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AMATH 584

Homework #1

1.1.) a.)

$$A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_5 A_3 A_2 B A_1 A_4 A_6 A_7$$

b.)

$$A = \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad ABC = A_5 A_3 A_2 B A_4 A_6 A_7$$

2.1.) Proof: Let A be a triangular and unitary matrix. Show A is diagonal. Let a_{ij} be an entry of A such that $i \neq j$. Then since A is triangular either $a_{ij} = 0$ or $a_{ji} = 0$. Since A is unitary we have $a_{ij} = \overline{a_{ji}}$. If $a_{ij} \neq 0$, then $a_{ij} = \overline{a_{ji}} = \overline{0} = 0$, so in all cases $a_{ij} = 0$. Thus all nondiagonal entries of A must be zero, so A is diagonal. $\boxtimes$

2.2.) Proof: Show $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$ where $i \in \{1, 2, \ldots, n\}$ and each x_i is orthogonal to every other x_i.

Base Case: $n=2$

$$\|x_1 + x_2\|^2 = (x_1+x_2)^*(x_1+x_2) = x_1^* x_1 + \cancelto{0}{x_1^* x_2} + \cancelto{0}{x_2^* x_1} + x_2^* x_2 = \|x_1\|^2 + \|x_2\|^2 \quad \square$$

Inductive Step: Assume $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$. Show $\|\sum_{i=1}^{n+1} x_i\|^2 = \sum_{i=1}^{n+1} \|x_i\|^2$.

$$\|\sum_{i=1}^{n+1} x_i\|^2 = \|\sum_{i=1}^{n} x_i + x_{n+1}\|^2 \overset{\text{Base Case}}{=} \|\sum_{i=1}^{n} x_i\|^2 + \|x_{n+1}\|^2 \overset{\text{Inductive Hypothesis}}{=} \sum_{i=1}^{n} \|x_i\|^2 + \|x_{n+1}\|^2 = \sum_{i=1}^{n+1} \|x_i\|^2 \quad \boxtimes$$

2.6.) Let $u, v \in \mathbb{C}^m$ and let $A = I + uv^*$ be nonsingular. Let $B = I + \alpha uv^*$. Then

$$AB = (I + uv^*)(I + \alpha uv^*) = II + uv^* + \alpha uv^* + uv^* \alpha uv^*$$

$$= I + uv^* + \alpha uv^* + \alpha(v^*u)uv^*$$

$$= I + uv^*(1 + \alpha + \alpha v^*u)$$

If $1 + \alpha + \alpha v^* u = 0$, then $B = A^{-1}$. This occurs when

$$1 + \alpha + \alpha v^* u = 0 \iff \alpha(1 + v^*u) = -1 \iff \alpha = \frac{-1}{1 + v^*u}$$

Thus if $\alpha = \frac{-1}{1+v^*u}$ then $I + \alpha uv^* = A^{-1}$.

This means if $v^*u = -1$ then A^{-1} does not exist, so A must be singular.

By definition $null(A) = \{x \in \mathbb{C}^m : Ax = 0\}$, so we see that if $v^*u = -1$ then

$$Ax = (I + uv^*)x = 0 \Rightarrow Ix + uv^*x = 0$$

$$\Rightarrow x + uv^*x = 0 \Rightarrow uv^*x = -x \Rightarrow uv^*x = v^*ux$$

Thus $null(I + uv^*) = \{x \in \mathbb{C}^m : uv^*x = v^*ux\}$.

```
A =

    1.0000   -1.0000    0.5000         0
         0    1.0000         0         0
         0   -1.0000    0.5000         0
         0   -1.0000         0    1.0000


C =

     0     0     0
     1     0     0
     0     1     1
     0     0     0


ABC =

     1     1     1
     2     2     2
    -1    -1    -1
     0     0     0


orig =

     2     2     2     2
     2     2     2     2
     2     2     2     2
     2     2     2     2


op1 =

     4     2     2     2
     4     2     2     2
     4     2     2     2
     4     2     2     2
```

```
op2 =

     4     2     2     2
     4     2     2     2
     2     1     1     1
     4     2     2     2


op3 =

     6     3     3     3
     4     2     2     2
     2     1     1     1
     4     2     2     2


op4 =

     3     3     3     6
     2     2     2     4
     1     1     1     2
     2     2     2     4


op5 =

     1     1     1     2
     2     2     2     4
    -1    -1    -1    -2
     0     0     0     0


op6 =

     1     1     1     1
     2     2     2     2
    -1    -1    -1    -1
     0     0     0     0
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```
op7 =

     1     1     1
     2     2     2
    -1    -1    -1
     0     0     0
```