Justin Hexem
AMATH 585

Homework #4

1.) Let us taylor expand $f(x-h)$ & $f(x+h)$ about x:

$$f(x-h) = f(x) + (x-h-x)\frac{f'(x)}{1!} + (x-h-x)^2\frac{f''(x)}{2!} + (x-h-x)^3\frac{f'''(x)}{3!} + O((x-h-x)^4)$$

$$= f(x) - hf'(x-h) + \frac{h^2}{2}f''(x-h) - \frac{h^3}{6}f'''(x-h) + O(h^4)$$

$$f(x+h) = f(x) + (x+h-x)\frac{f'(x)}{1!} + (x+h-x)^2\frac{f''(x)}{2!} + (x+h-x)^3\frac{f'''(x)}{3!} + O((x+h-x)^4)$$

$$= f(x) + hf'(x+h) + \frac{h^2}{2}f''(x+h) + \frac{h^3}{6}f'''(x+h) + O(h^4)$$

Adding these two equations gives

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

$$\Rightarrow f(x+h) - 2f(x) + f(x-h) - O(h^4) = h^2 f''(x)$$

$$\Rightarrow f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{O(h^4)}{h^2}$$

The error term for $f(x-h)$ and $f(x+h)$ is $\frac{f^{(4)}(\xi)}{4!}h^4$, so since we added $f(x-h)$ & $f(x+h)$ the error is now $2\frac{f^{(4)}(\xi)}{4!}h^4 = \frac{f^{(4)}(\xi)}{12}h^4$. This means for our approximation, the error is $-h^2\frac{f^{(4)}(\xi)}{12}$.

From my matlab code, it seems that the error is minimized when $h = 10^{-4}$. For $h = 10^{-1}, 10^{-2}, 10^{-3}$, we get a less precise result due to truncation error when we remove the higher order terms in the taylor series. For $h = 10^{-5}, \ldots, 10^{-8}$ we see the error increase due to roundoff error now dominating. After $h = 10^{-8}$, we get answers that make no sense due to the fact that the denominator squares h, so $h = (10^{-8})^2 = 10^{-16}$ which is the limit of a double-precision floating point number. This means the denominator causes the approximation to become extremely inaccurate.

h	finite difference	error
0.1	-0.49958	0.00041653
0.01	-0.5	4.1667e-06
0.001	-0.5	4.1674e-08
0.0001	-0.5	3.0387e-09
1e-05	-0.5	5.9648e-07
1e-06	-0.49993	6.6572e-05
1e-07	-0.49405	0.0059508
1e-08	-1.1102	0.61022
1e-09	55.511	56.011
1e-10	0	0.5
1e-11	0	0.5
1e-12	0	0.5
1e-13	5.5511e+09	5.5511e+09
1e-14	-5.5511e+11	5.5511e+11
1e-15	0	0.5
1e-16	-5.5511e+15	5.5511e+15

2.) $f(x) = e^{-x^2}$, $[0,1] \Rightarrow h = \frac{1}{n}$.

First, we will calculate the first five derivatives of f:

$$f'(x) = e^{-x^2}[-2x], \quad f''(x) = e^{-x^2}[-2x](-2x) + e^{-x^2}[-2] = e^{-x^2}[4x^2-2]$$

$$f'''(x) = e^{-x^2}[-2x](4x^2-2) + e^{-x^2}[8x] = e^{-x^2}[-8x^3+12x]$$

$$f^{(4)}(x) = e^{-x^2}[-2x](-8x^3+12x) + e^{-x^2}[-24x^2+12]$$

$$= e^{-x^2}[16x^4-24x^2-24x^2+12] = e^{-x^2}[16x^4-48x^2+12]$$

$$f^{(5)}(x) = e^{-x^2}[-2x](16x^4-48x^2+12) + e^{-x^2}[64x^3-96x]$$

$$= e^{-x^2}[-32x^5+96x^3-24x+64x^3-96x]$$

$$= e^{-x^2}[-32x^5+160x^3-120x]$$

a.) We know $E_n^T(f) = -\frac{1}{12}(b-a)h^2 f''(\xi) = -\frac{1}{12n^2}f''(\xi) \leq -\frac{1}{12n^2}\max_{x\in[0,1]} f''(x)$

$$f'''(x) = e^{-x^2}[-4x(-2x^2+3)] = 0$$

$$\Rightarrow x = 0,\ x = \frac{\pm\sqrt{24}}{-4} = \mp\frac{2\sqrt{6}}{4} = \mp\frac{\sqrt{6}}{2} = \frac{\sqrt{6}}{2} \quad \left(\frac{\sqrt{6}}{2} \notin [0,1]\right)$$

$$\Rightarrow f''(0) = 1\cdot[-2] = -2,\quad f''(1) = e^{-1}[4-2] = 2/e$$

Thus $\max_{x\in[0,1]} f''(x) = 2/e$, so $E_n^T(f) \leq -\frac{1}{12n^2}\cdot 2/e = -\frac{1}{6en^2}$. We require that the absolute error $\leq \frac{1}{2}\cdot 10^{-6}$, so

$$\frac{1}{6en^2} \leq \frac{1}{2}\cdot 10^{-6} \Rightarrow 2\cdot 10^6 \leq 6en^2 \Rightarrow \sqrt{\tfrac{1}{3e}\cdot 10^6} \leq n$$

$$\Rightarrow \sqrt{\tfrac{1}{3e}\cdot 10^6} \approx 350.2 \leq n \Rightarrow n = 351$$

Thus we require 351 subintervals to obtain $\int_0^1 e^{-x^2}dx$ to 6 correct decimal places using the composite trapezoidal rule.

Scanned with CamScanner

2.) b.) We know $E_n^S(f) = -\frac{1}{180}(b-a)h^4 f^{(4)}(\xi) = -\frac{1}{180n^4} f^{(4)}(\xi) \le -\frac{1}{180n^4} \max_{x\in[0,1]} f^{(4)}(x)$

$$f^{(5)}(x) = e^{-x^2}[-32x^5 + 160x^3 - 120x] = e^{-x^2}[-8x(4x^4 - 20x^2 + 15)]$$

$$\Rightarrow x=0,\ x^2 = \frac{20 \pm \sqrt{400-16\cdot 15}}{8} = \frac{20 \pm \sqrt{160}}{8} = \frac{20 \pm 4\sqrt{10}}{8} = \frac{5}{2} \pm \frac{\sqrt{10}}{2}$$

$$\Rightarrow x=0,\ x = \pm\sqrt{\frac{5}{2} \pm \frac{\sqrt{10}}{2}} = \sqrt{\frac{5}{2} \pm \frac{\sqrt{10}}{2}} \quad (\text{on } [0,1])$$

$$\Rightarrow f^{(4)}(0) = 1\cdot[12] = 12,\ f^{(4)}(1) = e^{-1}[16-48+12] = \frac{20}{e} \approx 7.36$$

$$f^{(4)}\left(\sqrt{\tfrac{5}{2}+\tfrac{\sqrt{10}}{2}}\right) = e^{-\frac{5}{2}-\frac{\sqrt{10}}{2}}\left[16\left(\tfrac{5}{2}+\tfrac{\sqrt{10}}{2}\right)^2 - 48\left(\tfrac{5}{2}+\tfrac{\sqrt{10}}{2}\right)+12\right]$$

$$= e^{-\frac{5}{2}-\frac{\sqrt{10}}{2}}\left[16\left(\tfrac{25}{4}+\tfrac{10}{4}+\tfrac{5\sqrt{10}}{2}\right) - 120 - 24\sqrt{10} + 12\right]$$

$$= e^{-\frac{5}{2}-\frac{\sqrt{10}}{2}}\left[140 + 40\sqrt{10} - 120 - 24\sqrt{10} + 12\right]$$

$$= e^{-\frac{5}{2}-\frac{\sqrt{10}}{2}}\left[32 + 16\sqrt{10}\right] \approx 1.40 \quad (\text{not in } [0,1])$$

$$f^{(4)}\left(\sqrt{\tfrac{5}{2}-\tfrac{\sqrt{10}}{2}}\right) = e^{-\frac{5}{2}+\frac{\sqrt{10}}{2}}\left[16\left(\tfrac{5}{2}-\tfrac{\sqrt{10}}{2}\right)^2 - 48\left(\tfrac{5}{2}-\tfrac{\sqrt{10}}{2}\right)+12\right]$$

$$= e^{-\frac{5}{2}+\frac{\sqrt{10}}{2}}\left[16\left(\tfrac{25}{4}+\tfrac{10}{4}-\tfrac{5\sqrt{10}}{2}\right) - 120 + 24\sqrt{10} + 12\right]$$

$$= e^{-\frac{5}{2}+\frac{\sqrt{10}}{2}}\left[140 - 40\sqrt{10} - 120 + 24\sqrt{10} + 12\right]$$

$$= e^{-\frac{5}{2}+\frac{\sqrt{10}}{2}}\left[32 - 16\sqrt{10}\right] \approx -7.42$$

Thus $\max_{x\in[0,1]} f^{(4)}(x) = 12$, so $E_n^S(f) \le -\frac{1}{180n^4}\cdot 12 = -\frac{1}{15n^4}$. We require the absolute error $\le \frac{1}{2}\cdot 10^{-6}$, so

$$\frac{1}{15n^4} \le \frac{1}{2}\cdot 10^{-6} \Rightarrow 2\cdot 10^6 \le 15n^4 \Rightarrow \sqrt[4]{\tfrac{2}{15}\cdot 10^6} \le n$$

$$\Rightarrow \sqrt[4]{\tfrac{2}{15}\cdot 10^6} \approx 19.1 \le n \Rightarrow n = 20$$

Thus we require 20 subintervals to obtain $\int_0^1 e^{-x^2}dx$ to 6 correct decimal places using the composite Simpson's rule.

3.) First we will construct the orthogonal polynomial $\pi_2(x) = x^2 - \rho_1 x + \rho_2$ that is orthogonal to $\{1, x\}$.

$$0 = \int_0^1 x\,\pi_2(x)\cdot 1\,dx = \int_0^1 (x^3 - \rho_1 x^2 + \rho_2 x)\,dx = \frac{1}{4} - \frac{1}{3}\rho_1 + \frac{1}{2}\rho_2$$

$$0 = \int_0^1 x\,\pi_2(x)\,x\,dx = \int_0^1 (x^4 - \rho_1 x^3 + \rho_2 x^2)\,dx = \frac{1}{5} - \frac{1}{4}\rho_1 + \frac{1}{3}\rho_2$$

$$\Rightarrow \begin{matrix} \frac{1}{3}\rho_1 - \frac{1}{2}\rho_2 = \frac{1}{4} \\ \frac{1}{4}\rho_1 - \frac{1}{3}\rho_2 = \frac{1}{5} \end{matrix} \quad \Rightarrow \begin{matrix} \frac{1}{3}\rho_1 = \frac{1}{2}\rho_2 + \frac{1}{4} \\ \rho_1 = \frac{3}{2}\rho_2 + \frac{3}{4} \end{matrix} \quad \Rightarrow \begin{matrix} \frac{1}{4}\left(\frac{3}{2}\rho_2 + \frac{3}{4}\right) - \frac{1}{3}\rho_2 = \frac{1}{5} \\ \frac{3}{8}\rho_2 + \frac{3}{16} - \frac{1}{3}\rho_2 = \frac{1}{5} \\ \frac{1}{24}\rho_2 = \frac{1}{80}, \quad \rho_2 = \frac{24}{80} = \frac{3}{10} \end{matrix}$$

$$\Rightarrow \rho_1 = \frac{3}{2}\cdot\frac{3}{10} + \frac{3}{4} = \frac{9}{20} + \frac{3}{4} = \frac{24}{20} = \frac{6}{5} \Rightarrow \rho_1 = \frac{6}{5},\ \rho_2 = \frac{3}{10}$$

$$\Rightarrow \pi_2(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$$

The roots of $\pi_2(x)$ give x_0, x_1

$$x = \frac{\frac{6}{5} \pm \sqrt{\frac{36}{25} - \frac{6}{5}}}{2} = \frac{3}{5} \pm \frac{\sqrt{6/25}}{2} = \frac{3}{5} \pm \frac{\sqrt{6}}{10} \Rightarrow x_0 = \frac{3}{5} - \frac{\sqrt{6}}{10},\ x_1 = \frac{3}{5} + \frac{\sqrt{6}}{10}$$

Now we will calculate the weights a_0, a_1 using the fact that this formula is exact for $\mathbb{P}_3$, but specifically $f(x) = 1$ & $f(x) = x$

$$\begin{matrix} a_0 + a_1 = \int_0^1 x\cdot 1\,dx = \frac{1}{2} \\ x_0 a_0 + x_1 a_1 = \int_0^1 x^2\,dx = \frac{1}{3} \end{matrix} \quad \Rightarrow \begin{matrix} a_0 + a_1 = \frac{1}{2} \\ \left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right)a_0 + \left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right)a_1 = \frac{1}{3} \end{matrix}$$

$$\Rightarrow a_0 = \frac{1}{2} - a_1 \Rightarrow \left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right)\left(\frac{1}{2} - a_1\right) + \left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right)a_1 = \frac{1}{3}$$

$$\Rightarrow \frac{3}{10} - \frac{\sqrt{6}}{20} - \left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right)a_1 + \left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right)a_1 = \frac{1}{3} \Rightarrow a_1\left(-\frac{3}{5} + \frac{\sqrt{6}}{10} + \frac{3}{5} + \frac{\sqrt{6}}{10}\right) = \frac{1}{3} - \frac{3}{10} + \frac{\sqrt{6}}{20}$$

$$\Rightarrow a_1\left(\frac{\sqrt{6}}{5}\right) = \frac{10-9}{30} + \frac{\sqrt{6}}{20} \Rightarrow a_1\left(\frac{\sqrt{6}}{5}\right) = \frac{1}{30} + \frac{\sqrt{6}}{20} \Rightarrow a_1 = \frac{5}{30\sqrt{6}} + \frac{1}{4} = \frac{5\sqrt{6}}{180} + \frac{1}{4}$$

$$\Rightarrow a_1 = \frac{1}{4} + \frac{\sqrt{6}}{36} \Rightarrow a_0 = \frac{1}{2} - \frac{1}{4} - \frac{\sqrt{6}}{36} = \frac{1}{4} - \frac{\sqrt{6}}{36}$$

Thus we obtain the Gaussian quadrature formula

$$\int_0^1 x f(x)\,dx \approx \left(\frac{1}{4} - \frac{\sqrt{6}}{36}\right) f\left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right) + \left(\frac{1}{4} + \frac{\sqrt{6}}{36}\right) f\left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right)$$

This formula has degree of exactness $2n-1 = 2\cdot 2 - 1 = 3$, so it is exact for all polynomials of degree 3 or less.

4.) We will first calculate the number of function evaluations are required for the composite trapezoidal rule. From a graphing calculator, we see that $\max_{x\in[0,1]} |f''(x)| \approx |-3.845| = 3.845$, so

$$E_n^T(f) = -\frac{1}{12}(b-a)h^2 f''(\xi) = \frac{1}{12n^2} f''(\xi) \leq \frac{1}{12n^2}(3.845) = \frac{3.845}{12n^2}$$

We want an accuracy of 10^{-12}, so this gives

$$\frac{3.845}{12n^2} \leq \frac{1}{2}\cdot 10^{-12} \Rightarrow 3.845 \cdot 2 \cdot 10^{12} \leq 12n^2 \Rightarrow 3.845 \cdot \tfrac{1}{6}\, 10^{12} \leq n^2$$

$$\Rightarrow \sqrt{3.845 \cdot \tfrac{1}{6} \cdot 10^{12}} \approx 800520.7 \leq n \Rightarrow n = 800{,}521.$$

Thus $800{,}521 + 1 = 800{,}522$ function evaluations are required for the composite trapezoidal rule to obtain an accuracy of 10^{-12}. Using Romberg integration, my matlab code only requires 33 function evaluations, and Clenshaw-Curtis quadrature only requires 19. We see that Clenshaw-Curtis quadrature required the least amount of function evaluations with 19, Romberg integration the second least with 33 evaluations, and the composite trapezoidal rule the most with 800,522 function evaluations.

```matlab
f = @g;
tol = 10e-12

a = 0;
b = 1;

q = 1/2;
h0 = 1;

trueSol = integral(f, 0, 1)

[prevA, fevals] = A(a, b, 1, f);
j = 1;

while (abs(trueSol - prevA(end)) > tol)
    [Ak0, fevals] = A(a, b, (q^j) * h0, f);
    nextA = [Ak0 zeros(1, j)];
    for k = 1:j
        nextA(k+1) = nextA(k) + ((nextA(k) - prevA(k)) / (4^k - 1));
    end
    prevA = nextA;
    j = j + 1;
end

fevals %counts total number of distinct function evaluations
intf = nextA(end)

chebf = chebfun('cos(x^2)', [0, 1]), chebintf = sum(chebf)

function y = g(x)
    y = cos(x.^2);
end

function [approx, fcount] = A(a, b, h, f)
    x = a:h:b;
    fx = f(x);
    fx(1) = fx(1) / 2;
    fx(end) = fx(end) / 2;
    approx = sum(fx) * h;
    fcount = length(fx);
end
```

```
>> OldProblem4

tol =

     9.999999999999999e-12


trueSol =

   0.904524237900272


fevals =

    33


intf =

   0.904524237900760


chebf =

   chebfun column (1 smooth piece)
       interval       length     endpoint values
[       0,       1]       19          1     0.54
vertical scale =   1


chebintf =

   0.904524237900272
```