Assignment 6.

Due Monday, Feb. 26, at 2:30pm PST.

Reading: Ch. 5 in [Gautschi].

1. Prove that the ODE IVP

$$y'(t) = \frac{1}{t^2 + 2y(t)^2}, \quad t \geq 1,$$
$$y(1) = \eta,$$

has a unique solution for all $t \geq 1$. Find a Lipschitz constant for this problem.

2. Consider the equation of harmonic motion:

$$u'' = -ku, \quad u(t_0) = u_0, \quad u'(t_0) = v_0.$$

Here $u(t)$ represents the distance from equilibrium and $k > 0$ is a spring constant. Write this as a system of two first-order differential equations, and show that the right-hand side of your system satisfies a Lipschitz condition on $\mathbf{R}^2$. Determine the (smallest possible) Lipschitz constant for the 1-norm, the 2-norm, and the ∞-norm.

3. Consider the one-step method

$$y_{n+1} = y_n + h[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})],$$

where $\theta \in [0,1]$ is given. Note that this method is *explicit* if $\theta = 1$ and otherwise it is implicit. Show that the local truncation error is $O(h^2)$ if $\theta = 1/2$ and otherwise it is $O(h)$.

4. Compute the leading term in the local truncation error of Heun's method:

$$\begin{aligned} \tilde{y}_{n+1} &= y_n + hf(t_n, y_n), \\ y_{n+1} &= y_n + \frac{h}{2}\left[f(t_{n+1}, \tilde{y}_{n+1}) + f(t_n, y_n)\right]. \end{aligned}$$

5. The initial value problem

$$y'(t) = y(t)^2 - \sin(t) - \cos^2(t), \quad y(0) = 1$$

has the solution $y(t) = \cos(t)$. Write a computer code to solve this problem up to time $T = 8$ with various different time steps $h = T/N$, with

$$N = 25,\ 50,\ 100,\ 200,\ 400,\ 800,\ 1600.$$

Do this using two different methods:

(a) Forward Euler.

(b) The classical fourth-order Runge-Kutta method.

Plot the true solution (with a solid line) and the approximate solution (with o's) for $N = 25$. Make a log-log plot of the errors for each method vs. the stepsize h.

6. Use a method of your choice or try `ode45` in Matlab to solve the Lotka-Volterra predator-prey equations:

$$\begin{aligned} R' &= (1 - .02F)R, \\ F' &= (-1 + .03R)F, \end{aligned}$$

starting with $R_0 = F_0 = 20$. Plot $R(t)$ and $F(t)$ vs. t on the same plot, labeling each curve. Present the solution in another way with a plot of F vs. R.