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AMATH 585

Homework #5

1.) a.) From 4.33 in the textbook with $a=4$, $b=5$, $tol=10^{-12}$, we see that

$$n=\left\lceil\frac{\log\left(\frac{b-a}{tol}\right)}{\log 2}\right\rceil=\left\lceil\frac{\log\left(\frac{1}{10^{-12}}\right)}{\log 2}\right\rceil=\left\lceil\frac{12}{\log 2}\right\rceil=\lceil 39.86\ldots\rceil=40$$

Thus 40 iterations are required to reduce the size of the interval to 10^{-12}.

b.) From my code, I only needed 3 iterations to satisfy $|f(x)|\leq 10^{-8}$, so an estimate for the number of steps required to obtain $|f(x)|\leq 10^{-16}$ would be 5. This is because the first iteration gave $|f(x)|\approx 10^{-1}$, the second gave $|f(x)|\approx 10^{-4}$, and the third gave $|f(x)|\approx 10^{-10}$. Since our accuracy seems to improve rapidly on the order of about 10^{-5} for this function, two more iterations should give $|f(x)|\approx 10^{-16}$.

c.) From my code, only 5 iterations were required to reduce $|f(x)|$ to $\approx 10^{-8}$. Each iteration gave $|f(x)|\approx 10^{1}$, $|f(x)|\approx 10^{0}$, $|f(x)|\approx 10^{-2}$, $|f(x)|\approx 10^{-5}$, $|f(x)|\approx 10^{-9}$. From this, I estimate that the next iteration will give $|f(x)|\approx 10^{-14}$, so two more iterations should be enough to give $|f(x)|\approx 10^{-16}$ in this specific case.

2) We have three different iterations $x_{n+1} = \varphi(x_n)$ for $f(x) = x^2 - a = 0$. Let $\varphi_1(x) = \frac{1}{2}(x + \frac{a}{x})$, $\varphi_2(x) = \frac{a}{x}$, and $\varphi_3(x) = 2x - \frac{a}{x}$. Theorem 4.7.1 gives us a way of calculating the order of convergence and the asymptotic error constant, so

$$\varphi_1'(x) = \frac{1}{2} - \frac{a}{2x^2}, \quad \varphi_1'(\sqrt{a}) = \frac{1}{2} - \frac{a}{2a} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\varphi_1''(x) = \frac{a}{x^3}, \quad \varphi_1''(\sqrt{a}) = \frac{a}{a^{3/2}} = \frac{1}{\sqrt{a}} \neq 0 \Rightarrow p = 2$$

$$c = \frac{1}{p!}\varphi_1^{(p)}(\sqrt{a}) = \frac{1}{2!}\varphi_1''(\sqrt{a}) = \frac{1}{2} \cdot \frac{1}{\sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Since $p = 2$ and $c = \frac{1}{2\sqrt{a}} > 0$, we see that $x_{n+1} = \varphi_1(x_n)$ converges with order of convergence 2.

$$\varphi_2'(x) = -\frac{a}{x^2}, \quad \varphi_2'(\sqrt{a}) = -\frac{a}{a} = -1 \Rightarrow p = 1$$

$$c = \frac{1}{p!}\varphi_2^{(p)}(\sqrt{a}) = \frac{1}{1!}(-1) = -1$$

Since $p = 1$ and $c = -1$, we see that $x_{n+1} = \varphi_2(x_n)$ cannot converge since its order of convergence is $p = 1$, which means c must be between 0 & 1, but $c = -1$.

$$\varphi_3'(x) = 2 + \frac{a}{x^2}, \quad \varphi_3'(\sqrt{a}) = 2 + \frac{a}{a} = 2 + 1 = 3 \Rightarrow p = 1$$

$$c = \frac{1}{p!}\varphi_3^{(p)}(\sqrt{a}) = \frac{1}{1!}(3) = 3$$

Since $p = 1$ and $c = 3$, we see that $x_{n+1} = \varphi_3(x_n)$ cannot converge since its order of convergence is $p = 1$, which means c must be between 0 & 1, but $c = 3$.

Therefore only $x_{n+1} = \varphi_1(x_n)$ converges with order of convergence 2.

3) If I enter a number into a calculator and repeatedly press the cosine button, I obtain $\alpha \approx 0.7391$. This means we have performed the fixed point iteration $x_{n+1} = \varphi(x_n)$ with $\varphi(x) = \cos x$. Since pressing the cosine button after obtaining α results in α, we see that $\alpha = \varphi(\alpha) = \cos\alpha$ so α is a fixed point for the problem $x = \cos x$. Now we will show that there are no other fixed points for $x = \cos x$.

We will first show that $\varphi(x) = \cos x$ is contractive on $[-1,1]$. Let $x, y \in [-1,1]$. Then the mean value theorem gives

$$\frac{f(x)-f(y)}{x-y} = f'(c) \text{ for some } c \text{ between } x \text{ \& } y$$

$$\Rightarrow \frac{\cos x - \cos y}{x-y} = -\sin c \Rightarrow \cos x - \cos y = (-\sin c)(x-y)$$

$$\Rightarrow |\cos x - \cos y| = |-\sin c||x-y|.$$

Since c is between x and y, $c \in [-1,1]$. We know $-\sin c = \pm 1$ only when $c = \frac{k\pi}{2}$ for some $k \in \mathbb{Z}, k \neq 0$. Since $\frac{k\pi}{2}$ is not in $[-1,1]$ for any $k \in \mathbb{Z}, k \neq 0$, we see that $|-\sin c| < 1$ for all $c \in [-1,1]$. Let $\gamma = \max_{c \in [-1,1]} |-\sin c|$. Then $0 < \gamma < 1$, and $|-\sin c| \leq \gamma$, so

$$|\cos x - \cos y| = |-\sin c||x-y| \leq \gamma|x-y|.$$

Thus $\varphi(x) = \cos x$ is contractive, so by theorem 4.9.1 the iteration $x_{n+1} = \cos x_n$ must converge to a unique fixed point in $[-1,1]$. Since $\alpha \in [-1,1]$, $x_{n+1} = \cos x_n$ must converge to $\alpha \approx 0.7391$. $\boxtimes$

4.) $\theta''(t) = -\sin(\theta(t)),\ 0 < t < T,\ \theta(0) = \alpha,\ \theta(1) = \beta$

$$\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i) = 0, \quad i = 1, 2, \ldots, n-1$$

$$\vec{\theta} = [\theta_1\ \theta_2\ \ldots\ \theta_{n-1}],\quad f^i(\vec{\theta}) = \frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i),\quad i = 1, 2, \ldots, n-1.$$

The Jacobian $\frac{\partial \vec{f}}{\partial \vec{\theta}} = \left[\frac{\partial f^i}{\partial \theta^j}(\vec{\theta})\right]_{i,j=1}^{n-1}$ is

$$\frac{\partial f^i}{\partial \theta^{i-1}}(\vec{\theta}) = \frac{\partial}{\partial \theta^{i-1}}(f^i(\vec{\theta})) = \frac{\partial}{\partial \theta^{i-1}}\left(\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i)\right) = \frac{1}{h^2}$$

$$\frac{\partial f^i}{\partial \theta^{i+1}}(\vec{\theta}) = \frac{\partial}{\partial \theta^{i+1}}(f^i(\vec{\theta})) = \frac{\partial}{\partial \theta^{i+1}}\left(\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i)\right) = \frac{1}{h^2}$$

$$\frac{\partial f^i}{\partial \theta^{i}}(\vec{\theta}) = \frac{\partial}{\partial \theta^{i}}(f^i(\vec{\theta})) = \frac{\partial}{\partial \theta^{i}}\left(\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i)\right) = -\frac{2}{h^2} + \cos(\theta_i)$$

$$\frac{\partial f^i}{\partial \theta^{j}}(\vec{\theta}) = \frac{\partial}{\partial \theta^{j}}(f^i(\vec{\theta})) = \frac{\partial}{\partial \theta^{j}}\left(\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i)\right) = 0 \quad \forall\ j \neq i-1, i, i+1$$

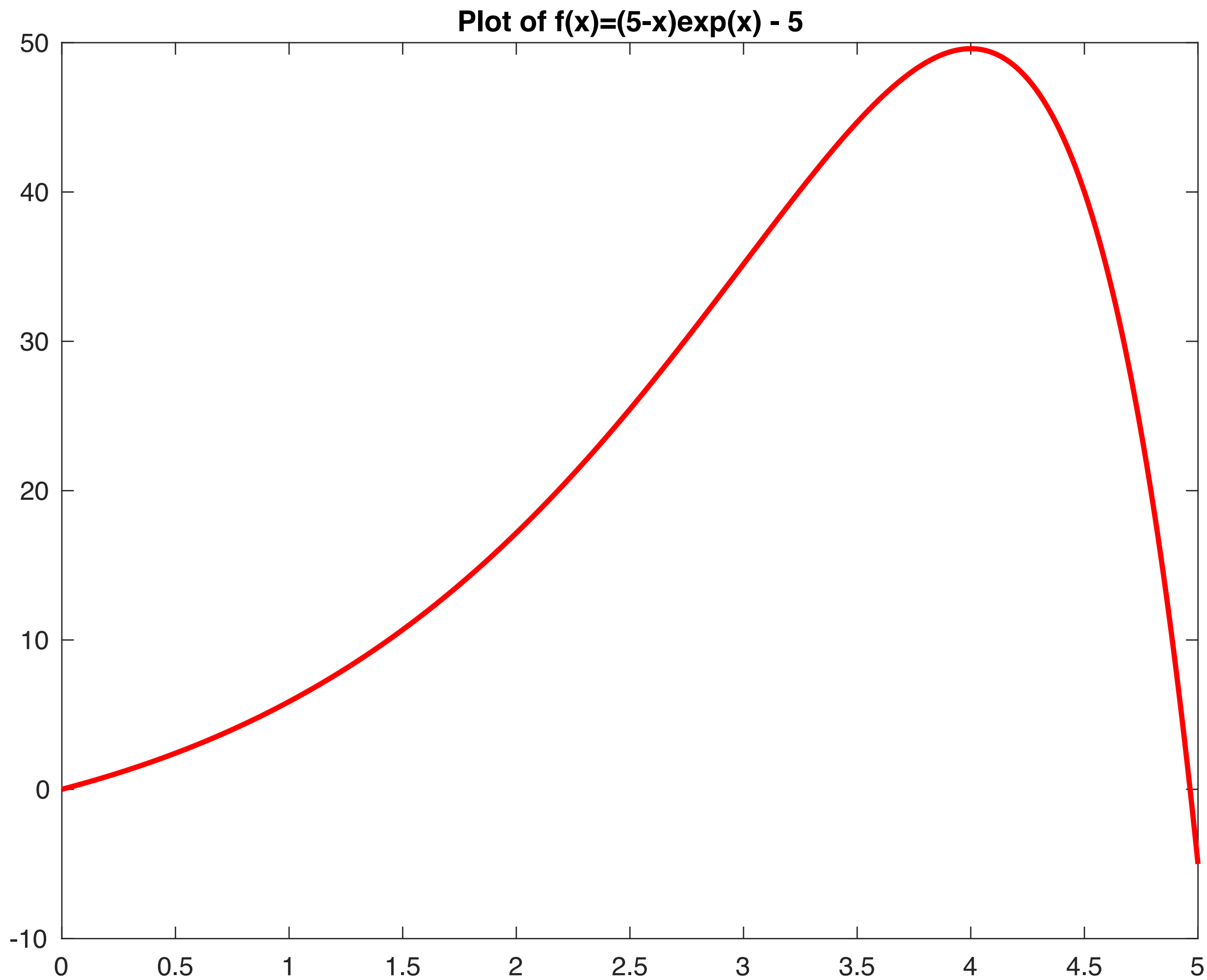
$$\frac{\partial \vec{f}}{\partial \vec{\theta}}(\vec{\theta}) = \begin{bmatrix} -\frac{2}{h^2} + \cos\theta_1 & \frac{1}{h^2} & 0 & \cdots & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} + \cos\theta_2 & \frac{1}{h^2} & \cdots & 0 & 0 \\ 0 & \frac{1}{h^2} & -\frac{2}{h^2} + \cos\theta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{2}{h^2} + \cos\theta_{n-2} & \frac{1}{h^2} \\ 0 & 0 & 0 & \cdots & \frac{1}{h^2} & -\frac{2}{h^2} + \cos\theta_{n-1} \end{bmatrix}.$$

The Newton iteration that I would use to solve this is:

$$\vec{\theta}^{(n+1)} = \vec{\theta}^{(n)} - \left[\frac{\partial \vec{f}}{\partial \vec{\theta}}(\vec{\theta}^{(n)})\right]^{-1} \cdot \vec{f}(\vec{\theta}^{(n)})$$

From my plots, we see that Newton iteration converges to results that make intuitive sense. Plots 1-3 all show a solution that starts at $\theta = 0.7$, swings to $\theta = -0.7$, then swings back to $\theta = 0.7$ despite different initial guesses. This makes since because we are not changing anything about the underlying BVP, so we expect to approach the same solution regardless of the initial guess.

Scanned with CamScanner


Plot of f(x)=(5-x)exp(x) - 5
50
40
30
20
10
0
-10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

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Problem1.m | Problem1a.m | Problem1b.m | Problem1c

```matlab
f = @(x) (5 - x)*exp(x) - 5;

a = 4;
b = 5;

tol = 10e-6;

format long;

[root, count] = bisection(a, b, f, tol)

function [x, count] = bisection(a, b, f, tol)
    ntol=ceil(log((b-a)/tol)/log(2));
    for j = 1:ntol
        x = (a+b)/2;
        fx = f(x);
        fa = f(a);

        if (fx*fa < 0)
            b = x;
        else
            a = x;
        end
        interval = [a, b]
    end
    count = ntol;
end
```

Command Window

```
interval =

   4.965110778808594   4.965118408203125


root =

   4.965110778808594


count =

    17
```

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Problem1.m | Problem1a.m | Problem1b.m | Problem1c.

```
f = @(x) (5 - x)*exp(x) - 5;
fd = @(x) (4 - x)*exp(x);

x0 = 5;

tol = 10e-8;

[root, count] = newton(x0, f, fd, tol)

function [x, count] = newton(x0, f, fd, tol)
    x = x0;
    fx = f(x);
    count = 0;
    while (abs(fx) > tol)
        fdx = fd(x);
        x = x - (fx / fdx)
        count = count + 1;
        fx = f(x)
    end
end
```

Command Window

```
    -2.012018060986165e-04


x =

   4.965114231746430


fx =

    -2.978897128969038e-10


root =

   4.965114231746430


count =

     3
```

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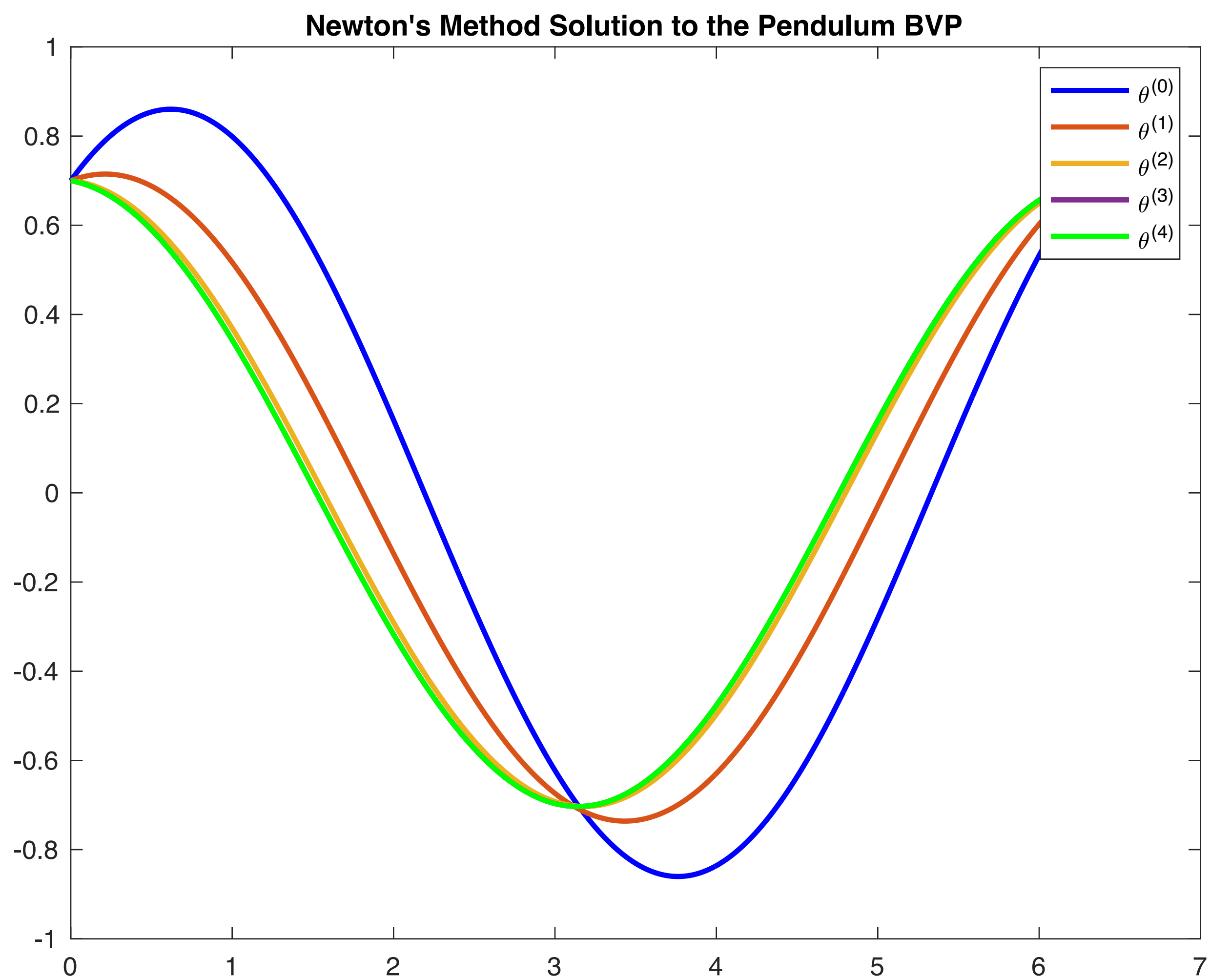
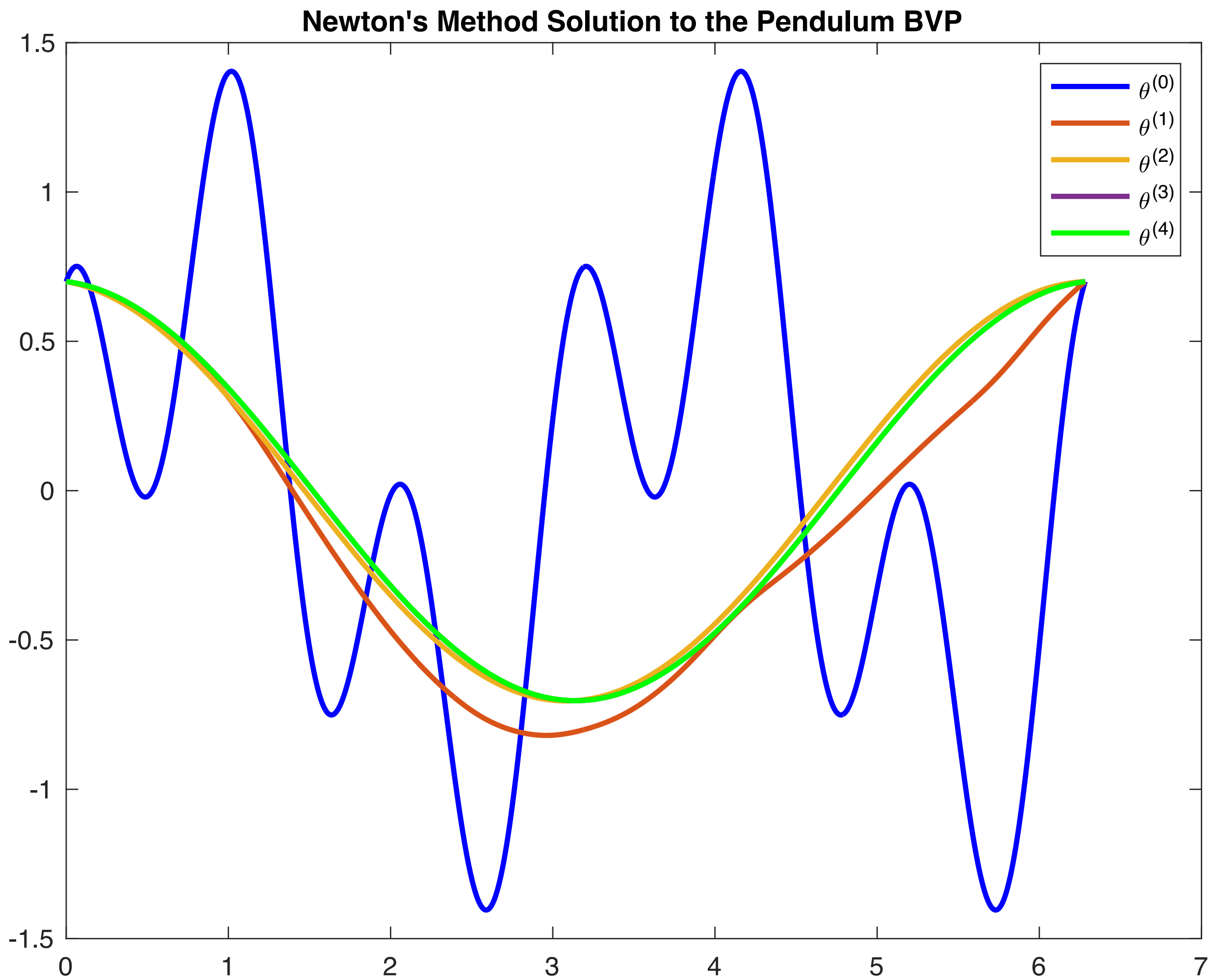
Problem1.m | Problem1a.m | Problem1b.m | Problem1c.m | Pr

```
f = @(x) (5 - x)*exp(x) - 5;

x0 = 4;
x1 = 5;

tol = 10e-8;

[root, count] = secant(x0, x1, f, tol)

function [x, count] = secant(x0, x1, f, tol)
    xprev = x0;
    x = x1;
    fx = f(x);
    count = 0;
    while (abs(fx) > tol)
        fxprev = f(xprev);
        xnext = x - ((x - xprev) / (fx - fxprev))*fx;
        xprev = x;
        x = xnext
        fx = f(x)
        count = count + 1;
    end
end
```

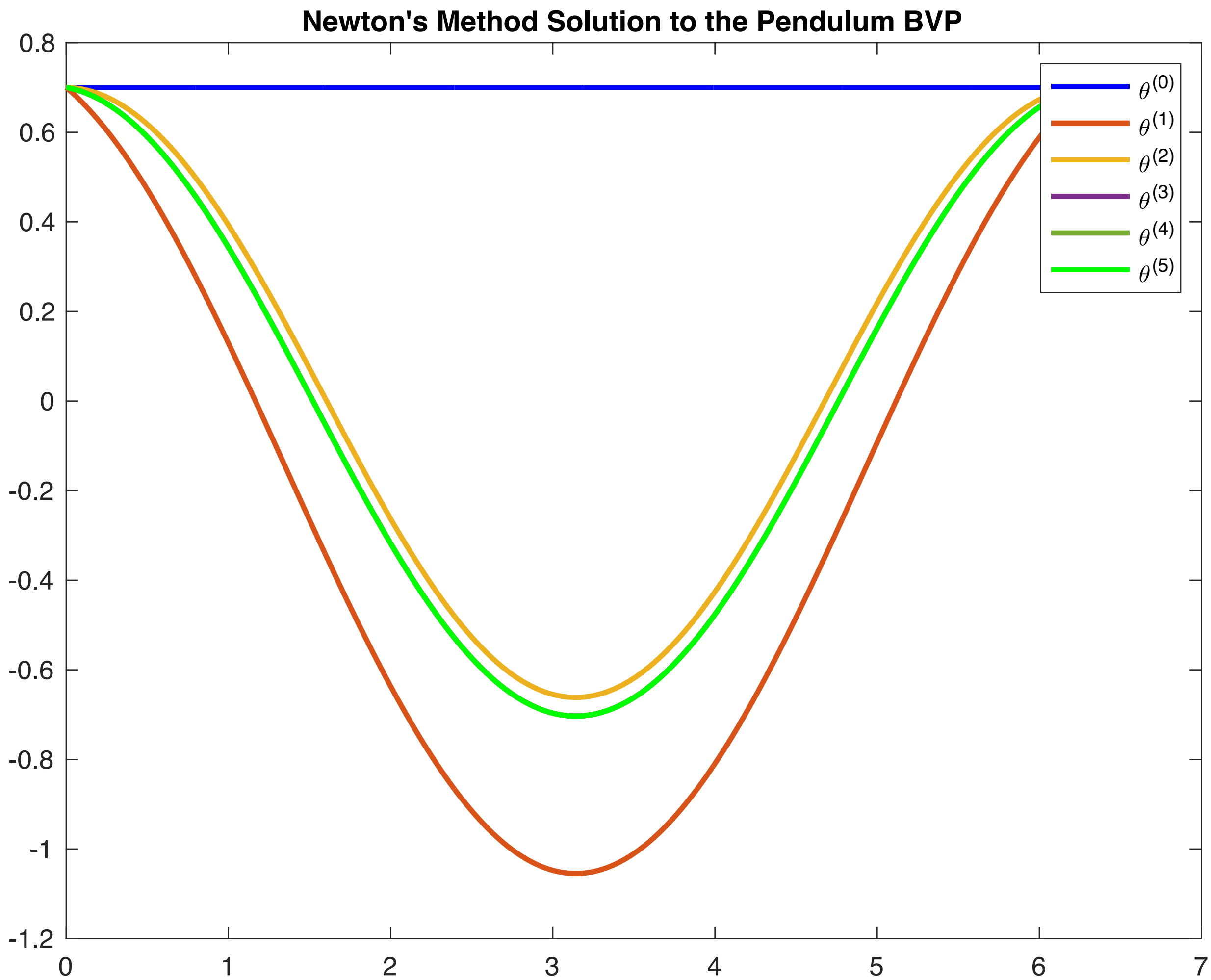
Command Window

```
x =

   4.965114231713327


fx =

     4.280998666672531e-09


root =

   4.965114231713327


count =

     5
```


Newton's Method Solution to the Pendulum BVP
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7
$\theta^{(0)}$
$\theta^{(1)}$
$\theta^{(2)}$
$\theta^{(3)}$
$\theta^{(4)}$



Newton's Method Solution to the Pendulum BVP
1.5
1
0.5
0
-0.5
-1
-1.5
0
1
2
3
4
5
6
7
$\theta^{(0)}$
$\theta^{(1)}$
$\theta^{(2)}$
$\theta^{(3)}$
$\theta^{(4)}$


Newton's Method Solution to the Pendulum BVP
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0
1
2
3
4
5
6
7
$\theta^{(0)}$
$\theta^{(1)}$
$\theta^{(2)}$
$\theta^{(3)}$
$\theta^{(4)}$
$\theta^{(5)}$

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Problem1.m | Problem1a.m | Problem1b.m | Problem1c.m | Problem4.m | +

```
n = 1000;

T = 2*pi;
alpha = 0.7*pi;
beta = 0.7*pi;
%alpha = 0.01*pi;
%beta = -0.85*pi;

h = T/n;
t = ((1:n-1)*h).';
tol = 10e-3;

theta0 = alpha*cos(t) + 0.5*sin(t);
%theta0 = 0.7*ones(n-1, 1);
%theta0 = (((beta - alpha) / (2*pi))*t + alpha).*cos(2*t);

tvals = [0 t.' T];
theta0Vals = [alpha theta0.' beta];

plot(tvals, theta0Vals, 'b', "LineWidth", 2)
title("Newton's Method Solution to the Pendulum BVP")
ylim([-5, 5])
hold on;

theta = nDnewton(n, alpha, beta, h, t, theta0, tol);

thetaVals = [alpha theta.' beta];
plot(tvals, thetaVals, 'g', "LineWidth", 2)
legend("\theta^{(0)}", "\theta^{(1)}", "\theta^{(2)}", "\theta^{(3)}", "\theta^{(4)}"
```

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Problem1.m | Problem1a.m | Problem1b.m | Problem1c.m | Problem4.m | +

```
legend('\theta_{(0)}', '\theta_{(1)}', '\theta_{(2)}', '\theta_{(3)}', '\theta_{(4)}', '\th

function theta = nDnewton(n, alpha, beta, h, t, theta0, tol)
    prevTheta = theta0;
    theta = theta0;
    error = norm(prevTheta);
    count = 0;
    tvals = [0 t.' 2*pi];
    while (error > tol)
        J = jacobian(theta, h, n);
        ftheta = f(theta, alpha, beta, h);
        Jinvftheta = J \ ftheta;

        theta = prevTheta - Jinvftheta;

        error = norm(theta - prevTheta);

        prevTheta = theta;
        count = count + 1;

        if (error > tol)
            thetaVals = [alpha theta.' beta];
            plot(tvals, thetaVals, "LineWidth", 2)
        end
    end
    count
end

function J = jacobian(theta, h, n)
    mainDiag = (-2/h^2)+cos(theta);
    otherDiags = (1/h^2)*ones(n-2, 1);
    J = diag(mainDiag, 0) + diag(otherDiags, -1) + diag(otherDiags, 1);
end

function ftheta = f(theta, alpha, beta, h)
    ftheta = zeros(length(theta), 1);
    for j = 1:length(theta)
        ftheta(j) = fj(theta, j, alpha, beta, h);
    end
end

function ftheta = fj(theta, j, alpha, beta, h)
    fullTheta = [alpha; theta; beta];
    ftheta = (1/h^2) * (fullTheta(j)-2*fullTheta(j+1)+fullTheta(j+2))+sin(fullTheta(j+1));
end
```