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AMATH 584

# Homework #2

3.2.) <u>Proof:</u> Let $A \in \mathbb{C}^{m \times m}$ and let $\rho(A) = |\lambda|$ where $\lambda$ is the largest eigenvalue by magnitude of $A$. Show $\rho(A) \leq \|A\|_{(m,m)}$.
We know $\|A\|_{(m,m)} = \sup_{\substack{x \in \mathbb{C}^m \\ x \neq 0}} \frac{\|Ax\|_m}{\|x\|_m}$ and $Av = \lambda v$ for some $v \in \mathbb{C}^m$.
This means $\|Av\|_m = \|\lambda v\|_m \iff \|Av\|_m = |\lambda| \, \|v\|_m \iff \frac{\|Av\|_m}{\|v\|_m} = |\lambda|$.
We know $\rho(A) = |\lambda|$, so $\rho(A) = \frac{\|Av\|_m}{\|v\|_m}$. By definition of supremum we obtain $\rho(A) = \frac{\|Av\|_m}{\|v\|_m} \leq \sup_{\substack{x \in \mathbb{C}^m \\ x \neq 0}} \frac{\|Ax\|_m}{\|x\|_m} = \|A\|_{(m,m)}$ since eigenvectors cannot be equal to zero. Thus $\rho(A) = \|A\|_{(m,m)}$. $\boxtimes$

4.4.) True. If $A, B \in \mathbb{C}^{m \times m}$ are unitarily equivalent, then for some unitary $Q \in \mathbb{C}^{m \times m}$ $A = QBQ^*$. We can take the SVD of $B$ as $B = U_B \Sigma_B V_B^*$. This means we have

$$A = QBQ^* = Q(U_B \Sigma_B V_B^*)Q^* = (QU_B)\Sigma_B(V_B^* Q^*) = (QU_B)\Sigma_B(QV_B)^*.$$

Since $U_B$, $V_B$, and $Q$ are unitary we know $QU_B$ and $QV_B$ are unitary as well. Now we can take the SVD of $A$ as $A = U_A \Sigma_A V_A^*$. We see that this gives

$$U_A \Sigma_A V_A^* = (QU_B)\Sigma_B(QV_B)^*.$$

By the uniqueness of singular values we must have $\Sigma_A = \Sigma_B$. Thus $A$ and $B$ must have the same singular values if $A$ and $B$ are unitarily equivalent.

If we assume $A$ & $B$ have the same singular values, then let $\Sigma$ be the diagonal matrix containing these singular values. Then we see that

$$\begin{matrix} A = U_A \Sigma V_A^* \\ B = U_B \Sigma V_B^* \end{matrix} \Rightarrow \begin{matrix} U_A^* A V_A = \Sigma \\ U_B^* B V_B = \Sigma \end{matrix} \Rightarrow U_A^* A V_A = U_B^* B V_B \Rightarrow A = U_A U_B^* B V_B V_A^*.$$

Since $U_A$, $U_B$, $V_A$, and $V_B$ are all unitary we know $U_A U_B^*$ & $V_B V_A^*$ must also be unitary. Thus $A$ and $B$ must be unitarily equivalent if they have the same singular values.

Therefore $A$ & $B$ are unitarily equivalent iff they have the same singular values.

5.3.) a.)

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \Rightarrow \det\left(\begin{bmatrix} 104-\lambda & -72 \\ -72 & 146-\lambda \end{bmatrix}\right)$$

$$\Rightarrow (104-\lambda)(146-\lambda) - (-72)^2 = 0 \Rightarrow \lambda^2 - 250\lambda + 10000 = 0$$

$$\Rightarrow (\lambda - 200)(\lambda - 50) = 0 \Rightarrow \lambda_1 = 200, \ \lambda_2 = 50$$

$$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{200} & 0 \\ 0 & \sqrt{50} \end{bmatrix} = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$$

$\underline{\lambda = 200:}$

$$\begin{bmatrix} 104-200 & -72 \\ -72 & 146-200 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -96 & -72 \\ -72 & -54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -96x_1 - 72x_2 = 0 \\ -72x_1 - 54x_2 = 0 \end{matrix}$$

$$\Rightarrow x_1 = -\tfrac{3}{4}x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} -\tfrac{3}{4} & 1 \end{bmatrix}^T \cdot \frac{1}{\|\vec{x}_1\|} = \begin{bmatrix} -\tfrac{3}{4} & 1 \end{bmatrix}^T \cdot \frac{1}{5/4} = \begin{bmatrix} -\tfrac{3}{5} & \tfrac{4}{5} \end{bmatrix}^T$$

$\underline{\lambda = 50:}$

$$\begin{bmatrix} 104-50 & -72 \\ -72 & 146-50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 54 & -72 \\ -72 & 96 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 54x_1 - 72x_2 = 0 \\ 72x_1 + 96x_2 = 0 \end{matrix}$$

$$\Rightarrow x_1 = \tfrac{4}{3}x_2 \Rightarrow \vec{x}_2 = \begin{bmatrix} \tfrac{4}{3} & 1 \end{bmatrix}^T \cdot \frac{1}{\|\vec{x}_2\|} = \begin{bmatrix} \tfrac{4}{3} & 1 \end{bmatrix}^T \cdot \frac{1}{5/3} = \begin{bmatrix} \tfrac{4}{5} & \tfrac{3}{5} \end{bmatrix}^T$$

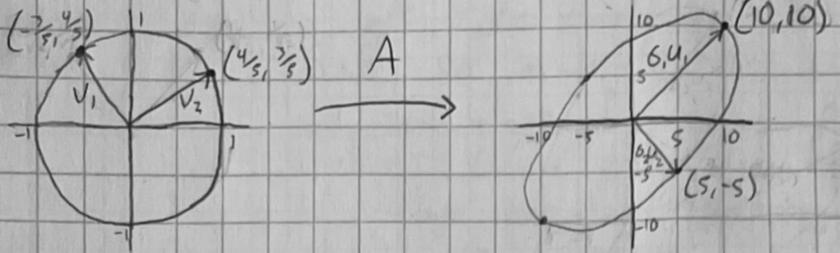
$$\Rightarrow V = \begin{bmatrix} -\tfrac{3}{5} & \tfrac{4}{5} \\ \tfrac{4}{5} & \tfrac{3}{5} \end{bmatrix} \Rightarrow A = U\Sigma V^* \Rightarrow AV = U\Sigma \Rightarrow AV\Sigma^{-1} = U$$

$$\Rightarrow U = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} -\tfrac{3}{5} & \tfrac{4}{5} \\ \tfrac{4}{5} & \tfrac{3}{5} \end{bmatrix}\begin{bmatrix} \tfrac{1}{10\sqrt{2}} & 0 \\ 0 & \tfrac{1}{5\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} -\tfrac{3}{50\sqrt{2}} & \tfrac{4}{25\sqrt{2}} \\ \tfrac{4}{50\sqrt{2}} & \tfrac{3}{25\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} & -\tfrac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow A = U\Sigma V^* = \begin{bmatrix} \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} & -\tfrac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}\begin{bmatrix} -\tfrac{3}{5} & \tfrac{4}{5} \\ \tfrac{4}{5} & \tfrac{3}{5} \end{bmatrix}$$

5.3.) b.) $\sigma_1 = 10\sqrt{2}$, $\sigma_2 = 5\sqrt{2}$, $u_1 = [\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}]^T$, $u_2 = [\frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$, $v_1 = [-\frac{3}{5}\ \frac{4}{5}]^T$, $v_2 = [\frac{4}{5}\ \frac{3}{5}]$

Unit ball in $\mathbb{R}^2$: Image of the unit ball under A:




$AV = U\Sigma$

c.) $\|A\|_1 = \max\{|-2|+|-10|, |11|+|5|\} = \max\{12, 16\} = 16$

$\|A\|_\infty = \max\{|-2|+|11|, |-10|+|5|\} = \max\{13, 15\} = 15$

$\|A\|_2 = \sigma_1 = 10\sqrt{2}$ (theorem 5.3)

$\|A\|_F = \sqrt{\sigma_1^2+\sigma_2^2} = \sqrt{(10\sqrt{2})^2+(5\sqrt{2})^2} = \sqrt{200+50} = 5\sqrt{10}$ (theorem 5.3)

d.)

$$A^{-1} = (U\Sigma V^*)^{-1} = (V^*)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^* = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} \frac{1}{20} & \frac{1}{20} \\ \frac{1}{10} & \frac{-1}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & \frac{-11}{100} \\ \frac{1}{10} & \frac{-1}{50} \end{bmatrix}$$

e.) $\det\left(\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \Rightarrow \det\left(\begin{bmatrix} -2-\lambda & 11 \\ -10 & 5-\lambda \end{bmatrix}\right) = 0 \Rightarrow (-2-\lambda)(5-\lambda)+110 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 100 = 0 \Rightarrow \lambda = \frac{3\pm\sqrt{9-4\cdot 100}}{2} = \frac{3\pm\sqrt{-391}}{2} = \frac{3}{2} \pm i\frac{\sqrt{391}}{2}$

$\Rightarrow \lambda_1 = \frac{3}{2} + i\frac{\sqrt{391}}{2}$, $\lambda_2 = \frac{3}{2} - i\frac{\sqrt{391}}{2}$

f.) $\det(A) = (-2)(5) - (11)(-10) = -10 - (-110) = 100$

$\lambda_1 \cdot \lambda_2 = \left(\frac{3}{2} + i\frac{\sqrt{391}}{2}\right)\left(\frac{3}{2} - i\frac{\sqrt{391}}{2}\right) = \frac{9}{4} + \frac{391}{4} = \frac{400}{4} = 100 = \det(A)$

$\sigma_1 \cdot \sigma_2 = 10\sqrt{2} \cdot 5\sqrt{2} = 10 \cdot 5 \cdot 2 = 100 = |\det(A)|$

g.) area $= \pi\sigma_1\sigma_2 = \pi \cdot 10\sqrt{2} \cdot 5\sqrt{2} = 100\pi$

4.) a.) The following table was generated in MATLAB:

| Matrix | Rank |
|---|---|
| A | 1 |
| B | 2 |
| C | 2 |
| D | 3 |
| E | 6 |

We see that all of these matrices are $6\times 6$, so $m=n=6$. For an arbitrary $X\in\mathbb{C}^{6\times 6}$ we see that if $X$ has $p$ nonzero singular values that only the first $p$ columns of $U$, the top-left $p\times p$ entries of $\Sigma$, and the first $p$ rows of $V^*$ are the only nonzero entries of these matrices if $X=U\Sigma V^*$.

This means we can remove all entries after the first $p$ of them for each of these matrices to obtain $U\in\mathbb{C}^{6\times p}$, $\Sigma\in\mathbb{C}^{p\times p}$, and $V^*\in\mathbb{C}^{p\times 6}$. Let $Z\in\mathbb{C}^{6\times p}$ be defined by $Z=U\Sigma$. Then $X=ZV^*$ where $Z\in\mathbb{C}^{6\times p}$ & $V\in\mathbb{C}^{6\times p}$, which means that $\text{rank}(X)=p$. This means $X$ is only low rank if $6\cdot\text{rank}(X)+6\cdot\text{rank}(X)<6\cdot 6 \Rightarrow 12\,\text{rank}(X)<36 \Rightarrow \text{rank}(X)<3$. Thus only flags A, B, and C are low rank.

b.) A is the only rank 1 flag and we see trivially that $a=[1\ 1\ 1\ 1\ 1\ 1]^T$ and $b=a$ for the low rank approximation $ab^T=aa^T=A$.

c.) B & C are the only rank 2 flags and we see that

$$B=ab^T+cd^T$$

$$a=\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\quad b=\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\quad c=\begin{bmatrix}1\\1\\0\\0\\1\\1\end{bmatrix}\quad d=\begin{bmatrix}-1\\-1\\0\\0\\-1\\-1\end{bmatrix}$$

$$C=ab^T+cd^T$$

$$a=\begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}\quad b=\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}\quad c=\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}\quad d=\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

4.) d.) We see that the low rank flags have many columns that are identical, and the full rank flag has no identical columns. The identity matrix also does not have any identical columns and is also full rank.

e.) The matrix seems to be full rank or at least close to it because it seems to have a diagonal structure, which makes it difficult for the columns to be linearly dependent. Also it seems that it is impossible for $B_{jk} = \frac{1}{x_j - y_k}$ to equal zero if $j=k$, so this leads me to believe that there are no zeros on the diagonal of B which implies $\det(B) \neq 0$. This means B must have full rank.

5.) a.) The plot suggests that the singular values of A seem to quickly tend to 0. The first singular value is about 535 and the fifth one is only $8.23 \cdot 10^{-5}$. They decay towards zero until they stop decreasing at $10^{-14}$ due to machine precision limitations.

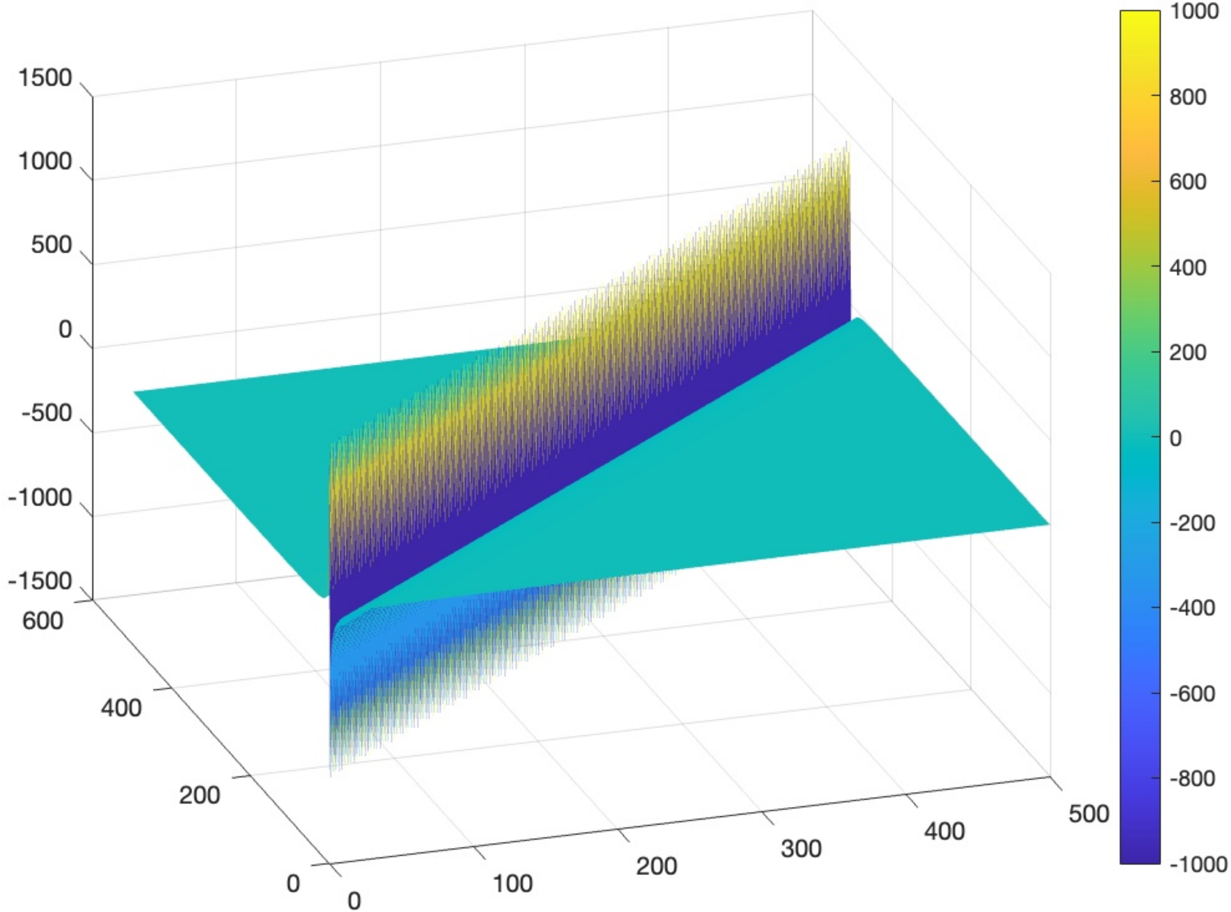
b.) The error plot seems to line up almost exactly with the magnitudes of the second through eleventh singular values of A. This makes sense due to theorem 5.8 which shows that the svd gives the best possible low rank approximation for A. The singular values tend to zero very quickly, which means that the svd low rank approximation will be able to capture most of the "energy" of A.

c.) I do not think there exists a rank 10 matrix that can effectively approximates B because the singular values remain of roughly equal magnitude for the first few hundred of them. By theorem 5.8 $A_v$ provides the best possible rank j approximation for B, so if $v=10$ we can find $\|A-A_v\|_2$ in MATLAB. This gives an error of 1570.8, which is the lower bound for the 2-norm error by theorem 5.8. According to MATLAB, the Frobenius norm error lower bound is 34676. From theorem 5.9 we know this is the lower bound of the rank 10 error in the Frobenius norm.

```matlab
%Problem 4 Part a

A = ones([6, 6]);
B = [0 0 1 1 0 0; 0 0 1 1 0 0; 1 1 1 1 1 1; 1 1 1 1 1 1; 0 0 1 1 0 0; 0 0 1 1 0 0];
C = [1 1 1 0 0 0; 1 1 1 1 1 1; 0 0 0 0 0 0; 1 1 1 1 1 1; 0 0 0 0 0 0; 1 1 1 1 1 1];
D = [1 0 0 0 0 1; 0 1 0 0 1 0; 0 0 1 1 0 0; 0 0 1 1 0 0; 0 1 0 0 1 0; 1 0 0 0 0 1];
E = [1 1 0 0 0 0; 1 1 1 0 0 0; 0 1 1 1 0 0; 0 0 1 1 1 0; 0 0 0 1 1 1; 0 0 0 0 1 1];

[Ua, Sa, Va] = svd(A);
[Ub, Sb, Vb] = svd(B);
[Uc, Sc, Vc] = svd(C);
[Ud, Sd, Vd] = svd(D);
[Ue, Se, Ve] = svd(E);

svAvec = diag(Sa);
svBvec = diag(Sb);
svCvec = diag(Sc);
svDvec = diag(Sd);
svEvec = diag(Se);

tol = 1e-14;
svA = svAvec(abs(svAvec)>tol);
svB = svBvec(abs(svBvec)>tol);
svC = svCvec(abs(svCvec)>tol);
svD = svDvec(abs(svDvec)>tol);
svE = svEvec(abs(svEvec)>tol);

Matrix = ["A"; "B"; "C"; "D"; "E"];
Rank = [length(svA); length(svB); length(svC); length(svD); length(svE)];

p4atable = table(Matrix, Rank)
```

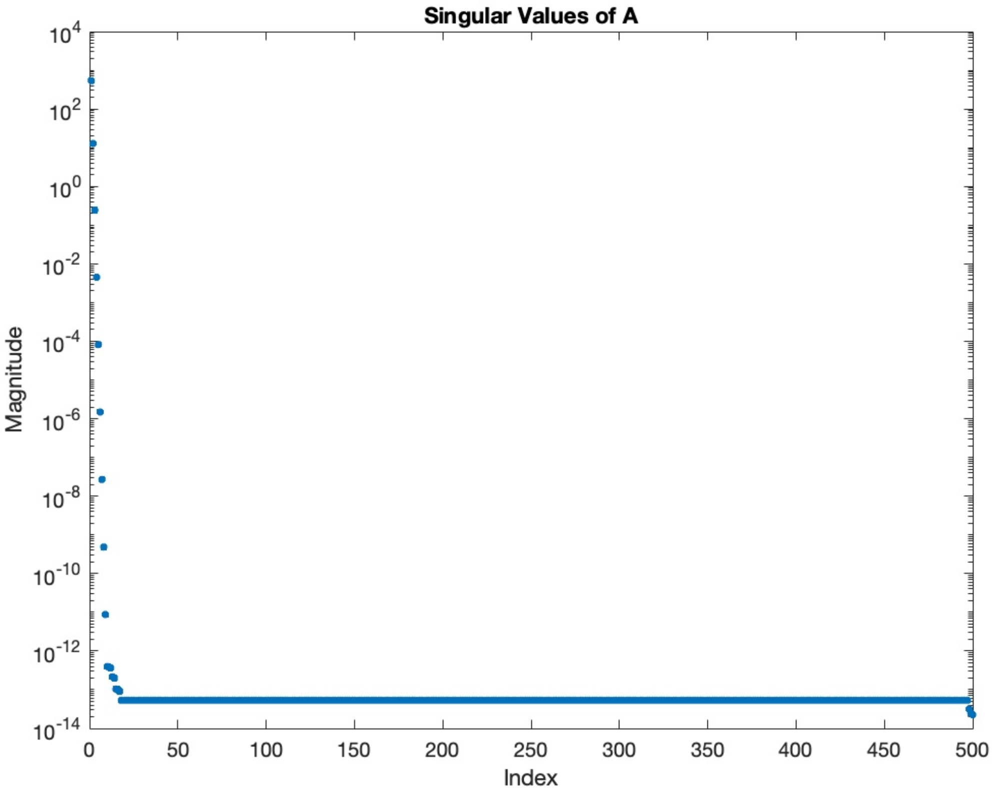
```
p4atable =

  5×2 table

    Matrix    Rank
    ______    ____

     "A"        1 
     "B"        2 
     "C"        2 
     "D"        3 
     "E"        6 
```

```
% Problem 4 Part e

m = 500;
x = zeros([m, 1]);
y = zeros([m, 1]);
for j = 1:m
    x(j) = (j - 1) / m;
    y(j) = (j + 1/2) / m;
end

B = zeros([m, m]);
for j = 1:m
    for k = 1:m
        B(j, k) = 1 / (x(j) - y(k));
    end
end

surf(B);
shading flat;
view(3), colorbar, shg
```



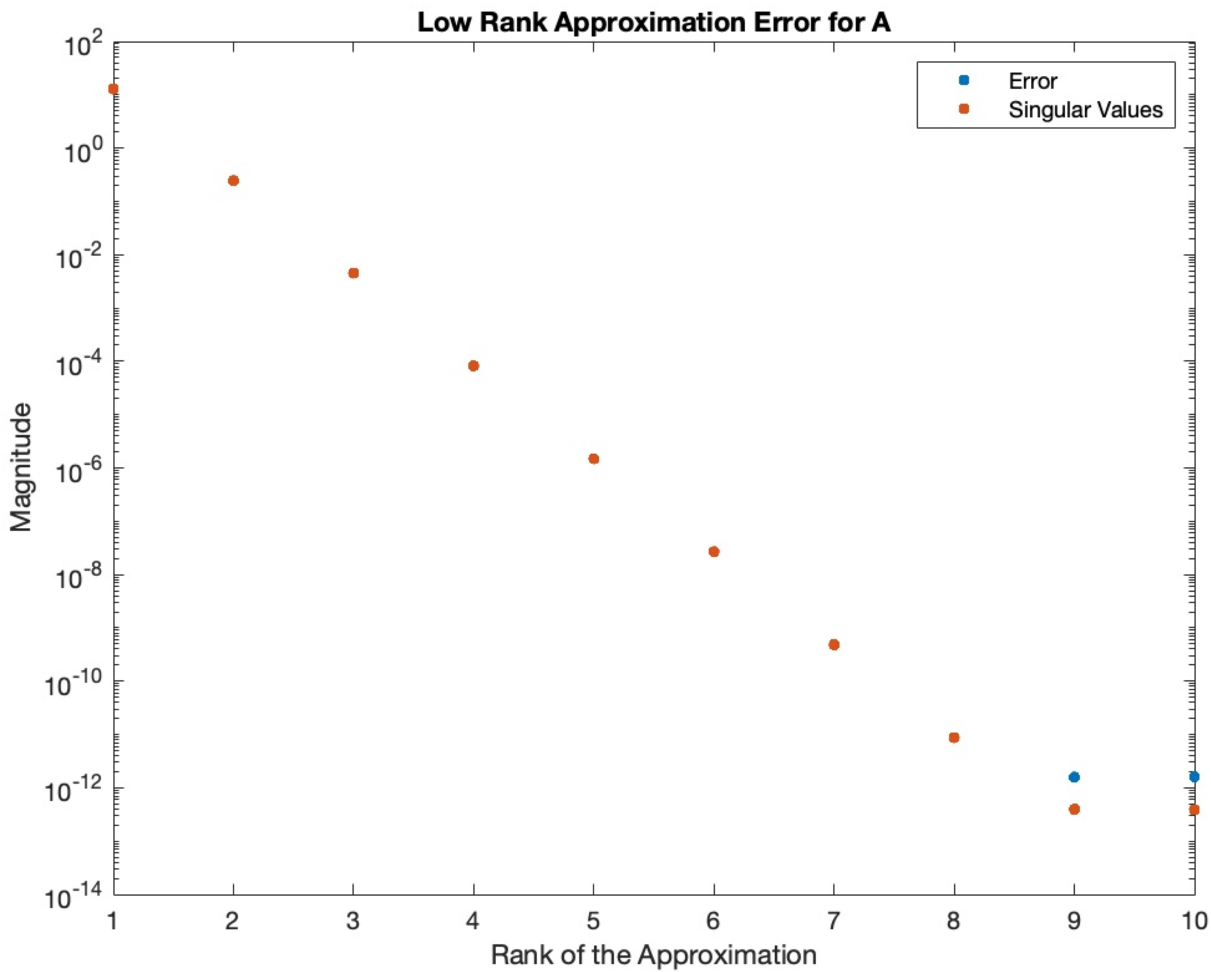
1500
1000
500
0
-500
-1000
-1500
600
400
200
0
0
100
200
300
400
500
1000
800
600
400
200
0
-200
-400
-600
-800
-1000

```matlab
%Problem 5 Part a

m = 500;
x = zeros([m, 1]);
y = zeros([m, 1]);

for j = 1:m
    x(j) = (j - 1) / (2 * m);
    y(j) = x(j) + (k + 1/2) / m;
end

A = zeros(m);
for j = 1:m
    for k = 1:m
        A(j,k) = 1 / (x(j) - y(k));
    end
end

[U, S, V] = svd(A);
s = diag(S);
idx = 1:m;
semilogy(idx, s, '.', 'MarkerSize', 10)
title('Singular Values of A')
xlabel('Index')
ylabel('Magnitude')
```



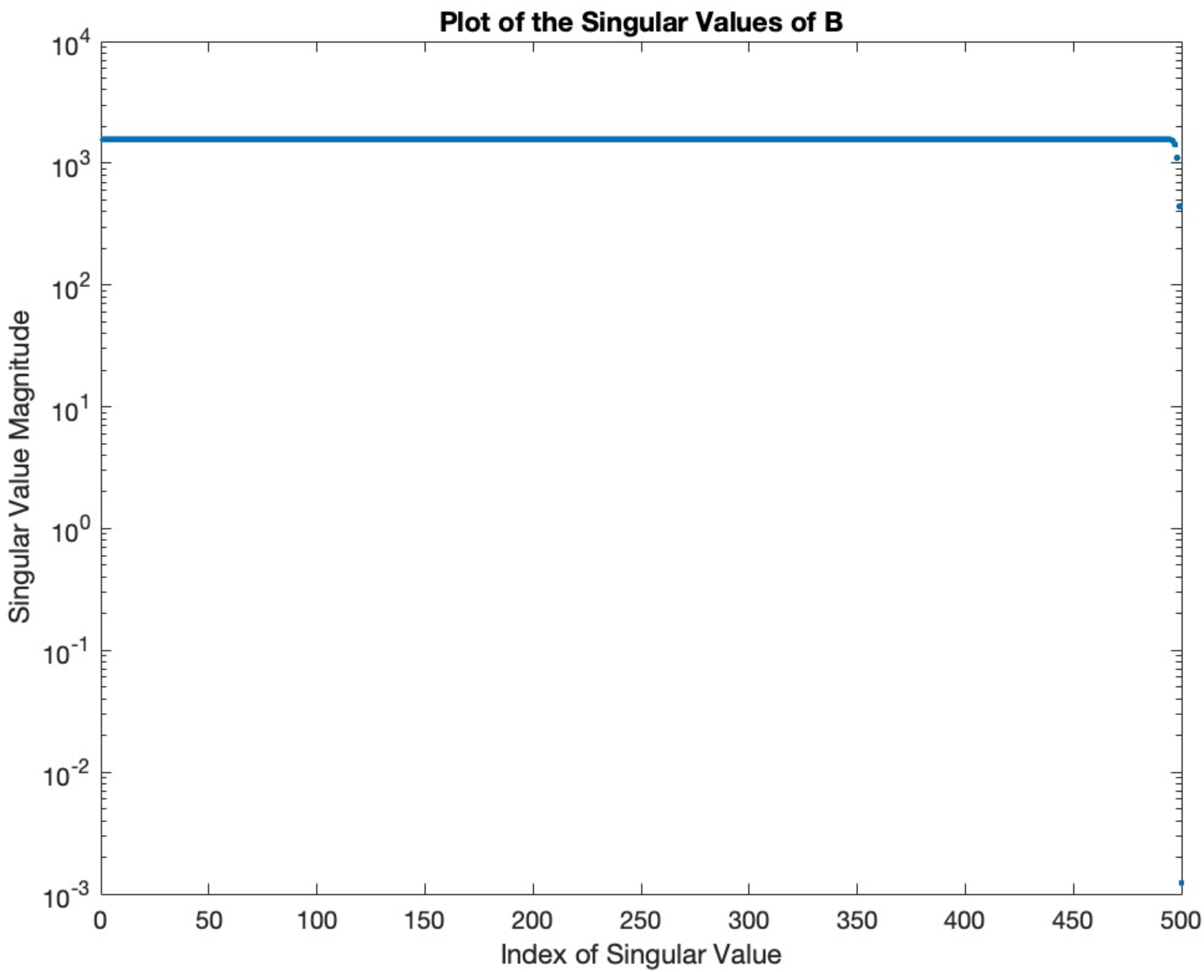
Singular Values of A
Magnitude
Index
10^4
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
10^-12
10^-14
0
50
100
150
200
250
300
350
400
450
500

```
%%
%Problem 5 Part b

E = zeros([10, 1]);
for j = 1:10
    Aj = U(:, 1:j) * S(1:j, 1:j) * V(:, 1:j)';
    E(j) = norm(A - Aj);
end

idx = 1:10;
clf
semilogy(idx, E, '.', 'MarkerSize', 12)
hold on
semilogy(idx, s(2:11), '.', 'MarkerSize', 12)
title('Low Rank Approximation Error for A')
legend('Error', 'Singular Values')
xlabel('Rank of the Approximation')
ylabel('Magnitude')
hold off
```



Low Rank Approximation Error for A
Error
Singular Values
Magnitude
Rank of the Approximation
$10^{2}$
$10^{0}$
$10^{-2}$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
1
2
3
4
5
6
7
8
9
10

```
%Problem 5 Part c

x = zeros([m, 1]);
y = zeros([m, 1]);
for j = 1:m
    x(j) = (j - 1) / m;
    y(j) = (j + 1/2) / m;
end

B = zeros([m, m]);
for j = 1:m
    for k = 1:m
        B(j, k) = 1 / (x(j) - y(k));
    end
end

[U, S, V] = svd(B);
s = diag(S);
idx = 1:500;
semilogy(idx, s, '.', 'MarkerSize', 6)
title('Plot of the Singular Values of B')
xlabel('Index of Singular Value')
ylabel('Singular Value Magnitude')

B10 = U(:, 1:10) * S(1:10, 1:10) * V(:, 1:10)';
ErrB = norm(B - B10)
FroErrB = norm(B - B10, "fro")
```


Plot of the Singular Values of B
Singular Value Magnitude
Index of Singular Value
10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
10^-3
0
50
100
150
200
250
300
350
400
450
500

```
ErrB =

   1.5708e+03


FroErrB =

   3.4676e+04
```