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AMATH 585

Homework #6

1.) $y'(t) = \frac{1}{t^2 + 2y(t)^2}$, $t \geq 1$, $y(1) = \eta$

We want to find $L \geq 0$ such that for all $y_1, y_2 \in \mathbb{R}$, $t \geq 1$,

$$\left| \frac{1}{t^2+2y_1(t)^2} - \frac{1}{t^2+2y_2(t)^2} \right| \leq L \left| y_1(t) - y_2(t) \right|$$

Let $f(t,y) = \frac{1}{t^2+2y^2}$. Then we have

$$\left| \frac{f(t,y_1) - f(t,y_2)}{y_1 - y_2} \right| \leq L.$$

By the mean value theorem we see that if $y_1 > y_2$ (WLOG),

$$\frac{f(t,y_1) - f(t,y_2)}{y_1 - y_2} = \frac{\partial f}{\partial y}(t,y) \text{ for some } y \text{ between } y_1 \text{ \& } y_2.$$

$$\frac{\partial f}{\partial y}(t,y) = \frac{\partial}{\partial y}\left(\frac{1}{t^2+2y^2} \right) = \frac{-4y}{(t^2+2y^2)^2}.$$

$$\text{Let } L = \max_{t \geq 1} \left| \frac{f(t,y_1) - f(t,y_2)}{y_1 - y_2} \right| = \max_{t \geq 1} \left| \frac{-4y}{(t^2+2y^2)^2} \right|$$

$$= \max_{t \geq 1} 4 \left| \frac{y}{(t^2+2y^2)^2} \right| \leq 4 \left| \frac{\eta}{(1+2\eta^2)^2} \right| = \frac{4|\eta|}{(1+2\eta^2)^2}$$

Then this gives

$$\left| \frac{f(t,y_1) - f(t,y_2)}{y_1 - y_2} \right| = \left| \frac{\partial f}{\partial y}(t,y) \right| = \max_{t \geq 1} \left| \frac{\partial f}{\partial y}(t,y) \right| \leq \frac{4|\eta|}{(1+2\eta^2)^2}$$

Thus we obtain the Lipschitz constant $\frac{4|\eta|}{(1+2\eta^2)^2}$ satisfying

$$\left| \frac{1}{t^2+2y_1(t)^2} - \frac{1}{t^2+2y_2(t)^2} \right| \leq \frac{4|\eta|}{(1+2\eta^2)^2} \left| y_1(t) - y_2(t) \right|.$$

2.) $u'' = -ku$, $u(t_0) = u_0$, $u'(t_0) = v_0$, $k > 0$.

Let $x(t) = u(t)$ & $y(t) = u'(t)$. Then we obtain the system

$$x'(t) = y(t), \quad y'(t) = -k x(t), \quad x(t_0) = u_0, \quad y(t_0) = v_0.$$

This is rewritten in matrix form as

$$\vec{y}' = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix} \vec{y}, \quad \vec{y} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{y}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}.$$

We want to show that the RHS satisfies a Lipschitz condition on $\mathbb{R}^2$, which means we want to find $L > 0$ such that for all $\vec{y}, \vec{y}^* \in \mathbb{R}^2$

$$\|A\vec{y} - A\vec{y}^*\| \leq L \|\vec{y} - \vec{y}^*\| \Rightarrow \frac{\|A(\vec{y} - \vec{y}^*)\|}{\|\vec{y} - \vec{y}^*\|} \leq L.$$

We see that

$$\frac{\|A(\vec{y} - \vec{y}^*)\|}{\|\vec{y} - \vec{y}^*\|} \leq \|A\|,$$

so define $L = \|A\|$. We will find L for the $1, 2$, and ∞ norms, and this will prove the RHS satisfies a Lipschitz condition on $\mathbb{R}^2$.

$$\|A\|_1 = \max\{|0| + |1|, |-k| + |0|\} = \max\{1, k\}$$

$$\|A\|_\infty = \max\{|0| + |-k|, |1| + |0|\} = \max\{1, k\}$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^*A)} = \max\{\sqrt{1}, \sqrt{k^2}\} = \max\{1, k\}$$

$$A^*A = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} = \begin{bmatrix} k^2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda = 1, k^2$$

3.) $y_{n+1} = y_n + h[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})]$

Let $u(t)$ be the reference solution. Then $u'(t_n) = y'(t_n) = f(t_n, y_n)$
We will start by calculating each difference quotient

$$\frac{y_{n+1} - y_n}{h} = \theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1}) = \theta u'(t_n) + (1-\theta) y'(t_{n+1})$$

$$= \theta u'(t_n) + (1-\theta)\left[y'(t_n) + h y''(t_n) + \frac{h^2}{2} y'''(t_n) + \mathcal{O}(h^3)\right]$$

$$= \theta u'(t_n) + (1-\theta)\left[u'(t_n) + h u''(t_n) + \frac{h^2}{2} u'''(t_n) + \mathcal{O}(h^3)\right]$$

$$= u'(t_n) + (1-\theta) h u''(t_n) + (1-\theta)\frac{h^2}{2} u'''(t_n) + \mathcal{O}(h^3)$$

$$\frac{u(t_{n+1}) - u(t_n)}{h} = \frac{u(t_n) + h u'(t_n) + \frac{h^2}{2} u''(t_n) + \frac{h^3}{6} u'''(t_n) + \mathcal{O}(h^4) - u(t_n)}{h}$$

$$= u'(t_n) + \frac{h}{2} u''(t_n) + \frac{h^2}{6} u'''(t_n) + \mathcal{O}(h^3)$$

$$\frac{y_{n+1} - y_n}{h} - \frac{u(t_{n+1}) - u(t_n)}{h} = u'(t_n) + (1-\theta) h u''(t_n) + (1-\theta)\frac{h^2}{2} u'''(t_n) - u'(t_n) - \frac{h}{2} u''(t_n) - \frac{h^2}{6} u'''(t_n) + \mathcal{O}(h^3)$$

$$= \left(\frac{1}{2} - \theta\right) h u''(t_n) + \left[(1-\theta)\frac{1}{2} - \frac{1}{6}\right] h^2 u'''(t_n) + \mathcal{O}(h^3)$$

$$= \left(\frac{1}{2} - \theta\right) h u''(t_n) + \left(\frac{1}{3} - \frac{\theta}{2}\right) h^2 u'''(t_n) + \mathcal{O}(h^3)$$

From this we see that if $\theta \neq 1/2$, then the $\mathcal{O}(h)$ term will be nonzero so the LTE for this method will be $\mathcal{O}(h)$. If $\theta = 1/2$, then the $\mathcal{O}(h)$ term vanishes, so the LTE becomes $\mathcal{O}(h^2)$ in this case.

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4) $\tilde{y}_{n+1} = y_n + h f(t_n, y_n)$

$y_{n+1} = y_n + \frac{h}{2}[f(t_{n+1}, \tilde{y}_{n+1}) + f(t_n, y_n)]$

Let $u(t)$ be the reference solution. Then $u'(t_n) = y'(t_n) = f(t_n, y_n)$

Let $f(t_n, y_n) = f$. Then

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2} f(t_{n+1}, \tilde{y}_{n+1}) + \frac{1}{2} f = \frac{1}{2} f(t_n + h, y_n + hf) + \frac{1}{2} f$$

Taylor expanding, we see that

$$f(t_n + h, y_n + hf) = f + hf_t + hff_y + \frac{h^2}{2} f_{tt} + h^2 f f_{ty} + \frac{h^2}{2} f^2 f_{yy} + \mathcal{O}(h^3)$$

$$\frac{u(t_{n+1}) - u(t_n)}{h} = \frac{1}{h}\left[u(t_n) + hu'(t_n) + \frac{h^2}{2} u''(t_n) + \frac{h^3}{6} u'''(t_n) + \mathcal{O}(h^4) - u(t_n)\right]$$

$$= u'(t_n) + \frac{h}{2} u''(t_n) + \frac{h^2}{6} u'''(t_n) + \mathcal{O}(h^3)$$

$$= f + \frac{h}{2}[f_t + ff_y] + \frac{h^2}{6}[f_{tt} + ff_{ty} + f^2 f_{yy} + f_y f_t] + \mathcal{O}(h^3)$$

$$\frac{y_{n+1} - y_n}{h} - \frac{u(t_{n+1}) - u(t_n)}{h} = \frac{1}{2} f + \frac{h}{2} f_t + \frac{h}{2} ff_y + \frac{h^2}{4} f_{tt} + \frac{h^2}{2} ff_{ty} + \frac{h^2}{4} f^2 f_{yy} + \mathcal{O}(h^3) + \frac{1}{2} f$$

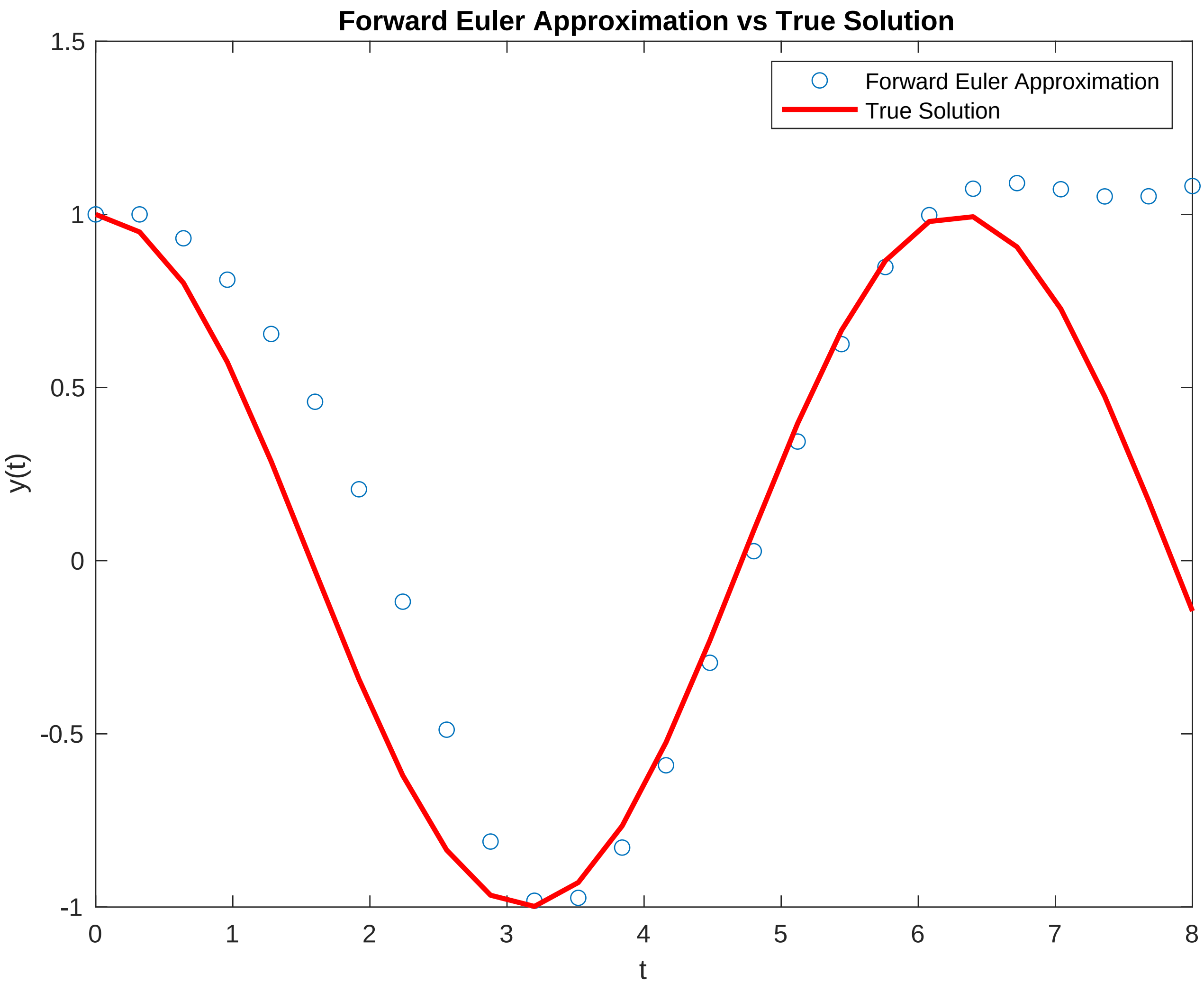
$$- f - \frac{h}{2} f_t - \frac{h}{2} ff_y - \frac{h^2}{6} f_{tt} - \frac{h^2}{6} ff_{ty} - \frac{h^2}{6} f^2 f_{yy} - \frac{h^2}{6} f_y f_t + \mathcal{O}(h^3)$$

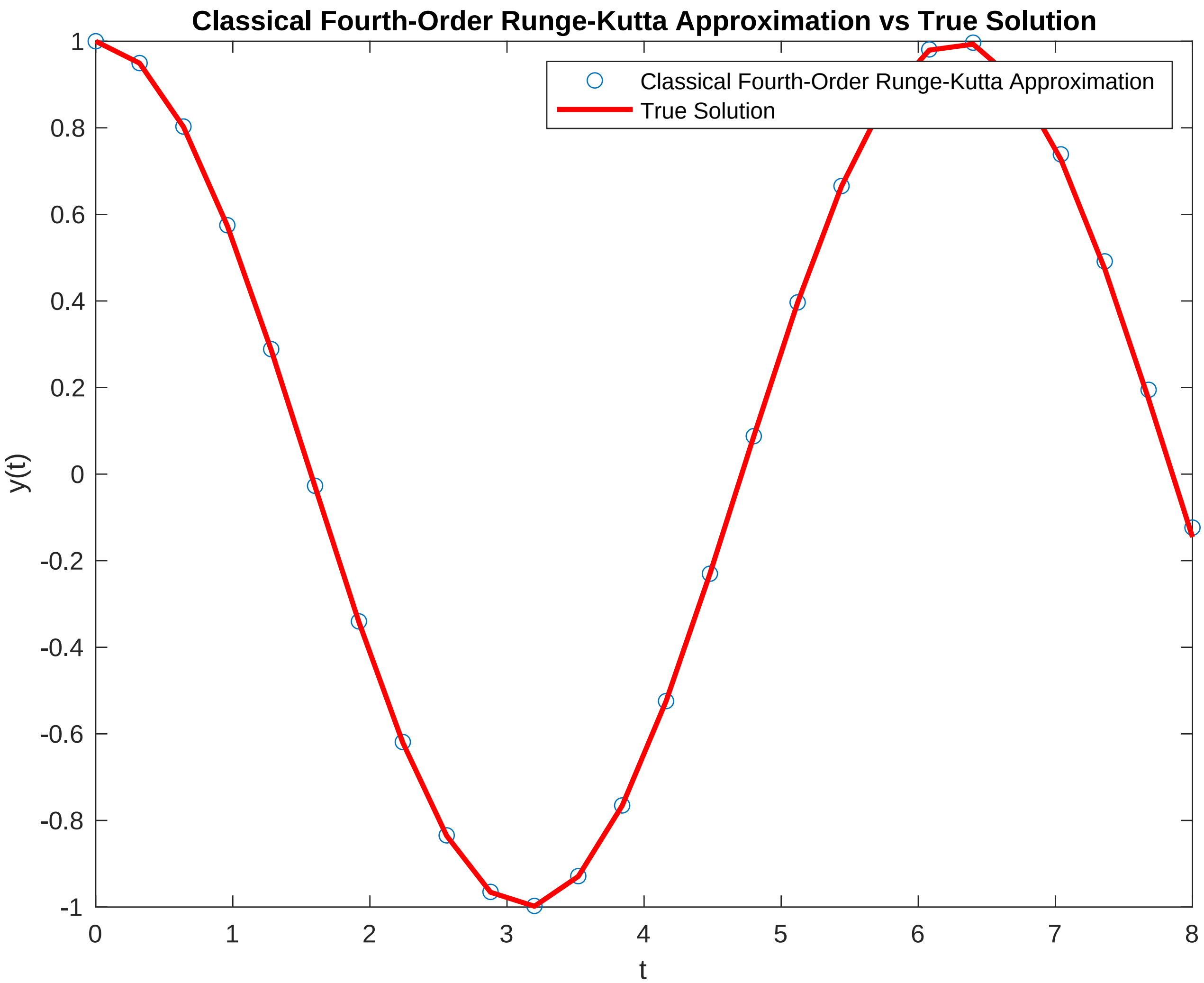
$$= \left[\frac{1}{4} - \frac{1}{6}\right] h^2 f_{tt} + \left(\frac{1}{2} - \frac{1}{6}\right] h^2 ff_{ty} + \left[\frac{1}{4} - \frac{1}{6}\right] h^2 f^2 f_{yy} - \frac{1}{6} h^2 f_y f_t + \mathcal{O}(h$$

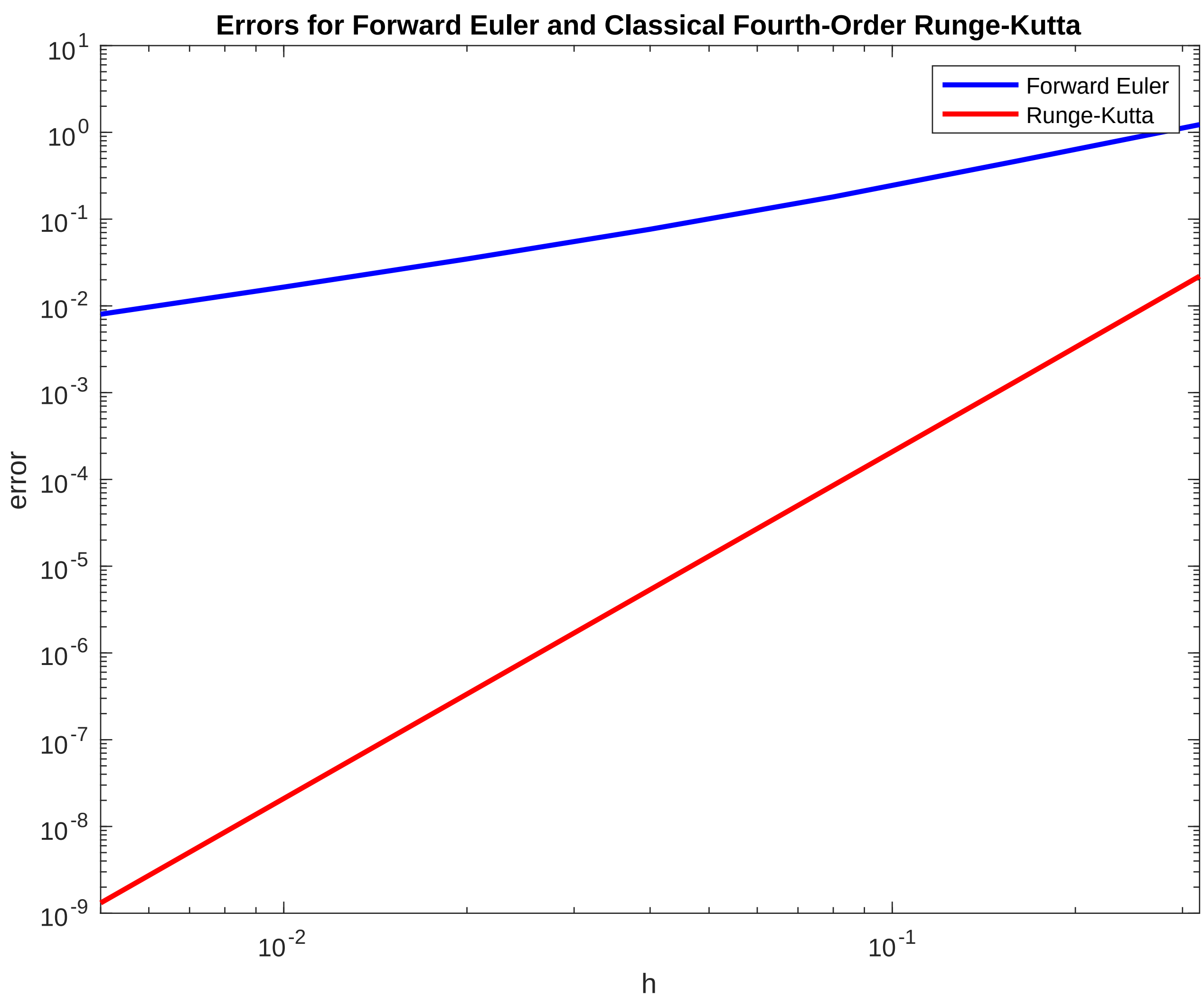
$$= \frac{1}{12} h^2 f_{tt} + \frac{1}{3} h^2 ff_{ty} + \frac{1}{12} h^2 f^2 f_{yy} - \frac{1}{6} h^2 f_y f_t + \mathcal{O}(h^3)$$

$$= \left[\frac{1}{12} f_{tt} + \frac{1}{3} ff_{ty} + \frac{1}{12} f^2 f_{yy} - \frac{1}{6} f_y f_t\right] h^2 + \mathcal{O}(h^3)$$

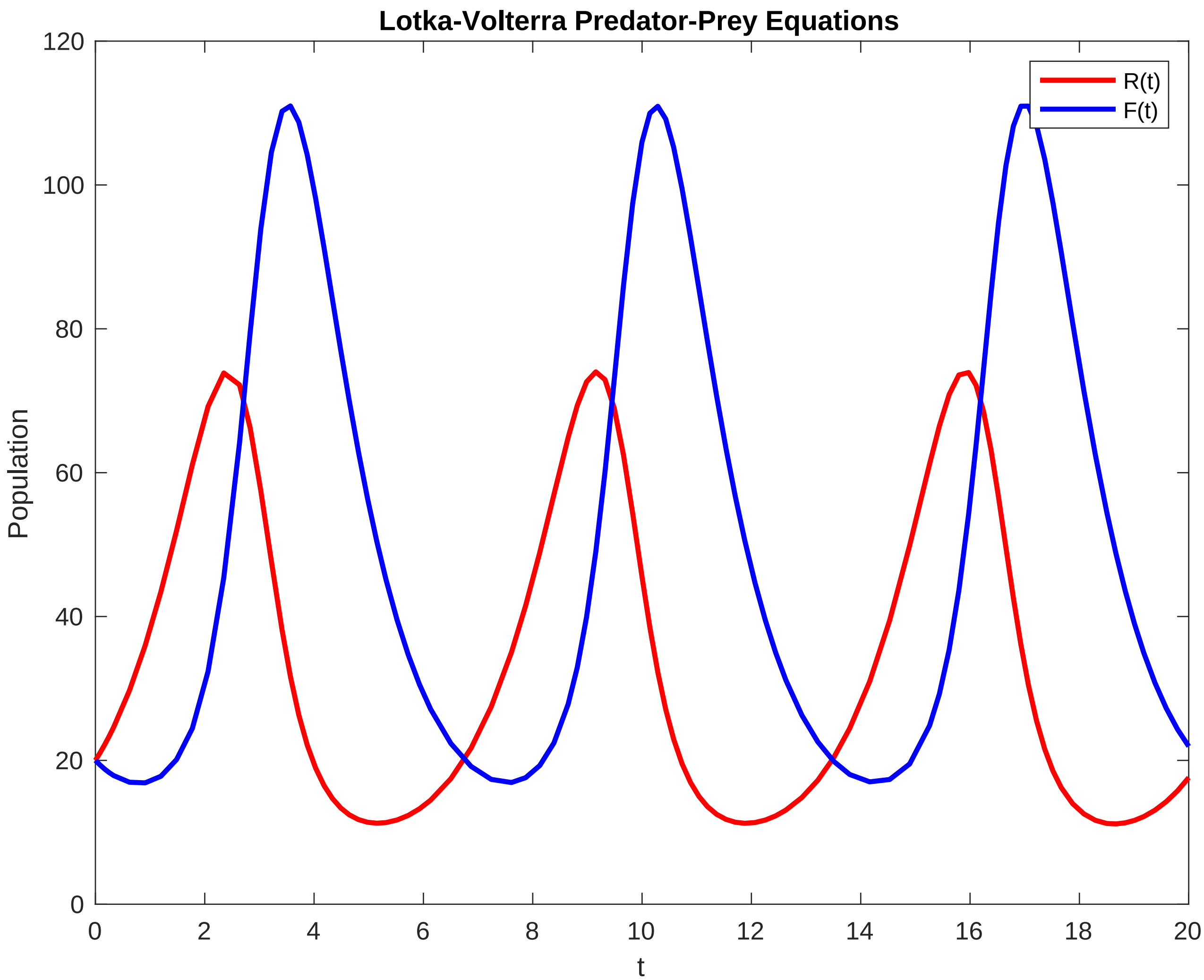
Thus the leading term in the local truncation error for Heun's method is $\left[\frac{1}{12} f_{tt} + \frac{1}{3} ff_{ty} + \frac{1}{12} f^2 f_{yy} - \frac{1}{6} f_y f_t\right] h^2$.

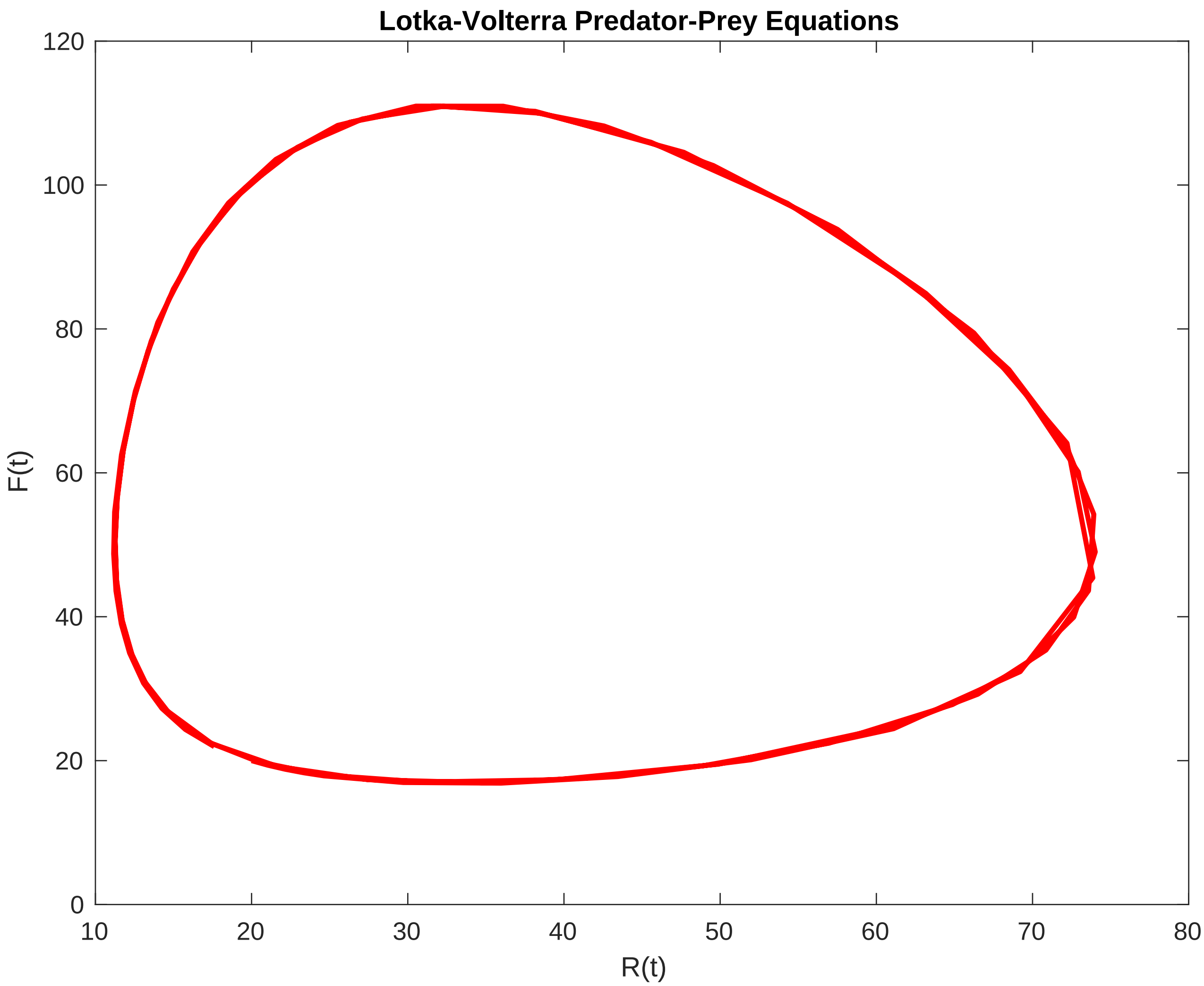

Forward Euler Approximation vs True Solution
Forward Euler Approximation
True Solution
y(t)
t
1.5
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8


Classical Fourth-Order Runge-Kutta Approximation vs True Solution
Classical Fourth-Order Runge-Kutta Approximation
True Solution
y(t)
t
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7
8



Errors for Forward Euler and Classical Fourth-Order Runge-Kutta
Forward Euler
Runge-Kutta
error
h
10^1
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
10^-9
10^-2
10^-1


Lotka-Volterra Predator-Prey Equations
R(t)
F(t)
Population
t
0
20
40
60
80
100
120
0
2
4
6
8
10
12
14
16
18
20



Lotka-Volterra Predator-Prey Equations
120
100
80
60
40
20
0
F(t)
10
20
30
40
50
60
70
80
R(t)