Assignment 4.

Due Friday, Feb. 9, at 2:30pm PST.

Reading: Ch 3 in the Gautschi book.

1. Using Taylor series, derive the following second order accurate approximation to $f''(x)$, assuming that $f \in C^4[x-h, x+h]$:

$$f(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{1}$$

   Show that the error term is $-(h^2/12)f''''(\xi)$, for some point $\xi \in [x-h, x+h]$.

   Use Matlab (or a language of your choice) to evaluate (1) for $f(x) = \sin(x)$ and $x = \pi/6$. Try $h = 10^{-1},\ 10^{-2}, \ldots, 10^{-16}$, and make a table of values of h, the computed finite difference quotient, and the error (which you can compute analytically since $f''(x) = -\sin(x)$ and $\sin(\pi/6) = 1/2$). Explain your results. In particular, explain why you obtained the greatest accuracy with a particular value of h and why you get totally wrong answers for some values of h.

2. p. 202, problem 11 in [Gautschi].

3. Construct the 2-point weighted Gauss quadrature formula for the interval $[0, 1]$ with weight function $w(x) = x$; that is, find a formula of the form

$$\int_0^1 x f(x)\, dx \approx a_0 f(x_0) + a_1 f(x_1)$$

   that is exact for all polynomials of degree 3 or less.

4. Write a code to compute the integral of a given function using Romberg integration and to count the number of function evaluations needed to obtain a given level of accuracy. Use it to compute

$$\int_0^1 \cos(x^2)\, dx.$$

   Turn in a listing of your code. Determine (experimentally) how many function evaluations are required in order to obtain an error of size 10^{-12}. Compare this with the number of function evaluations that would be required by the composite trapezoidal rule (which you can determine analytically, as in problem 2). You may also compare your results with results from `chebfun`, which uses Clenshaw-Curtis quadrature. To do this, you can type `f = chebfun('cos(x^2)',[0,1]), intf = sum(f)`. The package will inform you of the length of `f`, which is 1 plus the degree of the polynomial that it has used in order to compute this integral to about the machine precision. How does this compare to the number of function evaluations that you used?