}(\alpha)\frac{1}{\mu_\odot+\mu_\oplus}, \\ S(\mu_\odot,\mu_\oplus,\alpha) = \frac{1}{4}\tilde{\omega} P_{11}(\alpha) F \frac{\mu_\odot\mu_\oplus}{\mu_\odot+\mu_\oplus}, \end{cases}$$

where $\tilde{\omega}$ and P_{11} are the single-scattering albedo and phase function, respectively.

The unit outward surface normal vector is

$$\hat{n} = \frac{\bar{J}(\vartheta,\varphi)}{J(\vartheta,\varphi)} = \frac{D\bar{x}}{\sqrt{\bar{x}^T D^2 \bar{x}}} = \frac{\sqrt{D}\,\hat{r}}{\sqrt{\hat{r}^T D \hat{r}}},$$

where $\sqrt{D} = \mathrm{diag}(\frac{1}{a},\frac{1}{b},\frac{1}{c})$, $\hat{r} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)^T$.

The cosines $\mu_\odot$ and $\mu_\oplus$ are

$$\begin{cases} \mu_\odot = \hat{e}_\odot \cdot \hat{m} \\ \mu_\oplus = \hat{e}_\oplus \cdot \hat{m} \end{cases} \Leftrightarrow \begin{cases} \mu_\odot = \dfrac{\hat{e}_\odot^T \sqrt{D}\, \hat{r}}{\sqrt{\hat{r}^T D \hat{r}}} \\[2ex] \mu_\oplus = \dfrac{\hat{e}_\oplus^T \sqrt{D}\, \hat{r}}{\sqrt{\hat{r}^T D \hat{r}}} \end{cases}$$

The angular regime Ω_+ is determined by the conditions

$$\begin{cases} \mu_\odot \geq 0 \\ \mu_\oplus \geq 0 \end{cases} \Leftrightarrow \begin{cases} \hat{e}_\odot^T \sqrt{D}\, \hat{r} \geq 0 \\ \hat{e}_\oplus^T \sqrt{D}\, \hat{r} \geq 0 \end{cases}$$

Let R be a rotation to a new reference frame K': we have

$$\begin{cases} \hat{e}_\odot' = R\hat{e}_\odot \\ \hat{e}_\oplus' = R\hat{e}_\oplus \\ \hat{r}' = R\hat{r} \end{cases} \Leftrightarrow \begin{cases} \hat{e}_\odot = R^T \hat{e}_\odot' \\ \hat{e}_\oplus = R^T \hat{e}_\oplus' \\ \hat{r} = R^T \hat{r}' \end{cases}$$

The disk-integrated brightness

$$L = \int_{\Omega_+'} d\gamma' d\psi' abc \sin\gamma' \; \frac{1}{4}\tilde{\omega} P_{11}(\alpha) F \cdot$$

$$\frac{(\hat{e}_\odot'^T R\sqrt{D} R^T \hat{r}')(\hat{e}_\oplus'^T R\sqrt{D} R^T \hat{r}')}{\hat{e}_\odot'^T R\sqrt{D} R^T \hat{r}' + \hat{e}_\oplus'^T R\sqrt{D} R^T \hat{r}'}$$

$$= \frac{1}{4}\tilde{\omega} P_{11}(\alpha) F abc \int_{\Omega_+'} d\gamma' d\psi' \sin\gamma' \cdot$$

$$\frac{[(R\sqrt{D} R^T \hat{e}_\odot')^T \hat{r}'][(R\sqrt{D} R^T \hat{e}_\oplus')^T \hat{r}']}{(R\sqrt{D} R^T \hat{e}_\odot')^T \hat{r}' + (R\sqrt{D} R^T \hat{e}_\oplus')^T \hat{r}'},$$

where Ω_+' is the angular regime in γ', ψ'.

In order for the integral to be analytically calculated, the rotation R needs to be such, ~~for~~ example, that the normal vector at the position $\hat{e}'_{\odot}$, denoted by $\hat{n}'_{\odot}$, equals

$$\hat{n}'_{\odot} = \hat{e}_{x'}$$

and, similarly

$$\hat{n}'_{\oplus} = \cos\alpha' \hat{e}_{x'} + \sin\alpha' \hat{e}_{y'}, \quad \cos\alpha' = \hat{n}'_{\oplus} \cdot \hat{n}'_{\odot} = \hat{n}_{\oplus} \cdot \hat{n}_{\odot}.$$

Note that α' is not the phase angle. Denote further

$$\begin{cases} \bar{N}'_{\odot} = R\sqrt{D}R^T \hat{e}'_{\odot} \uparrow\uparrow \hat{n}'_{\odot}, & \bar{N}'^T_{\odot}\bar{N}'_{\odot} = \hat{e}'^T_{\odot} R\sqrt{D}R^T R\sqrt{D}R^T \hat{e}'_{\odot} = \hat{e}^T_{\odot} D \hat{e}_{\odot} \\ \bar{N}'_{\oplus} = R\sqrt{D}R^T \hat{e}'_{\oplus} \uparrow\uparrow \hat{n}'_{\oplus}, & \bar{N}'^T_{\oplus}\bar{N}'_{\oplus} = \hat{e}^T_{\oplus} D \hat{e}_{\oplus} \end{cases}$$

The integration bounds are as follows. For γ', the integration spans across $0 \le \gamma' \le \pi$. For ψ', the bounds are set by

$$\begin{cases} (R\sqrt{D}R^T \hat{e}'_{\odot})^T \begin{pmatrix} \cos\psi' \\ \sin\psi' \\ 0 \end{pmatrix} = 0 \\ (R\sqrt{D}R^T \hat{e}'_{\oplus})^T \begin{pmatrix} \cos\psi' \\ \sin\psi' \\ 0 \end{pmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} \bar{N}'^T_{\odot} \begin{pmatrix} \cos\psi' \\ \sin\psi' \\ 0 \end{pmatrix} = 0 \\ \bar{N}'^T_{\oplus} \begin{pmatrix} \cos\psi' \\ \sin\psi' \\ 0 \end{pmatrix} = 0 \end{cases}.$$

$$\Leftrightarrow \begin{cases} \cos\psi' = 0 \\ \cos\alpha'\cos\psi' + \sin\alpha'\sin\psi' = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \cos\psi' = 0 \\ \cos(\psi' - \alpha') = 0 \end{cases} \Leftrightarrow \begin{cases} \psi' = \pm\frac{\pi}{2} \\ \psi' - \alpha' = \pm\frac{\pi}{2} \end{cases}$$

We have

$$L = \frac{1}{4}\tilde{\omega} P_{11}(\alpha) F abc \underbrace{\int_0^{\pi} d\gamma' \sin^2\gamma'}_{=\frac{\pi}{2}} \int_{\alpha'-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi'$$

$$\frac{(N'_{\odot x}\cos\psi' + \overset{=0}{N'_{\odot y}}\sin\psi')(N'_{\oplus x}\cos\psi' + N_{\oplus y}\sin\psi')}{(N'_{\odot x}+N'_{\oplus x})\cos\psi' + (\underset{=0}{N'_{\odot y}} + N'_{\oplus y})\sin\psi'}$$

$$= \frac{\pi}{8}\tilde{\omega} P_{11}(\alpha) F abc \int_{\alpha'-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi' \; \frac{N'_{\odot x}N'_{\oplus x}\cos^2\psi' + N'_{\odot x}N'_{\oplus y}\cos\psi'\sin\psi'}{A\cos(\psi'-\lambda)},$$

where

$$\begin{cases} A^2 = (N'_{\odot x}+N'_{\oplus x})^2 + N'^{\,2}_{\oplus y} = \hat{e}_{\odot}^{T} D \hat{e}_{\odot} + \hat{e}_{\oplus}^{T} D \hat{e}_{\oplus} + 2\sqrt{\hat{e}_{\odot}^{T} D \hat{e}_{\odot}}\sqrt{\hat{e}_{\oplus}^{T} D \hat{e}_{\oplus}}\cos\alpha' \\ \cos\lambda = \dfrac{N'_{\odot x}+N'_{\oplus x}}{A} = \dfrac{1}{A}\left(\sqrt{\hat{e}_{\odot}^{T} D \hat{e}_{\odot}} + \sqrt{\hat{e}_{\oplus}^{T} D \hat{e}_{\oplus}}\cos\alpha'\right) \\ \sin\lambda = \dfrac{N'_{\oplus y}}{A} = \dfrac{1}{A}\sqrt{\hat{e}_{\oplus}^{T} D \hat{e}_{\oplus}}\sin\alpha' \end{cases}$$

Then, $\varphi \equiv \psi' - \lambda$,

$$L = \frac{\pi}{8}\tilde{\omega} P_{11}(\alpha) \frac{F abc N'_{\odot x}}{A} \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \; \frac{N'_{\oplus x}\cos^2(\varphi+\lambda) + N'_{\oplus y}\cos(\varphi+\lambda)\sin(\varphi+\lambda)}{\cos\varphi}.$$

$$= \frac{\pi}{8}\tilde{\omega} P_{11}(\alpha) \frac{F abc}{A} |\bar{N}'_{\odot}||\bar{N}'_{\oplus}| \underbrace{\int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \; \frac{\cos\alpha'\cos^2(\varphi+\lambda) + \sin\alpha'\cos(\varphi+\lambda)\sin(\varphi+\lambda)}{\cos\varphi}}_{\equiv I}$$

where

$$I = \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \; \frac{\cos(\varphi+\lambda)\cos(\varphi+\lambda-\alpha')}{\cos\varphi}$$

$$= \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} \frac{d\varphi}{\cos\varphi}\cdot\Big[\cos^2\varphi\cos\lambda\cos(\lambda-\alpha') - \cos\varphi\sin\varphi\cos\lambda\sin(\lambda-\alpha') - \sin\varphi\cos\varphi\sin\lambda\cos(\lambda-\alpha') + \sin^2\varphi\sin\lambda\sin(\lambda-\alpha')\Big]$$

$$= \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} \frac{d\varphi}{\cos\varphi} \left[\cos^2\varphi \cos(\lambda+(\lambda-\alpha')) - \cos\varphi \sin\varphi \sin(\lambda+(\lambda-\alpha')) + \sin\lambda \sin(\lambda-\alpha') \right]$$

$$= \left[\sin\left(\frac{\pi}{2}-\lambda\right) - \sin\left(\alpha'-\frac{\pi}{2}-\lambda\right)\right] \cos(2\lambda-\alpha') + \left[\cos\left(\frac{\pi}{2}-\lambda\right) - \cos\left(\alpha'-\frac{\pi}{2}-\lambda\right)\right] \sin(2\lambda-\alpha')$$

$$+ \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} d\varphi \, \frac{1}{\cos\varphi} \cdot \sin\lambda \sin(\lambda-\alpha')$$

$$= \left[\cos\lambda + \cos(\lambda-\alpha')\right] \cos(2\lambda-\alpha') + \left[\sin\lambda + \sin(\lambda-\alpha')\right] \sin(2\lambda-\alpha')$$

Note: Correction 20/6/2014

$$+ \sin\lambda \sin(\lambda-\alpha') \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} \frac{d\varphi}{\cos\varphi}$$

$$= \cos(\lambda-2\lambda+\alpha') + \cos(\lambda-\alpha'-2\lambda+\alpha') + \sin\lambda \sin(\lambda-\alpha') \int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} \frac{d\varphi}{\cos\varphi}$$

$$= \cos(\lambda-\alpha') + \cos\lambda + \sin\lambda \sin(\lambda-\alpha') \underbrace{\int_{\alpha'-\frac{\pi}{2}-\lambda}^{\frac{\pi}{2}-\lambda} \frac{d\varphi}{\cos\varphi}}_{\equiv I_0}$$

In order to calculate I_0, we note that

$$\frac{d}{d\varphi} \ln\left(\cot\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\right) = \frac{1}{\cot\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)} \frac{-1}{\sin^2\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)} \cdot \frac{-1}{2} = \frac{2\cdot\frac{1}{2}}{\sin\left(\frac{\pi}{2}-\varphi\right)} = \frac{1}{\cos\varphi}$$

Thus

$$I_0 = \ln\left[\cot\left(\frac{\pi}{4}-\left(\frac{\pi}{4}-\frac{\lambda}{2}\right)\right)\right] - \ln\left[\cot\left(\frac{\pi}{4}-\left(\frac{\alpha'}{2}-\frac{\pi}{4}-\frac{\lambda}{2}\right)\right)\right]$$

$$= \ln\left[\frac{\cot\frac{\lambda}{2}}{\cot\left(\frac{\pi}{2}+\frac{\lambda}{2}-\frac{\alpha'}{2}\right)}\right]$$

The disk-integrated brightness is thus

$$L = \frac{\pi}{8}\tilde{\omega} P_{11}(\alpha)\,\frac{F\,abc}{A}\,|\bar{N}'_{\odot}|\,|\bar{N}'_{\oplus}| \cdot$$

$$\left\{ \cos(\lambda-\alpha') + \cos\lambda + \sin\lambda\,\sin(\lambda-\alpha')\,\ln\left[\frac{\cot\frac{\lambda}{2}}{\cot\left(\frac{\pi}{2}+\frac{\lambda}{2}-\frac{\alpha'}{2}\right)}\right]\right\}$$

$$= \frac{\pi}{8}\tilde{\omega} P_{11}(\alpha)\,\frac{F\,abc}{A}\,\sqrt{\hat{e}_{\odot}^{T} D \hat{e}_{\odot}}\,\sqrt{\hat{e}_{\oplus}^{T} D \hat{e}_{\oplus}} \cdot$$

$$\left\{ \cos(\lambda-\alpha') + \cos\lambda + \sin\lambda\,\sin(\lambda-\alpha')\,\ln\left[\cot\frac{\lambda}{2}\cdot\cot\left(\frac{\alpha'-\lambda}{2}\right)\right]\right\}$$

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$$I = [\cos\lambda + \cos(\lambda-\alpha')]\cos(2\lambda-\alpha') + [\sin\lambda + \sin(\lambda-\alpha')]\sin(2\lambda-\alpha')$$
$$+ \sin\lambda\sin(\lambda-\alpha')\,I_0$$

$$= \cos(\lambda-(2\lambda-\alpha')) + \cos(\lambda-\alpha'-(2\lambda-\alpha')) + \sin\lambda\sin(\lambda-\alpha')\,I_0$$

$$= \cos(-\lambda+\alpha') + \cos(-\lambda) + \sin\lambda\sin(\lambda-\alpha')\,I_0$$

$$= \cos(\lambda-\alpha') + \cos\lambda + \sin\lambda\sin(\lambda-\alpha')\,I_0$$