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AMATH 583

Final Exam

1.) a.) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma}(x-\mu)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2} = g(z)$ where $g(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2}$, $z = \frac{1}{\sigma\sqrt{2}}(x-\mu)$

From the notes, we see that the shifting & scaling properties

$$\hat{f}(k) = \sigma\sqrt{2} \cdot e^{-ik\mu} \cdot \hat{g}(k)$$

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2} e^{-ikx} dx = \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-(x^2+ikx)} dx = \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-(x^2+ikx-k^2/4+k^2/4)} dx$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-(x+\frac{ik}{2})^2} e^{-k^2/4} dx = \frac{e^{-k^2/4}}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-(x+ik/2)^2} dx \quad , \quad \begin{array}{l} u = (x+\frac{ik}{2})^2 \\ du = 2(x+\frac{ik}{2})dx \Rightarrow dx = \frac{1}{2\sqrt{u}} du \end{array}$$

$$= \frac{e^{-k^2/4}}{2\pi\sigma} \int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{e^{-u}}{\sqrt{u}} du = \frac{e^{-k^2/4}}{4\pi\sigma} \int_{-\infty}^{\infty} \frac{e^{-u}}{\sqrt{u}} du = \frac{e^{-k^2/4}}{4\pi\sigma} \Gamma\left(\tfrac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}e^{-k^2/4}}{4\pi\sigma}}$$

b.) $f(t) = \sin\omega_0 t$, ω_0 constant

$$f(t) = \sin\omega_0 t = \frac{1}{2i}\left[e^{i\omega_0 t} - e^{-i\omega_0 t}\right] = \frac{1}{2i}\left[e^{i\omega_0 t} - e^{i(-\omega_0)t}\right]$$

From lecture we know $\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt$, so

$$\mathcal{F}[f(t)] = \mathcal{F}\left[\frac{1}{2i}\left(e^{i\omega_0 t} - e^{i(-\omega_0)t}\right)\right] = \frac{1}{2i}\left(\mathcal{F}\left[e^{i\omega_0 t}\right] - \mathcal{F}\left[e^{i(-\omega_0)t}\right]\right)$$

$$= -\frac{i}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\omega_0-k)t} dt - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(-\omega_0-k)t} dt\right] = -\frac{i}{2\sqrt{2\pi}}\left[2\pi\delta(\omega_0-k) - 2\pi\delta(-\omega_0-k)\right]$$

$$= \boxed{-i\frac{\sqrt{2\pi}}{2}\left(\delta(k-\omega_0) - \delta(k+\omega_0)\right)}$$

c.) $f(x) = e^{-a|x|}$, $a > 0$

Let $g(x) = \begin{cases} e^{-ax} , & x>0 \\ 0 , & x<0 \end{cases}$ & $h(x) = \begin{cases} e^{ax} , & x<0 \\ 0 , & x>0 \end{cases}$. Then $f(x) = g(x) + h(x)$.

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(a+ik)x} dx \quad , \quad \begin{array}{l} u = -(a+ik)x \qquad u(0)=0, u(\infty)=-\infty \\ du = -(a+ik)dx \Rightarrow dx = -\left(\frac{1}{a+ik}\right)du \end{array}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{-\infty} -\left(\frac{1}{a+ik}\right)e^u du = \frac{1}{\sqrt{2\pi}(a+ik)}\int_{-\infty}^{0} e^u du = \frac{1}{\sqrt{2\pi}(a+ik)}[1-0] = \frac{1}{\sqrt{2\pi}(a+ik)}$$

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{ax}e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a-ik)x} dx \quad , \quad \begin{array}{l} u = (a-ik)x \qquad u(-\infty)=-\infty, u(0)=0 \\ du = (a-ik)dx \Rightarrow dx = -\left(\frac{1}{a+ik}\right)du \end{array}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \frac{1}{(a-ik)} e^u du = \frac{1}{\sqrt{2\pi}(a-ik)}\int_{-\infty}^{0} e^u du = \frac{1}{\sqrt{2\pi}(a-ik)}[1-0] = \frac{1}{\sqrt{2\pi}(a-ik)}$$

$$\mathcal{F}[f(x)] = \mathcal{F}[g(x)+h(x)] = \mathcal{F}[g(x)] + \mathcal{F}[h(x)] = \frac{1}{\sqrt{2\pi}(a+ik)} + \frac{1}{\sqrt{2\pi}(a-ik)} = \boxed{\sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}}$$

1.) d.) $f(t) = \delta(t)$

Since $\delta(t) \cdot g(x) = g(0)$ for some function $g(x)$, we see that

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(t) e^{-ikt}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik\cdot 0}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt = \boxed{\frac{1}{\sqrt{2\pi}}}$$

odd on symmetric interval

2.) $p(t) = \begin{cases} 0, & t<0 \\ 1, & 0<t<1 \\ 0, & t>1 \end{cases}$, $q(t) = \begin{cases} 0, & t<0 \\ 1-t, & 0<t<1 \\ 0, & t>0 \end{cases}$

$$p \odot q = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p^*(\tau) q(t+\tau)\, d\tau = \frac{1}{\sqrt{2\pi}}\int_0^1 1\cdot(1-(t+\tau))\, d\tau = \frac{1}{\sqrt{2\pi}}\int_0^1 (1-t-\tau)\, d\tau$$

$$= \frac{1}{\sqrt{2\pi}}\left[\tau - \tau t - \tfrac{1}{2}\tau^2\right]_{\tau=0}^{1} = \frac{1}{\sqrt{2\pi}}\left(1 - t - \tfrac{1}{2}\right) = \boxed{\frac{\frac{1}{2}-t}{\sqrt{2\pi}}}$$

3.) $p(t) = \begin{cases} 0, & t<0 \\ 1, & 0<t<1 \\ 0, & t>0 \end{cases}$

$$p \odot p = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p^*(\tau) p(t+\tau)\, d\tau = \frac{1}{\sqrt{2\pi}}\int_0^1 1\cdot 1\, d\tau = \boxed{\frac{1}{\sqrt{2\pi}}}$$

4.) $\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}$

a.) Let $T(x,0) = f(x)$ & $\mathcal{F}[T(x,t)] = \tau(k,t)$. We see that

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} \xRightarrow{FT} \frac{\partial \tau}{\partial t} = -Kk^2\tau(k,t) \Rightarrow \frac{\partial \tau}{\partial t} + Kk^2\tau(k,t) = 0$$ ← first-order linear ODE, $p(k,t) = Kk^2$, $q(k,t) = 0$

$$\Rightarrow \mu(k,t) = e^{\int Kk^2 dt} = e^{Kk^2 t} \Rightarrow \tau(k,t) = \frac{\int e^{Kk^2 t}\cdot 0\, dt + C}{e^{Kk^2 t}} = Ce^{-Kk^2 t}$$

$$\Rightarrow \tau(k,0) = Ce^{-Kk^2\cdot 0} = C = \hat{f}(k) \Rightarrow \tau(k,t) = \hat{f}(k)e^{-Kk^2 t}$$

Let $\hat{g}(k,t) = e^{-Kk^2 t}$. Then $T(x,t) = f(x) * g(k,t)$. We see that

$$g(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-Kk^2 t} e^{ikx}\, dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-Kt\left(k^2 - \frac{ix}{Kt}\right)}\, dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-Kt\left(k^2 - 2\frac{ix}{Kt} - \frac{x^2}{4K^2t^2} + \frac{x^2}{4K^2t^2}\right)}\, dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-Kt\left(k-\frac{ix}{2Kt}\right)^2} e^{-\frac{x^2}{4Kt}}\, dk = \frac{e^{-\frac{x^3}{4Kt}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-Kt\left(k-\frac{ix}{2Kt}\right)^2}\, dk,$$ $u = k - \frac{ix}{2Kt}$, $du = dk$; $v^2 = Ktu^2$, $v = \sqrt{Kt}u$, $dv = \sqrt{Kt}\,du$

$$= \frac{e^{-x^2/4Kt}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-Ktu^2}\, du = \frac{e^{-x^2/4Kt}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{Kt}} e^{-v^2}\, dv = \frac{e^{-x^2/4Kt}}{\sqrt{2\pi Kt}}\int_{-\infty}^{\infty} e^{-v^2}\, dv$$

$$= \frac{e^{-x^3/4Kt}}{\sqrt{2\pi Kt}}\cdot\sqrt{\pi} = \boxed{\frac{e^{-x^3/4Kt}}{\sqrt{2Kt}}}$$

$$\Rightarrow T(x,t) = f(x) * \frac{e^{-x^3/4Kt}}{\sqrt{2K\tau}} = \frac{1}{\sqrt{2Kt}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^3/4Kt}\, dy$$

4.) b.) $k = 10^3$, $T(x,0) = \begin{cases} 0, & |x| \geq 1 \\ 100, & |x| \leq 1 \end{cases}$

$$T(x,t) = \frac{1}{\sqrt{2kt}} \int_{-\infty}^{\infty} T(y,0)\, e^{-(x-y)^2/4kt}\, dy = \frac{1}{\sqrt{2000t}} \int_{-1}^{1} 100 e^{-(x-y)^2/4000t}\, dy$$

$$= \frac{100}{\sqrt{2000t}} \int_{-1}^{1} e^{-\left(\frac{x-y}{\sqrt{4000t}}\right)^2} dy \qquad u = \frac{x-y}{\sqrt{4000t}} \quad du = -\frac{1}{\sqrt{4000t}}dy \Rightarrow dy = -\sqrt{4000t}\,du$$

$$= \frac{100}{\sqrt{2000t}} \int_{\frac{x+1}{\sqrt{4000t}}}^{\frac{x-1}{\sqrt{4000t}}} -\sqrt{4000t}\, e^{-u^2} du = \frac{100\sqrt{4000t}}{\sqrt{2000t}} \int_{\frac{x-1}{\sqrt{4000t}}}^{\frac{x+1}{\sqrt{4000t}}} e^{-u^2} du$$

$$= 100\sqrt{2} \cdot \left[\int_{\frac{x-1}{\sqrt{4000t}}}^{0} e^{-u^2} du + \int_{0}^{\frac{x+1}{\sqrt{4000t}}} e^{-u^2} du\right]$$

$$= 100\sqrt{2}\left[-\int_{0}^{\frac{x-1}{\sqrt{4000t}}} e^{-u^2} du + \int_{0}^{\frac{x+1}{\sqrt{4000t}}} e^{-u^2} du\right]$$

$$= 100\sqrt{2}\left[-\frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{x-1}{\sqrt{4000t}}\right) + \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{x+1}{\sqrt{4000t}}\right)\right]$$

$$= \boxed{\frac{100\sqrt{2\pi}}{2}\left(-\operatorname{erf}\left(\frac{x-1}{\sqrt{4000t}}\right) + \operatorname{erf}\left(\frac{x+1}{\sqrt{4000t}}\right)\right)}$$