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AMATH 584

Homework #4

10.1.) The Householder reflector $F \in \mathbb{C}^{m\times m}$ is given by $F = I - 2\frac{vv^*}{v^*v}$ where $v \perp H$ and H is a hyperplane in $\mathbb{C}^m$.

a.) Find the eigenvalues of F.

Let $x \in \mathbb{C}^m$ and let $\mathcal{B} = \{\hat{v}, h_2, h_3, \dots, h_m\}$ be an orthonormal basis of $\mathbb{C}^m$ where $h_2, h_3, \dots, h_m \in H$ & $\hat{v} = \frac{v}{\|v\|_2}$. Then $x = \alpha_1 v + \alpha_2 h_2 + \alpha_3 h_3 + \dots + \alpha_m h_m$. This means

$$Fx = F(\alpha_1 \hat{v} + \alpha_2 h_2 + \alpha_3 h_3 + \dots + \alpha_m h_m)$$

$$= F\alpha_1 \hat{v} + F\alpha_2 h_2 + F\alpha_3 h_3 + \dots + F\alpha_m h_m$$

$$= \alpha_1 F\hat{v} + \alpha_2 F h_2 + \alpha_3 h_3 + \dots + \alpha_m F h_m$$

$$\alpha_1 F\hat{v} = \alpha_1 F \frac{v}{\|v\|_2} = \alpha_1 \frac{1}{\|v\|_2} Fv = \alpha_1 \frac{1}{\|v_2\|}\left(I - 2\frac{vv^*}{v^*v}\right)v = \alpha_1 \frac{1}{\|v\|_2}\left(v - 2\frac{vv^*v}{v^*v}\right)$$

$$= \alpha_1 \frac{1}{\|v\|_2}\left(v - 2v\frac{v^*v}{v^*v}\right) = \alpha_1 \frac{1}{\|v\|_2}(v - 2v) = \alpha_1 \frac{1}{\|v\|_2}(-v) = \alpha_1\left(-\frac{v}{\|v\|_2}\right) = \alpha_1(-\hat{v})$$

$$\Longrightarrow \alpha_1 F\hat{v} = \alpha_1(-\hat{v}) \Rightarrow F\hat{v} = -\hat{v} \Rightarrow \lambda_1 = -1$$

Let $2 \le i \le m$. Then

$$\alpha_i F h_i = \alpha_i\left(I - 2\frac{vv^*}{v^*v}\right)h_i = \alpha_i\left(h_i - 2\frac{vv^*h_i}{v^*v}\right) = \alpha_i\left(h_i - 2\frac{v\cdot 0}{v^*v}\right)$$

$$= \alpha_i(h_i - 0) = \alpha h_i$$

$$\Longrightarrow \alpha_i F h_i = \alpha_i h_i \Rightarrow F h_i = h_i \Rightarrow \lambda_i = 1$$

Therefore F has eigenvalues $\{-1, \overbrace{1, 1, \dots, 1}^{m-1}\}$.

Geometrically, we see that F reflects a vector x across the hyperplane H with normal vector v. If $x \in H$, then the reflected vector Fx will remain the same after the reflection F is applied so $Fx = x$. H has dimension $m-1$, so every basis vector of H has eigenvalue 1. If $x \notin H$, then $x = \alpha v$ for some $\alpha \in \mathbb{C}$. Since $v \perp H$, the reflected vector Fx

will be of equal magnitude and will still be normal to H. However, Fx will now point in the opposite direction of x so $Fx = -x$. Thus $x = \alpha v$ is an eigenvector for F with eigenvalue -1. Therefore F has eigenvalues $\{-1, \underbrace{1, 1, \ldots, 1}_{m-1}\}$.

b.) We know $\det(F) = \prod_{i=1}^{m} \lambda_i$ where λ_i are the eigenvalues of F. From part a we know $\lambda_1 = -1$ & $\lambda_2 = \lambda_3 = \ldots = \lambda_m = 1$, so $\det(F) = (-1) \cdot \prod_{j=2}^{m} 1 = -1$.

c.) From exercise 6.1 we know $F = I - 2\frac{vv^*}{v^*v}$ is unitary since $\frac{vv^*}{v^*v}$ is a projector. This means $F^*F = I$, so by theorem 5.4 the nonzero singular values of F are the square roots of the nonzero eigenvalues of $F^*F = I$. I has m eigenvalues all equal to 1, so the singular values of F are all equal to $\sqrt{1} = 1$. Thus F has singular values $\{\underbrace{1, 1, \ldots, 1}_{m}\}$.

2.) a) This circmatvec(c,v) function computes the matrix vector product Cv where C has first column c and C is circulant. It computes Cv by using the fact that $C = F^{-1}\Lambda F$ where F is the discrete Fourier transform matrix and $\Lambda = \text{diag}(Fc)$. The line "d=fft(c)" computes Fc so that we know Λ, and the line "y=ifft(d.*fft(v))" computes $F^{-1}\Lambda F v$.

Proof: Let C be a circulant matrix and let v be a vector of compatible dimension with C. Show that Cv = circmatvec(c,v). We know circmatvec(c,v) computes $F^{-1}\Lambda F v$, so since $C = F^{-1}\Lambda F$ we see that $Cv = F^{-1}\Lambda F v$. Thus in infinite precision Cv = circmatvec(c,v). $\boxtimes$

b.) We see that if $C_T = [T\ B; B\ T]$, then if C_T is to be circulant we first require that it is Toeplitz. This means the lower-triangular part of B must be equal to the upper-triangular part of T. Also, the upper-triangular part of B must be equal to the lower-triangular part of T. An example in the 4x4 case is given below:

$$T = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -3 & -2 & -1 \\ 3 & 0 & -3 & -2 \\ 2 & 3 & 0 & -3 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

no pun intended

2.) b.) continued.) The pseudocode for my toepmatvec function is as follows:

$y = $ toepmatvec(ct, rt, v)

% All we need is the first column of [T B; B T], so
% we see that if ct is the first column of T then
% Bct is [rt(1); flip(rt(2:end))], which is the first column
% of B.

- Form Bct, the first column of B, from rt.
- Form c by concatenating ct & Bct
- Form a new vector v that concatenates length(v) zeros onto the end of v.
- Set y = circmatvec(c, v)
- Remove the last length(v) entries from y
- Return y

c.) <u>Proof</u>: Let $T \in \mathbb{C}^{m \times n}$ be a Toeplitz matrix. Show T^* is also Toeplitz. Assume to the contrary that T^* is not Toeplitz. Then there exists an entry of T^* that is different from the other entries on its diagonal. Let us call this entry t_{ij} and let the entries on the rest of the diagonal <u>have</u> the value x. Then we see that T has the element $\bar{t}_{ij}$ in its jth row and ith column with the rest of its elements on that diagonal equal to $\bar{x}$. Since T is Toeplitz, we know $\bar{t}_{ij} = \bar{x}$, which implies $t_{ij} = x$. This contradicts our earlier assumption that $t_{ij} \neq x$. ※ Thus T^* must be Toeplitz. ⊠

d.) The pseudocode for my toepsvd function is as follows:

[u, s, v] = toepsvd(ct, rt, k)

- Generate a random test matrix G.
- Calculate $T \cdot G$ using toepmatvec to approx. range(T).
- QR decompose Y to obtain an orthonormal basis for range(T).
- Calculate $B = Q \cdot T$ to obtain a smaller approx. of T.
- Get [uu, s, v] by taking the svd of B.
- Set $u = Q \cdot uu$ and remove all "silent" entries of u, s, and v.
- Return u, s, and v.

2.)d.) continued.) The computational complexity is:

- Calculating $T \cdot G$ is $\mathcal{O}(m\log(m) \cdot k)$.
- QR decomposition is $\mathcal{O}(mk^2)$.
- Calculating Q^*T is $\mathcal{O}(m\log(m) \cdot k)$.
- Taking the svd of B is $\mathcal{O}(mk^2)$.
- Calculating $Q \cdot uu$ is $\mathcal{O}(mk^2)$.

This means the overall computational complexity is $\mathcal{O}(m\log(m)k + mk^2)$.

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>> Problem2d

rk =

    80


totalTimeToepSVD =

    0.2509


totalTimeSVD =

    3.1840


Error =

   7.8245e-15
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