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AMATH 585

Homework #2

1.) From the Lagrange interpolation formula we obtain the formula of the quadratic that interpolates $(x_k, f(x_k))$, $(x_{k-1}, f(x_{k-1}))$, and $(x_{k-2}, f(x_{k-2}))$ as

$$q_k(x) = f(x_{k-2})\frac{(x-x_{k-1})(x-x_k)}{(x_{k-2}-x_{k-1})(x_{k-2}-x_k)} + f(x_{k-1})\frac{(x-x_{k-2})(x-x_k)}{(x_{k-1}-x_{k-2})(x_{k-1}-x_k)} + f(x_k)\frac{(x-x_{k-2})(x-x_{k-1})}{(x_k-x_{k-2})(x_k-x_{k-1})}$$

If $f(x) = x^3-2$, $x_0=0$, $x_1=1$, and $x_2=2$, then $f(x_0)=-2$, $f(x_1)=-1$, and $f(x_2)=6$. The root of $q_2(x)$ closest to x_2 will give us x_3.

$$q_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= (-2)\frac{(x-1)(x-2)}{(0-1)(0-2)} + (-1)\frac{(x-0)(x-2)}{(1-0)(1-2)} + (6)\frac{(x-0)(x-1)}{(2-0)(2-1)}$$

$$= -(x-1)(x-2) + x(x-2) + 3x(x-1)$$

$$= -(x^2-3x+2) + x^2-2x+3x^2-3x = -x^2+3x-2+4x^2-5x$$

$$= 3x^2-2x-2$$

$$\Rightarrow 3x^2-2x-2=0 \Rightarrow x = \frac{2\pm\sqrt{4-4(3)(-2)}}{6} = \frac{2\pm\sqrt{28}}{6} = \frac{2\pm2\sqrt{7}}{6} = \frac{1}{3}\pm\frac{\sqrt{7}}{3}$$

Now we will find which root is closest to $x_2=2$

$$\left|2-\left(\frac{1}{3}+\frac{\sqrt{7}}{3}\right)\right| = \left|2-\frac{1}{3}-\frac{\sqrt{7}}{3}\right| = \left|\frac{5}{3}-\frac{\sqrt{7}}{3}\right|$$

$$\left|2-\left(\frac{1}{3}-\frac{\sqrt{7}}{2}\right)\right| = \left|2-\frac{1}{3}+\frac{\sqrt{7}}{3}\right| = \left|\frac{5}{3}+\frac{\sqrt{7}}{3}\right|$$

Clearly $x=\frac{1}{3}+\frac{\sqrt{7}}{3}$ is closest to $x_2=2$, so $x_3=\frac{1}{3}+\frac{\sqrt{7}}{3}$.

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2.) We are looking for ℓ that maps $[-1,1]$ to $[a,b]$, so $\ell(-1)=a$ and $\ell(1)=b$. Let $x\in[-1,1]$. We see that $\ell(x)\in[a,b]$, so we will first stretch $[-1,1]$ to be the same width as $[a,b]$ by multiplying x by $\left(\frac{b-a}{2}\right)$. This gives $\left(\frac{b-a}{2}\right)x$. Now we will shift x to start at a by adding $\left(\frac{b+a}{2}\right)$. This gives $\left(\frac{b-a}{2}\right)x+\left(\frac{b+a}{2}\right)$. Define $\ell:[-1,1]\to[a,b]$ by $\ell(x)=\left(\frac{b-a}{2}\right)x+\left(\frac{b+a}{2}\right)$. Then we see that

$$\ell(-1)=\left(\frac{b-a}{2}\right)(-1)+\left(\frac{b+a}{2}\right)=-\frac{b}{2}+\frac{a}{2}+\frac{b}{2}+\frac{a}{2}=a$$
$$\ell(1)=\left(\frac{b-a}{2}\right)(1)+\left(\frac{b+a}{2}\right)=\frac{b}{2}-\frac{a}{2}+\frac{b}{2}+\frac{a}{2}=b.$$

Thus ℓ maps $[-1,1]$ linearly onto $[a,b]$.

To find the interpolation points to use for $[a,b]$ we simply apply ℓ to all of the Chebyshev interpolation points $x_j=\cos\left(\frac{j\pi}{n}\right)$ since $x_j\in[-1,1]$. This gives the points

$$x_i=\left(\frac{b-a}{2}\right)\cos\left(\frac{i\pi}{n}\right)+\left(\frac{b+a}{2}\right).$$

3.) (see plots)

4.) In the plot of $\lambda_n(x)$ for part a we see that as n increases that the maximum value of $\lambda_n(x)$ grows extremely rapidly. Equation (2.59a) in the text shows us that Λ_n gives us an upper bound for the error $\|f-p_n(f;\cdot)\|_\infty$ and $\Lambda_n=\|\lambda_n(x)\|_\infty$, so in this case Λ_n corresponds to the maximum value of $\lambda_n(x)$ on $[-1,1]$. The points for part a are evenly spaced, which may not be the best choice if we would like p_n to closely resemble f on this interval. If we instead choose points like in part b that are more concentrated near the ends of the interval, we notice that the maximum value of $\lambda_n(x)$ grows much more slowly. This means that Λ_n also grows much slower than in part a, so our error bound is much smaller. This means we achieve a much better approximation of. Intuitively, we see that in our plot of $\lambda_n(x)$ for part a that the largest values of $\lambda_n(x)$ lie close to the endpoints -1 and 1, so by clustering more points around this area we greatly improve our approximation.

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5.) <u>Proof:</u> Show that the power x^n on the interval $-1 \leq x \leq 1$ can be uniformly approximated by a linear combination of powers $1, x, x^2, \ldots, x^{n-1}$ with error $\leq 2^{-(n-1)}$. We know that the Chebyshev polynomials $T_n(x)$ have leading coefficient 2^{n-1}, so let

$$T_n(x) = 2^{n-1} x^n - p_{n-1}(x) \quad \text{for some } p_{n-1} \in \mathbb{P}_{n-1}.$$

Then we see that

$$\frac{1}{2^{n-1}} T_n(x) = x^n - \frac{1}{2^{n-1}} p_{n-1}(x).$$

Define $q_{n-1} \in \mathbb{P}_{n-1}$ such that $q_{n-1}(x) = \frac{1}{2^{n-1}} p_{n-1}(x)$. Then we have

$$2^{-(n-1)} T_n(x) = x^n - q_{n-1}(x).$$

We know that $T_n(x) = \cos(n \cdot \cos^{-1} x)$ as well, so this means

$$|T_n(x)| = |\cos(n \cdot \cos^{-1} x)| \leq 1.$$

Thus we obtain

$$|x^n - q_{n-1}(x)| = |2^{-(n-1)} T_n(x)| = 2^{-(n-1)} |T_n(x)| = 2^{-(n-1)} |\cos(n \cdot \cos^{-1} x)|$$
$$\leq 2^{-(n-1)} \cdot 1 = 2^{-(n-1)}.$$

Therefore there exists a linear combination of powers $1, x, x^2, \ldots, x^{n-1}$ $(q_{n-1}(x))$ that uniformly approximates x^n on $-1 \leq x \leq 1$ with error $\leq 2^{-(n-1)}$. $\boxtimes$

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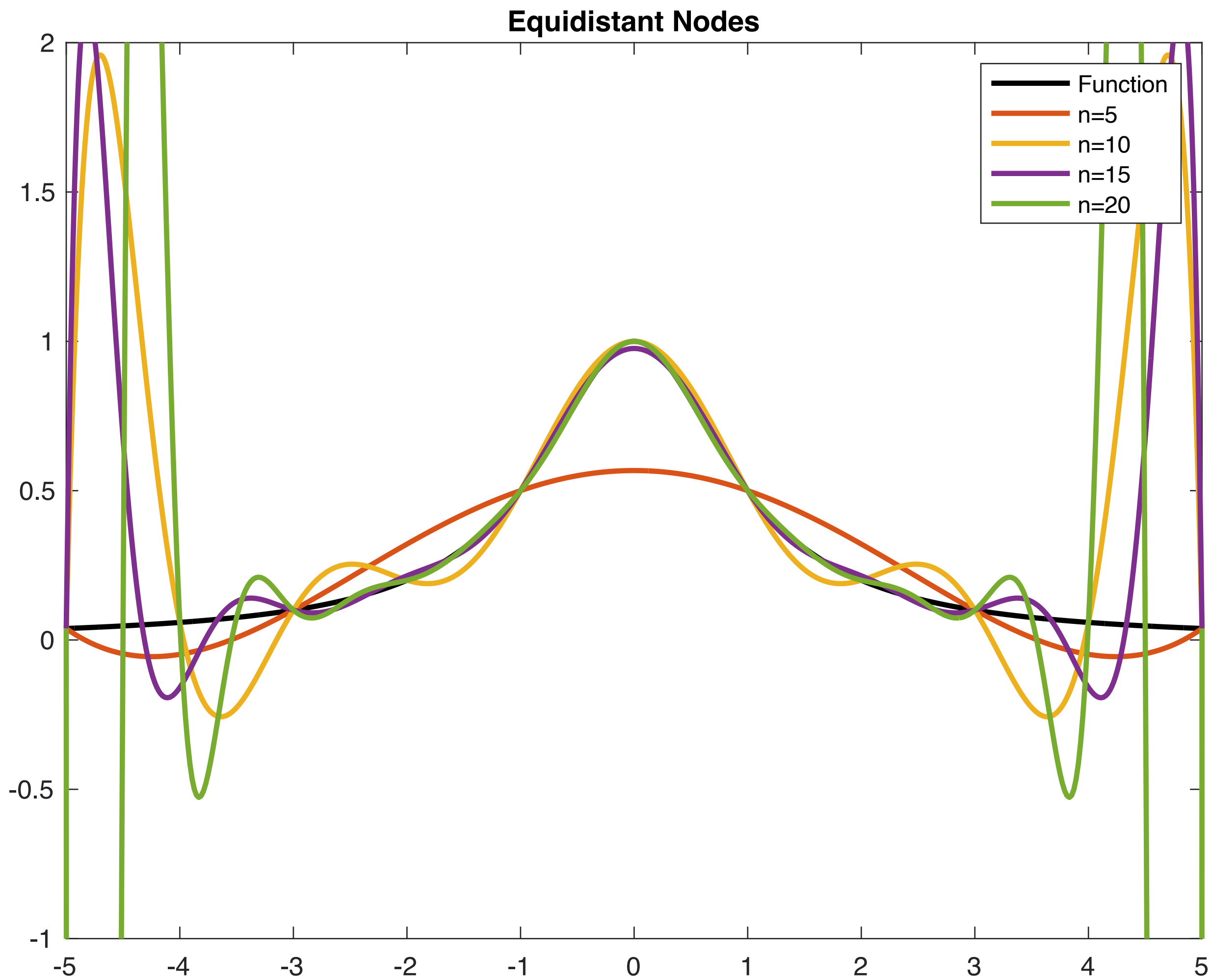
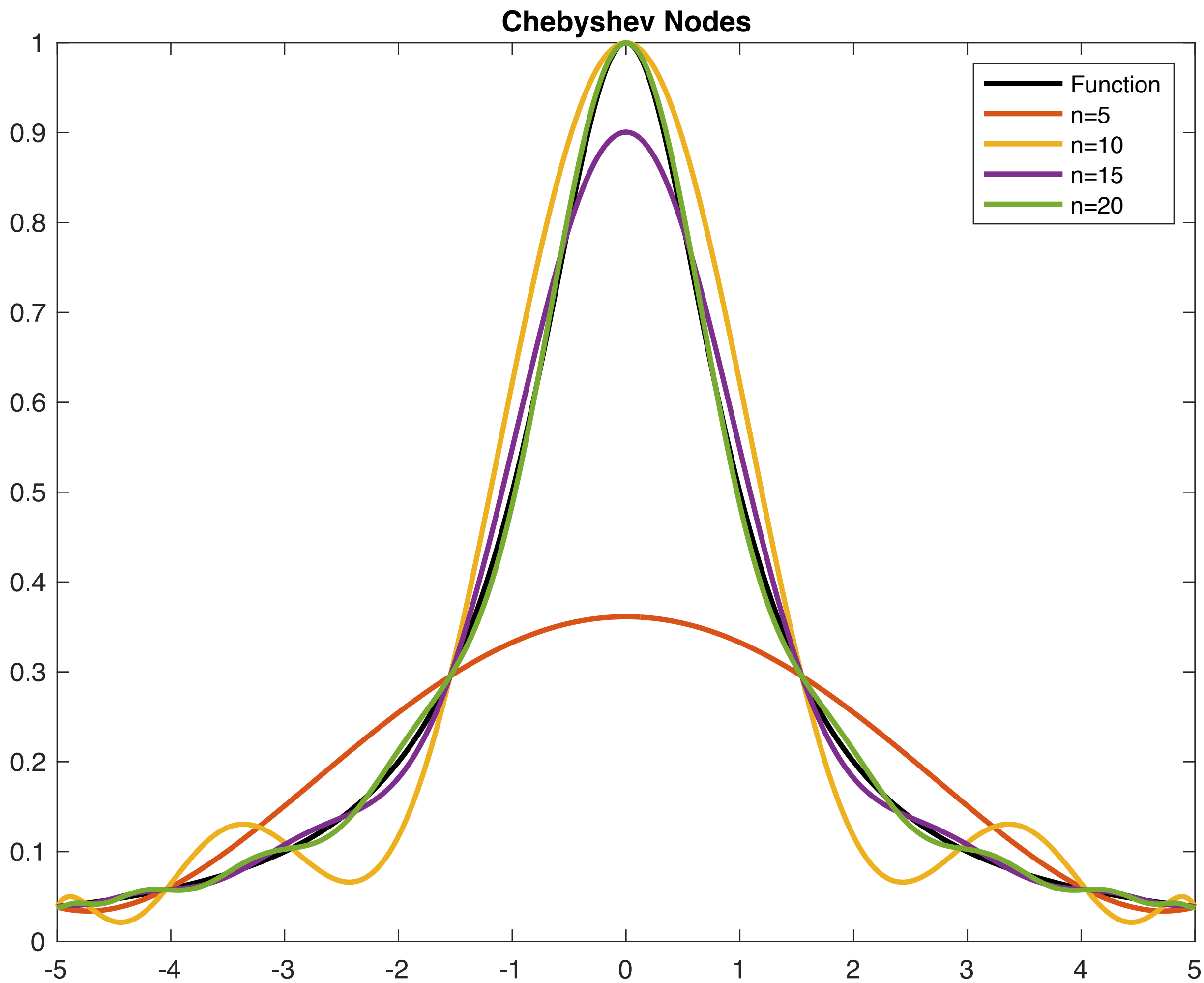
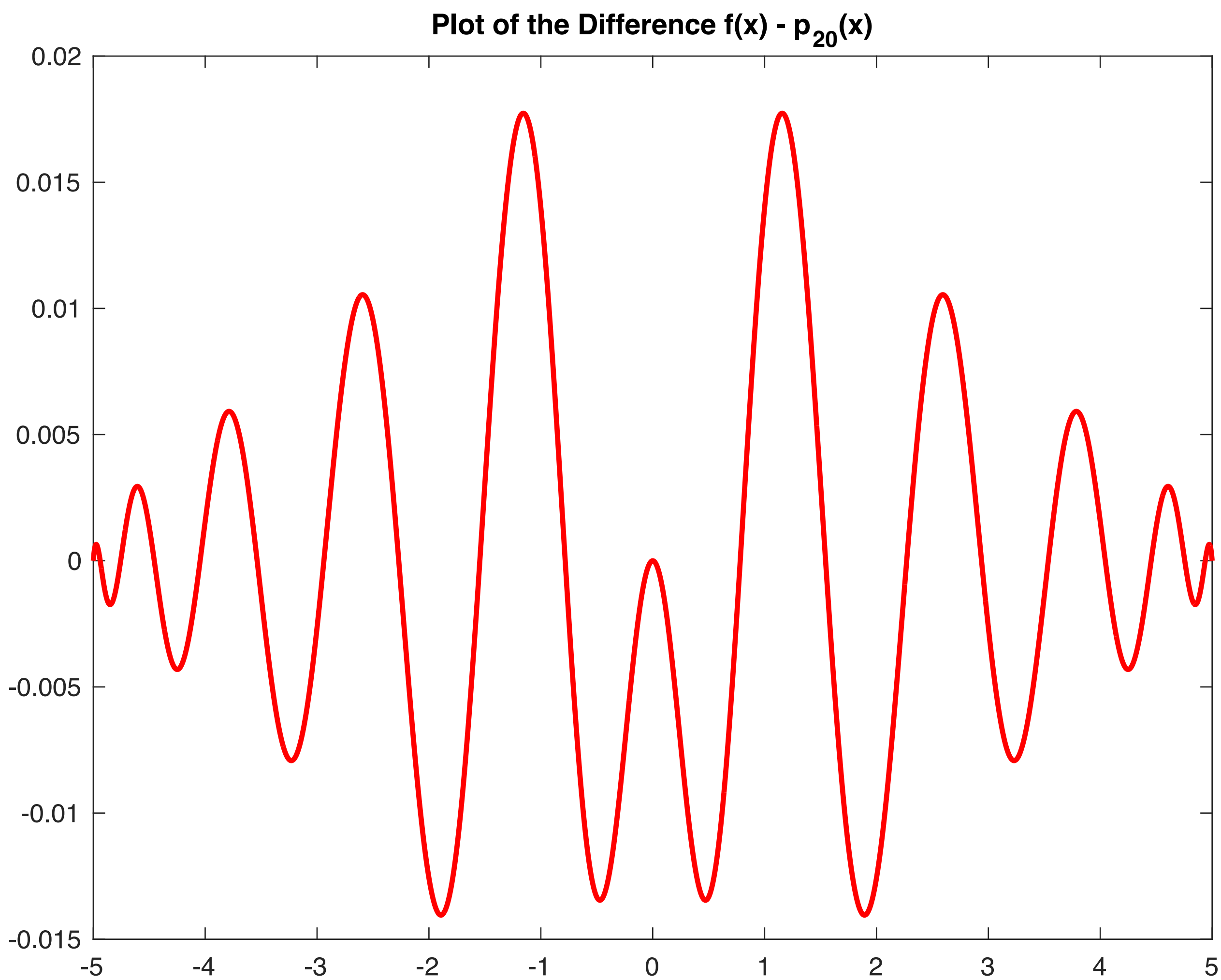
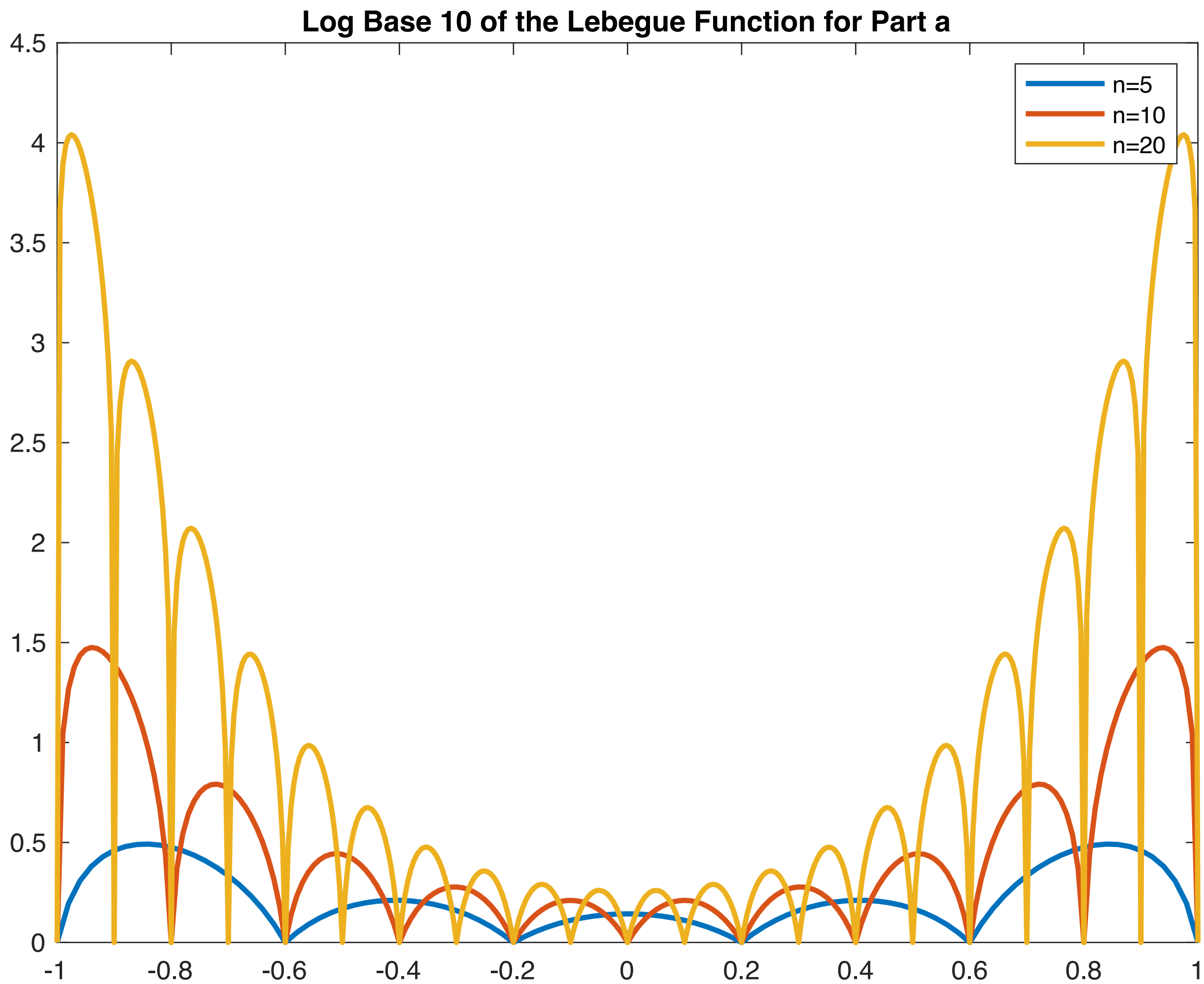
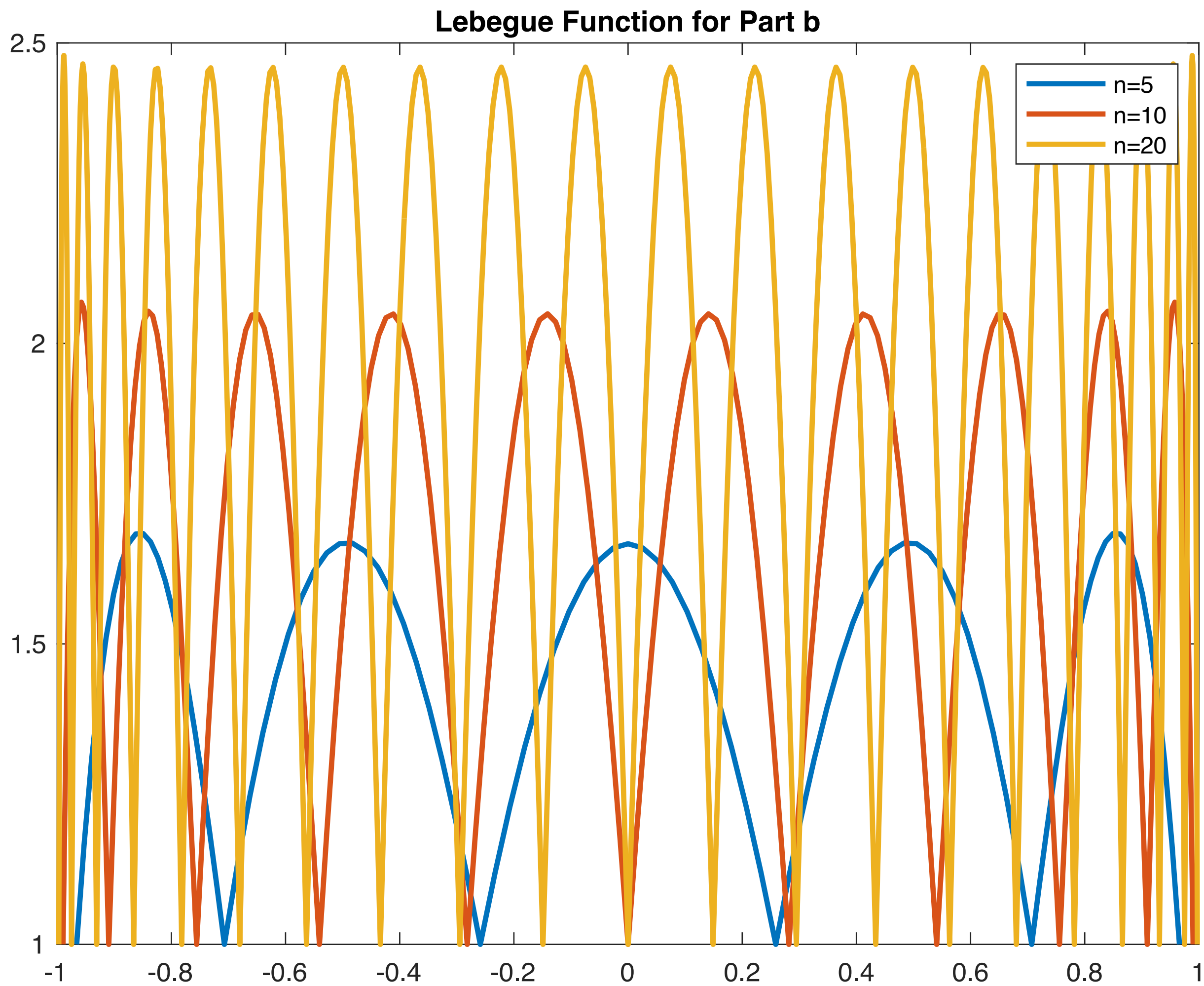

Equidistant Nodes
Function
n=5
n=10
n=15
n=20
2
1.5
1
0.5
0
-0.5
-1
-5
-4
-3
-2
-1
0
1
2
3
4
5



Chebyshev Nodes
Function
n=5
n=10
n=15
n=20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
-5
-4
-3
-2
-1
0
1
2
3
4
5


Plot of the Difference f(x) - p20(x)
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-5
-4
-3
-2
-1
0
1
2
3
4
5


Log Base 10 of the Lebegue Function for Part a
n=5
n=10
n=20
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



Lebegue Function for Part b
2.5
2
1.5
1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
n=5
n=10
n=20