Assignment 7.

Due Wednesday, March 6, at 2:30pm PST.

Reading: Ch. 11 from my book with Tim Chartier; available on the Canvas web page next to "Numerical solution of the initial value problem for ODE's", under heading Syllabus.

1. Determine the coefficients b_0, b_1, b_2 for the third order, 2-step Adams-Moulton method:

$$y_{k+2} = y_{k+1} + h[b_0 f(t_k, y_k) + b_1 f(t_{k+1}, y_{k+1}) + b_2 f(t_{k+2}, y_{k+2})].$$

   Do this in two different ways:

   (a) Using the theorem that for a general linear m-step method,

$$\sum_{\ell=0}^{m} a_\ell y_{k+\ell} = h \sum_{\ell=0}^{m} b_\ell f(t_{k+\ell}, y_{k+\ell}), \quad a_m = 1,$$

   the LTE is of order $p \geq 1$ if and only if

$$\sum_{\ell=0}^{m} a_\ell = 0 \quad \text{and} \quad \sum_{\ell=0}^{m} \ell^j a_\ell = j \sum_{\ell=0}^{m} \ell^{j-1} b_\ell, \quad j = 1, \ldots, p.$$

   (b) Using the relation

$$y(t_{k+2}) = y(t_{k+1}) + \int_{t_{k+1}}^{t_{k+2}} f(s, y(s))\, ds,$$

   and replacing f in the integral by a quadratic polynomial $p(s)$ that takes the values $f(t_k, y_k)$, $f(t_{k+1}, y_{k+1})$, and $f(t_{k+2}, y_{k+2})$ at the points t_k, t_{k+1}, and t_{k+2}.

2. What is the order of the local truncation error for each of the following linear multistep methods, and which of these methods are *convergent*? Justify your answers.

   (a) $y_k - y_{k-2} = h[f(t_k, y_k) - 3f(t_{k-1}, y_{k-1}) + 4f(t_{k-2}, y_{k-2})]$.

   (b) $y_k - 2y_{k-1} + y_{k-2} = h[f(t_k, y_k) - f(t_{k-1}, y_{k-1})]$.

   (c) $y_k - y_{k-1} - y_{k-2} = h[f(t_k, y_k) - f(t_{k-1}, y_{k-1})]$.

3. (Computational Problem.) Consider the system of equations

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} -1000 & 1 \\ 0 & -1/10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$u_1(0) = 1, \quad u_2(0) = 2,$$

   whose exact solution is

$$u_1(t) = \frac{9979}{9999} e^{-1000t} + \frac{20}{9999} e^{-t/10}, \quad u_2(t) = 2e^{-t/10}.$$

(a) Use the classical fourth-order Runge-Kutta method to solve this system of equations, integrating out to $T = 1$. What size time step is necessary to achieve a reasonably accurate approximate solution? Turn in a plot of $u_1(t)$ and $u_2(t)$ that shows what happens if you choose the time step too large, and also turn in a plot of $u_1(t)$ and $u_2(t)$ once you have found a good size time step.

(b) Now solve this system of ODEs using MATLAB's `ode23s` routine (which uses a second-order implicit method). How many time steps does it require? Can you explain why a second-order method can solve this problem accurately using fewer time steps than the fourth-order Runge-Kutta method?

4. Show that the implicit midpoint method

$$y_{k+1} = y_k + hf(t_{k+1/2}, (y_k + y_{k+1})/2), \quad t_{k+1/2} = t_k + h/2$$

is A-stable.

5. Consider the A-stable trapezoidal method applied to the test problem $y' = \lambda y$:

$$y_{k+1} = y_k + \frac{h}{2}\lambda[y_k + y_{k+1}].$$

Of course, one can easily solve this linear equation for y_{k+1}, but suppose we used *fixed point iteration* to solve for y_{k+1}: Taking $w^{(0)}$ as an initial guess for y_{k+1}, set

$$w^{(j+1)} = y_k + \frac{h}{2}\lambda[y_k + w^{(j)}], \quad j = 0, 1, \ldots.$$

What conditions must $h\lambda$ satisfy so that $w^{(j)}$ will converge to y_{k+1}? [Note that if we use the trapezoidal rule but solve for y_{k+1} using fixed point iteration, we lose the benefits of A-stability!]