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AMATH 585

Homework #7

1.) $\frac{1}{h}[y_{k+2} - y_{k+1}] = b_0 f(t_k, y_k) + b_1 f(t_{k+1}, y_{k+1}) + b_2 f(t_{k+2}, y_{k+2})$

a.) $a_2 = 1, a_1 = -1, a_0 = 0 \Rightarrow 1 - 1 + 0 = 0$ ✓

$2 \cdot a_2 + 1 \cdot a_1 + 0 \cdot a_0 = b_2 + b_1 + b_0 \Rightarrow 1 = b_2 + b_1 + b_0$
$2^2 \cdot a_2 + 1^2 \cdot a_1 + 0^2 \cdot a_0 = 2(2 \cdot b_2 + 1 \cdot b_1 + 0 \cdot b_0) \Rightarrow 3 = 4b_2 + 2b_1$
$2^3 \cdot a_2 + 1^3 \cdot a_1 + 0^3 \cdot a_0 = 3(2^2 \cdot b_2 + 1^2 \cdot b_1 + 0^2 \cdot b_0) \Rightarrow 7 = 12b_2 + 3b_1$

$3 = 4b_2 + 2b_1 \Rightarrow b_1 = 3/2 - 2b_2$

$\Rightarrow 7 = 12b_2 + 3(3/2 - 2b_2) = 12b_2 + 9/2 - 6b_2 = 9/2 + 6b_2$

$\Rightarrow 5/2 = 6b_2 \Rightarrow b_2 = 5/12 \Rightarrow b_1 = 3/2 - 2(5/12) = 9/6 - 5/6 = 4/6 = 2/3$

$\Rightarrow 1 = 5/12 + 2/3 + b_0 \Rightarrow b_0 = 1 - 5/12 - 2/3 = \frac{12-5-8}{12} = -\frac{1}{12}$

$\boxed{b_2 = 5/12, \; b_1 = 2/3, \; b_0 = -\frac{1}{12}}$

Thus this Adam's-Moulton method is of order 3 since it satisfies $\sum_{\ell=0}^{m} a_\ell = 0$, $\sum_{\ell=0}^{m} \ell^j a_\ell = j \sum_{\ell=0}^{m} \ell^{j-1} b_\ell$, $j = 1, 2, 3$.

b.) $y(t_{k+2}) = y(t_{k+1}) + \int_{t_{k+1}}^{t_{k+2}} f(s, y(s))\,ds$

$$p(s) = \left(\frac{s - t_{k+1}}{t_{k+2} - t_{k+1}}\right)\left(\frac{s - t_k}{t_{k+2} - t_k}\right) f(t_{k+2}, y_{k+2}) + \left(\frac{s - t_{k+2}}{t_{k+1} - t_{k+2}}\right)\left(\frac{s - t_k}{t_{k+1} - t_k}\right) f(t_{k+1}, y_{k+1})$$
$$+ \left(\frac{s - t_{k+2}}{t_k - t_{k+2}}\right)\left(\frac{s - t_{k+1}}{t_k - t_{k+1}}\right) f(t_k, y_k)$$

$$\Rightarrow b_2 = \frac{1}{h}\int_{t_{k+1}}^{t_{k+2}} \left(\frac{s - t_{k+1}}{t_{k+2} - t_{k+1}}\right)\left(\frac{s - t_k}{t_{k+2} - t_k}\right) ds$$
$$b_1 = \frac{1}{h}\int_{t_{k+1}}^{t_{k+2}} \left(\frac{s - t_{k+2}}{t_{k+1} - t_{k+2}}\right)\left(\frac{s - t_k}{t_{k+1} - t_k}\right) ds$$
$$b_0 = \frac{1}{h}\int_{t_{k+1}}^{t_{k+2}} \left(\frac{s - t_{k+2}}{t_k - t_{k+2}}\right)\left(\frac{s - t_{k+1}}{t_k - t_{k+1}}\right) ds$$

1.) b.) continued.)

$$b_2 = \frac{1}{h}\int_{t_{k+1}}^{t_{k+2}} \left(\frac{s-t_{k+1}}{h}\right)\left(\frac{s-t_k}{2h}\right) ds = \frac{1}{2h^3}\int_{t_{k+1}}^{t_{k+2}} \left(s^2 - (t_{k+1}+t_k)s + t_{k+1}t_k\right) ds$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}s^3 - \frac{1}{2}(t_{k+1}+t_k)s^2 + t_{k+1}t_k s\right]_{t_{k+1}}^{t_{k+2}}, \quad t_{k+2} = t_{k+1}+h, \quad t_k = t_{k+1}-h$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}s^3 - \frac{1}{2}(2t_{k+1}-h)s^2 + (t_{k+1}^2 - ht_{k+1})s\right]_{t_{k+1}}^{t_{k+1}+h}$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}\left((t_{k+1}+h)^3 - t_{k+1}\right) - \frac{1}{2}(2t_{k+1}-h)\left((t_{k+1}+h)^2 - t_{k+1}\right) + (t_{k+1}^2 - ht_{k+1})(t_{k+1}+h-t_{k+1})\right]$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}(3ht_{k+1}^2 + 3h^2t_{k+1} + h^3) - \frac{1}{2}(2t_{k+1}-h)(2ht_{k+1}+h^2) + (t_{k+1}^2 - ht_{k+1})h\right]$$

$$= \frac{1}{2h^3}\left[ht_{k+1}^2 + h^2t_{k+1} + h^3/3 - \frac{1}{2}(4ht_{k+1}^2 - h^3) + ht_{k+1}^2 - h^2t_{k+1}\right]$$

$$= \frac{1}{2h^3}\left[t_{k+1}^2(h-2h+h) + t_{k+1}(h^2-h^2) + h^3/3 + h^3/2\right]$$

$$= \frac{1}{2h^3}\cdot\frac{5h^3}{6} = \boxed{5/12}$$

$$b_1 = \frac{1}{h}\int_{t_{k+1}}^{t_{k+2}} \left(\frac{s-t_{k+2}}{-h}\right)\left(\frac{s-t_k}{h}\right) ds = -\frac{1}{h^3}\int_{t_{k+1}}^{t_{k+2}} \left(s^2 - (t_{k+2}+t_k)s + t_{k+2}t_k\right) ds$$

$$= -\frac{1}{h^3}\left[\frac{1}{3}s^3 - \frac{1}{2}(t_{k+2}+t_k)s^2 + t_{k+2}t_k s\right]_{t_{k+1}}^{t_{k+2}}, \quad t_{k+2} = t_{k+1}+h, \quad t_k = t_{k+1}-h$$

$$= -\frac{1}{h^3}\left[\frac{1}{3}s^3 - t_{k+1}s^2 + (t_{k+1}^2 - h^2)s\right]_{t_{k+1}}^{t_{k+1}+h}$$

$$= -\frac{1}{h^3}\left[\frac{1}{3}\left((t_{k+1}+h)^3 - t_{k+1}^3\right) - t_{k+1}\left((t_{k+1}+h)^2 - t_{k+1}^2\right) + (t_{k+1}^2 - h^2)(t_{k+1}+h-t_{k+1})\right]$$

$$= -\frac{1}{h^3}\left[ht_{k+1}^2 + h^2t_{k+1} + h^3/3 - t_{k+1}(2ht_{k+1}+h^2) + ht_{k+1}^2 - h^3\right]$$

$$= -\frac{1}{h^3}\left[t_{k+1}^2(h-2h+h) + t_{k+1}(h^2-h^2) + h^3/3 - h^3\right] = \left(-\frac{1}{h^3}\right)\left(-\frac{2h^3}{3}\right) = \boxed{2/3}$$

$$b_0 = \frac{1}{h}\int_{t_{k+1}}^{t_{k+2}} \left(\frac{s-t_{k+2}}{-2h}\right)\left(\frac{s-t_{k+1}}{-h}\right) ds = \frac{1}{2h^3}\int_{t_{k+1}}^{t_{k+2}} \left(s^2 - (t_{k+2}+t_{k+1})s + t_{k+2}t_{k+1}\right) ds$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}s^3 - \frac{1}{2}(t_{k+2}+t_{k+1})s^2 + t_{k+2}t_{k+1}s\right]_{t_{k+1}}^{t_{k+2}}, \quad t_{k+2} = t_{k+1}+h, \quad t_k = t_{k+1}-h$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}s^3 - \frac{1}{2}(2t_{k+1}+h)s^2 + (t_{k+1}^2 + ht_{k+1})s\right]_{t_{k+1}}^{t_{k+1}+h}$$

$$= \frac{1}{2h^3}\left[\frac{1}{3}\left((t_{k+1}+h)^3 - t_{k+1}^3\right) - \frac{1}{2}(2t_{k+1}+h)\left((t_{k+1}+h)^2 - t_{k+1}^2\right) + (t_{k+1}^2 + ht_{k+1})h\right]$$

$$= \frac{1}{2h^3}\left[ht_{k+1}^2 + h^2t_{k+1} + h^3/3 - \frac{1}{2}(2t_{k+1}+h)(2ht_{k+1}+h^2) + ht_{k+1}^2 + h^2t_{k+1}\right]$$

$$= \frac{1}{2h^3}\left[t_{k+1}^2(h-2h+h) + t_{k+1}(h^2-h^2-h^2+h^2) + h^3/3 - \frac{h^3}{2}\right] = \frac{1}{2h^3}\cdot\left(-\frac{h^3}{6}\right) = \boxed{-\frac{1}{12}}$$

2.) a) $y_k - y_{k-2} = h[f(t_k, y_k) - 3f(t_{k-1}, y_{k-1}) + 4f(t_{k-2}, y_{k-2})]$

$\Rightarrow a_2 = 1, a_1 = 0, a_0 = -1, b_2 = 1, b_1 = -3, b_0 = 4$

By theorem 11.3.1 we see that:

$a_2 + a_1 + a_0 = 1 + 0 - 1 = 0$ ✓

$2a_2 + 1a_1 + 0a_0 = 2$, $b_2 + b_1 + b_0 = 1 - 3 + 4 = 2$ ✓

$2^2 a_2 + 1^2 a_1 + 0^2 a_0 = 4$, $2(2b_2 + 1b_1 + 0b_0) = 2(2-3) = -2$ X

Thus the LTE is $O(h)$. Its characteristic polynomial is

$\chi(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) \Rightarrow \lambda_1 = 1, \lambda_2 = -1.$

This means the root condition is satisfied since $\lambda_1 = 1$ & $\lambda_2 = -1$ are both simple. By the Dahlquist Equivalence theorem since $p = 1$ and $\chi(\lambda)$ satisfies the root condition we know this LMM is convergent.

b) $y_k - 2y_{k-1} + y_{k-2} = h[f(t_k, y_k) - f(t_{k-1}, y_{k-1})]$

$\Rightarrow a_2 = 1, a_1 = -2, a_0 = 1, b_2 = 1, b_1 = -1, b_0 = 0$

By theorem 11.3.1 we see that:

$a_2 + a_1 + a_0 = 1 - 2 + 1 = 0$ ✓

$2a_2 + 1a_1 + 0a_0 = 2 - 2 = 0$, $b_2 + b_1 + b_0 = 1 - 1 + 0 = 0$ ✓

$2^2 a_2 + 1^2 a_1 + 0^2 a_0 = 4 - 2 = 2$, $2(2b_2 + 1b_1 + 0b_0) = 2(2-1) = 2$ ✓

$2^3 a_2 + 1^3 a_1 + 0^3 a_0 = 8 - 2 = 6$, $3(2^2 b_2 + 1^2 b_1 + 0^2 b_0) = 3(4-1) = 9$ X

Thus the LTE is $O(h^2)$. Its characteristic polynomial is

$\chi(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \Rightarrow \lambda = 1$

Since $\lambda = 1$ is not simple, so the root condition is not satisfied.

2.) b.) continued.)

Since the root condition is not satisfied, the Dahlquist Equivalence Theorem tells us that this LMM is not convergent.

c.) $y_k - y_{k-1} - y_{k-2} = h[f(t_k, y_k) - f(t_{k-1}, y_{k-1})]$

$\Rightarrow a_2 = 1, a_1 = -1, a_0 = -1, b_2 = 1, b_1 = -1, b_0 = 0$

$a_2 + a_1 + a_0 = 1 - 1 - 1 = -1 \neq 0$ X

By theorem 11.3.1 this LMM has LTE of order $p < 1$. From the Dahlquist Equivalence Theorem this means this LMM is not convergent.

3.) a.) For my code, I did not get a reasonably accurate approximate solution until I used a time step of $h = \frac{1}{400}$. Since we integrate to $T = 1$, this means I needed 400 time steps.

b.) Using ode23s, only 51 time steps were needed. This is much less than the classical fourth-order Runge-Kutta method. This is because even though ode23s is only a second order method, the problem is stiff, so the explicit classical fourth-order Runge-Kutta method takes much longer to converge than the implicit ode23s method. Implicit methods are much better at solving stiff problems than explicit methods even when the order is smaller. We can tell this problem is stiff by seeing the true solution decays at two very different rates.

4.) **Proof:** Let $y_{k+1} = y_k + hf(t_{k+\frac{1}{2}}, \frac{(y_k + y_{k+1})}{2})$, $t_{k+\frac{1}{2}} = t_k + h/2$ and let us consider the test equation $y' = \lambda y$. Then $f(t,y) = \lambda y$, so

$$y_{k+1} = y_k + h\lambda\left(\frac{y_k + y_{k+1}}{2}\right) = y_k + \frac{h\lambda}{2}y_k + \frac{h\lambda}{2}y_{k+1}$$

$$\Rightarrow y_{k+1} - \frac{h\lambda}{2}y_{k+1} = y_k + \frac{h\lambda}{2} \Rightarrow \left(1 - \frac{h\lambda}{2}\right)y_{k+1} = \left(1 + \frac{h\lambda}{2}\right)y_k$$

$$\Rightarrow y_{k+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}y_k \Rightarrow y_k = \left(\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}\right)^k y_0$$

This means this method has a region of absolute stability

$$\left|\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}\right| < 1 \Rightarrow \left|1 + \frac{h\lambda}{2}\right| < \left|1 - \frac{h\lambda}{2}\right|.$$

This is the same region of absolute stability as the trapezoidal method in the textbook, so this condition is only satisfied when $Re(\lambda) < 0$. Thus the implicit midpoint method $y_{k+1} = y_k + hf(t_{k+\frac{1}{2}}, \frac{(y_k + y_{k+1})}{2})$ must be A-stable. $\otimes$

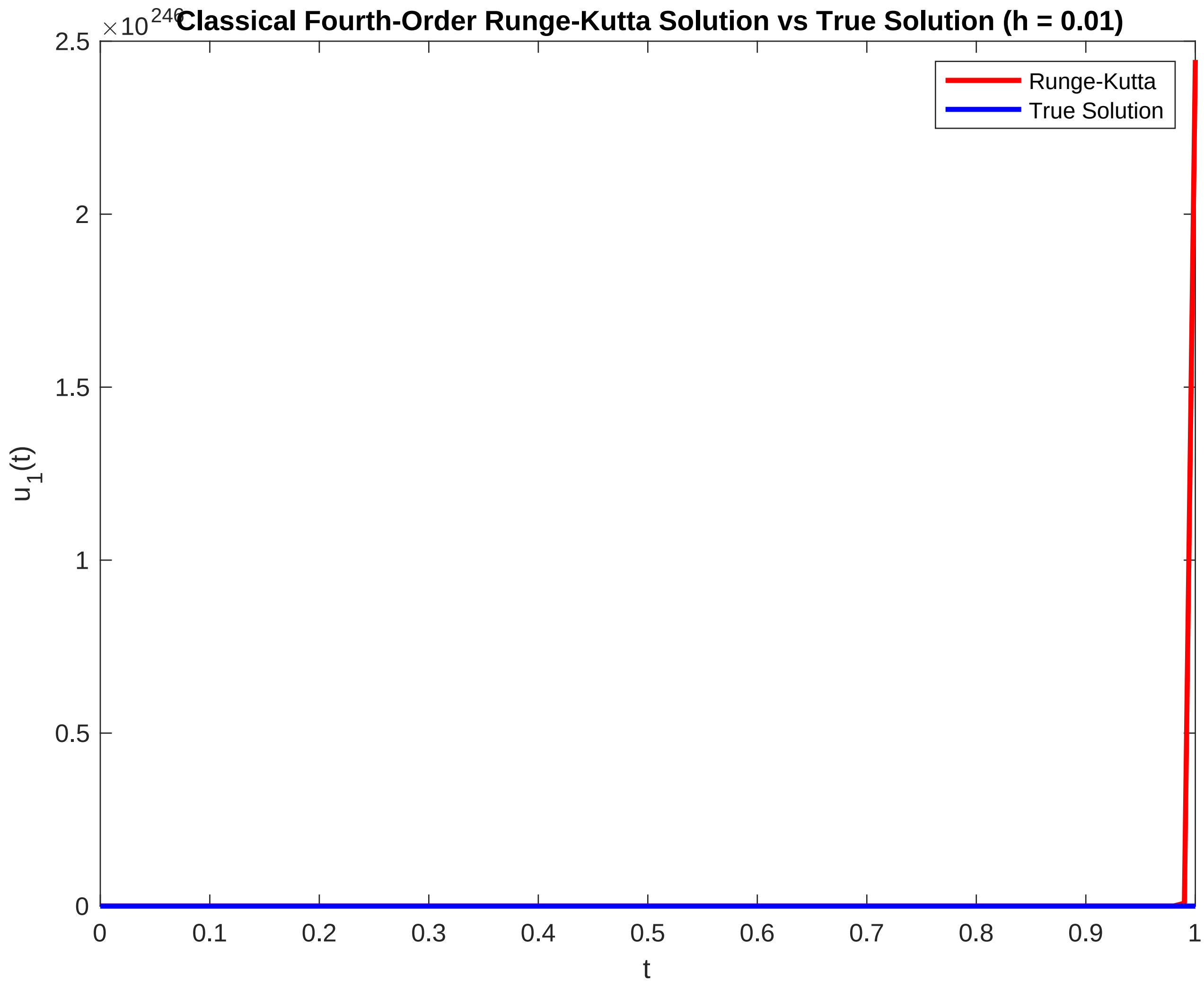
5.) $y_{k+1} = y_k + \frac{h\lambda}{2}(y_k + y_{k+1})$, $w^{(j+1)} = h\left[\frac{\lambda}{2}y_k + \frac{\lambda}{2}w^{(j)}\right] + y_k$, $j = 0, 1, 2, \ldots$.

We see that $g(w) = \frac{\lambda}{2}y_k + \frac{\lambda}{2}w$. Let $u, v \in \mathbb{R}$. Then we see that

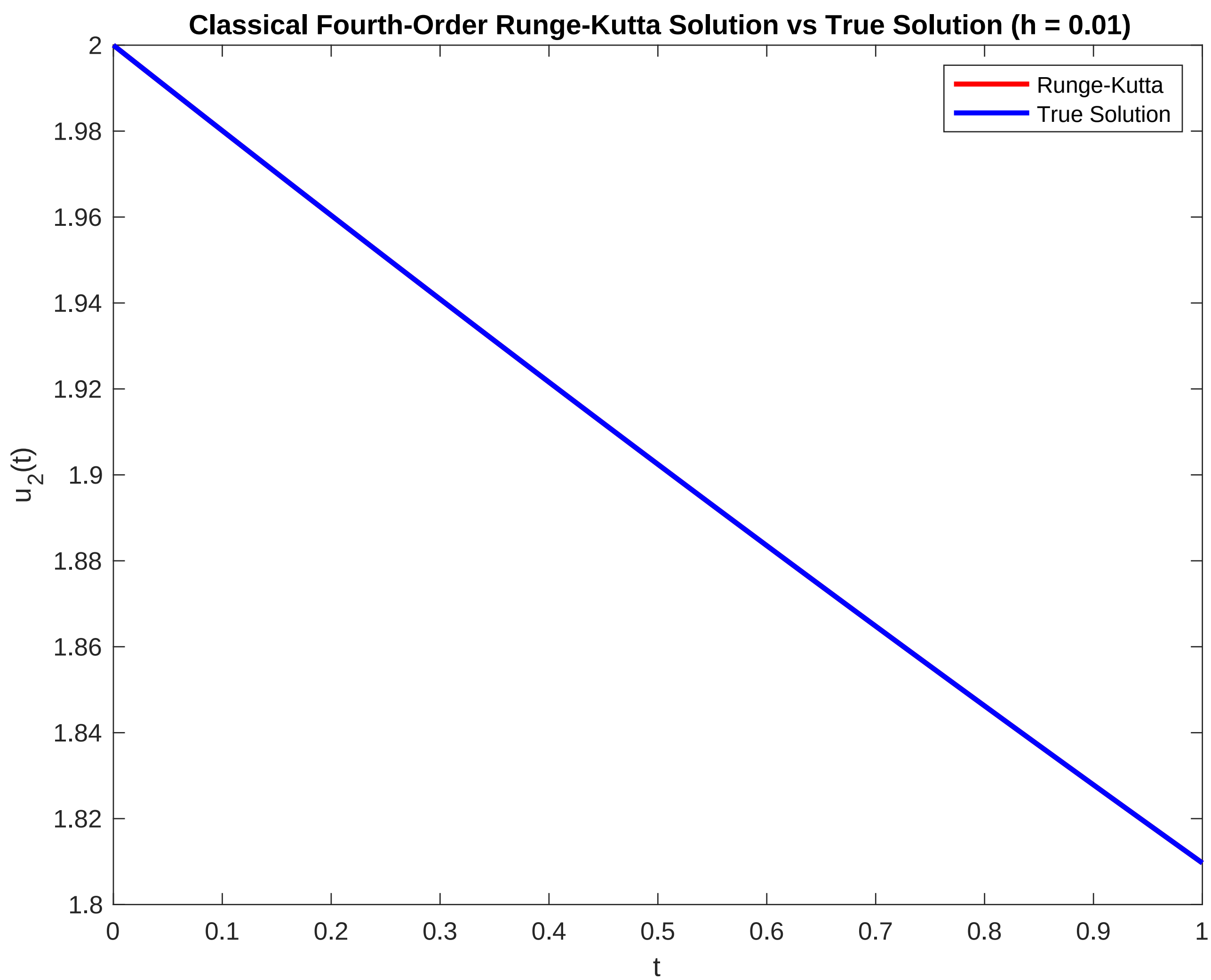
$$h|g(u) - g(v)| = |hg(u) - hg(v)| = \left|\left(\frac{h\lambda}{2}y_k + \frac{h\lambda}{2}u\right) - \left(\frac{h\lambda}{2}y_k + \frac{h\lambda}{2}v\right)\right|$$

$$= \left|\frac{h\lambda}{2}u - \frac{h\lambda}{2}v\right| = \left|\frac{h\lambda}{2}\right||u - v|.$$

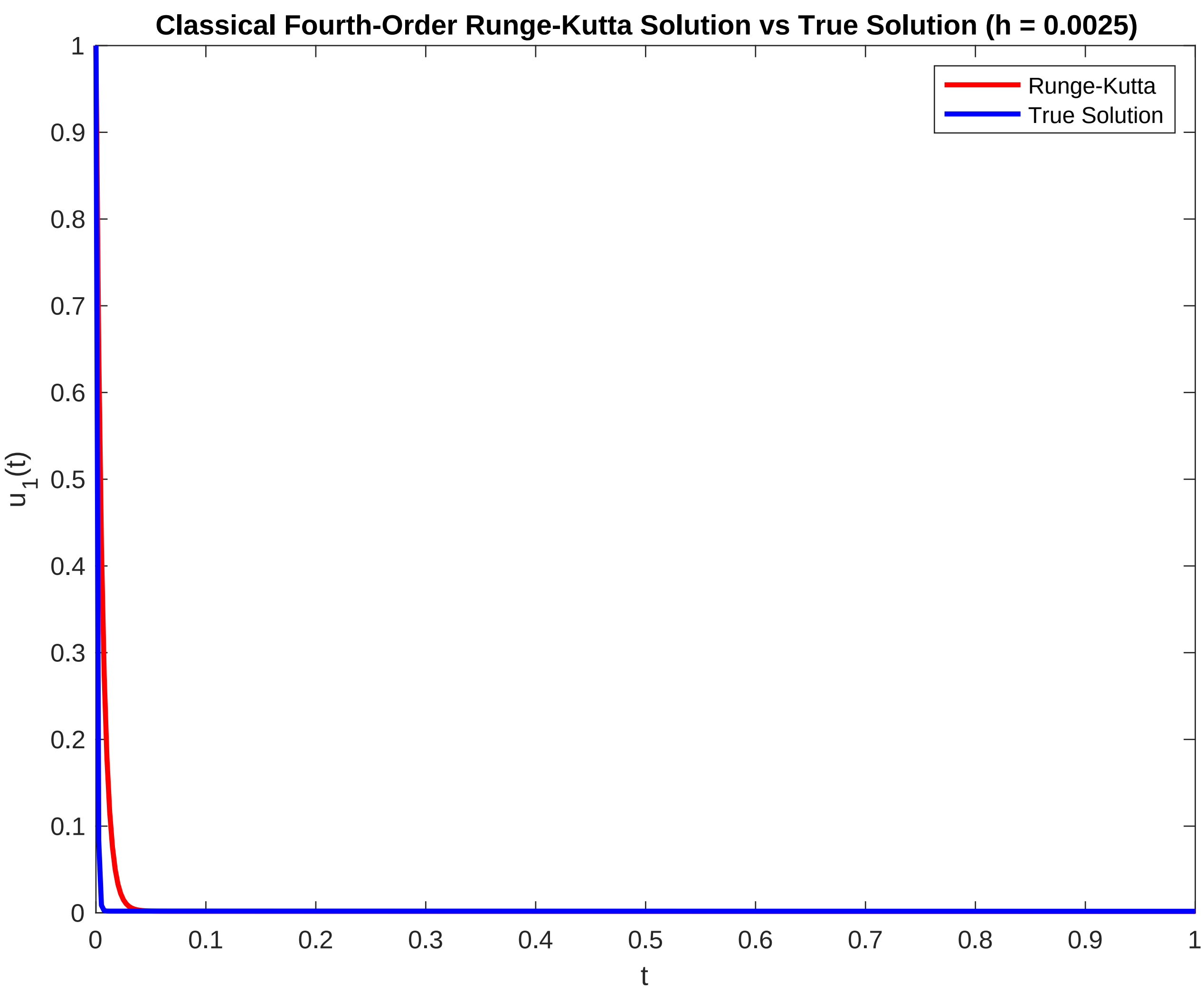
Theorem 11.5.1 tells us that if y_{k+1} is a solution for $w^{(j+1)} = h\left[\frac{\lambda}{2}y_k + \frac{\lambda}{2}w^{(j)}\right] + y_k$, then $w^{(j)}$ will converge to y_{k+1} if $\alpha = \left|\frac{h\lambda}{2}\right| < 1$ for any $w^{(0)} \in \mathbb{R}$. This means $h\lambda$ must satisfy $|h\lambda| < 2$ so that $w^{(j)}$ will converge to y_{k+1}.



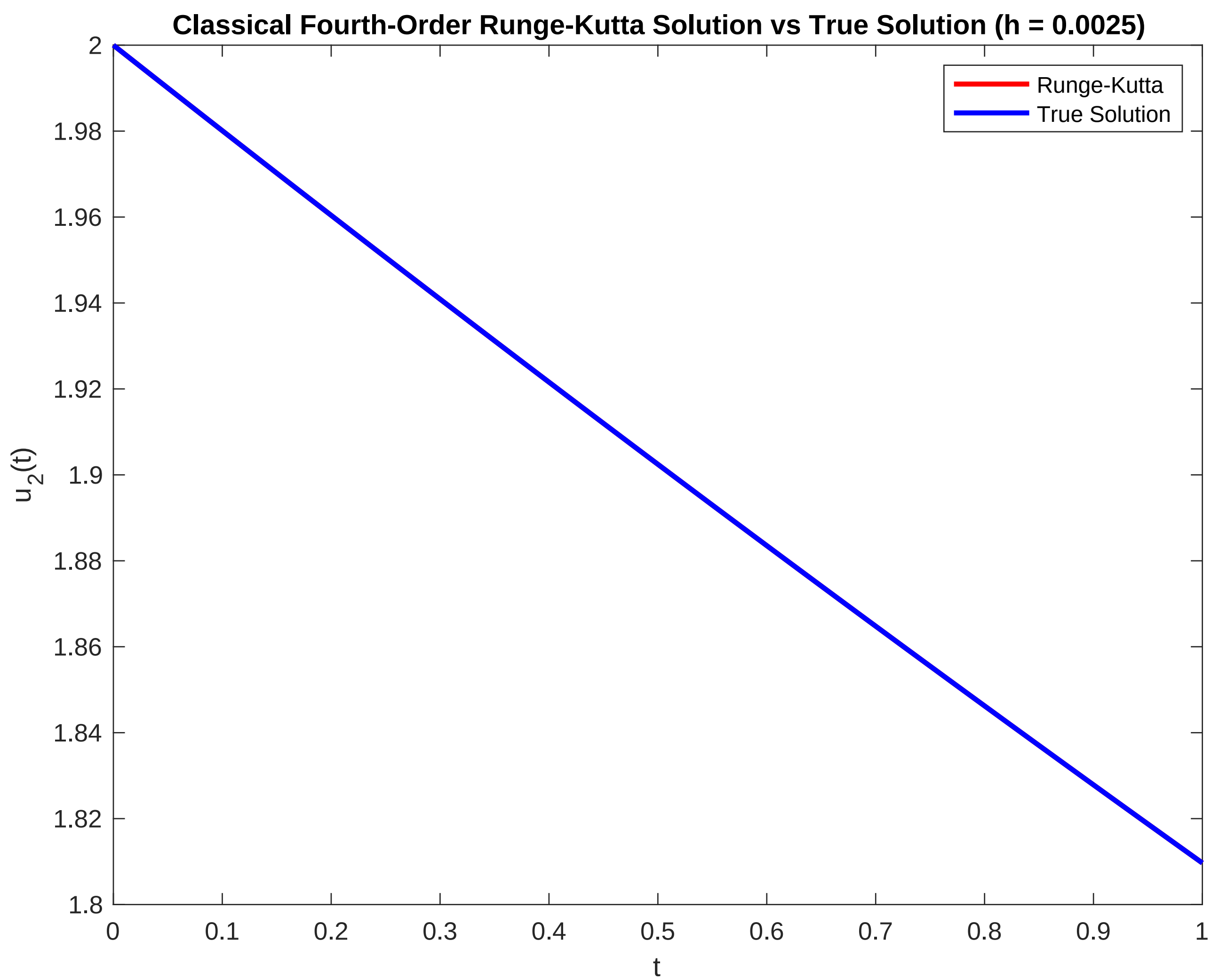
Classical Fourth-Order Runge-Kutta Solution vs True Solution (h = 0.01)
×10^246
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
u_1(t)
Runge-Kutta
True Solution



Classical Fourth-Order Runge-Kutta Solution vs True Solution (h = 0.01)
Runge-Kutta
True Solution
2
1.98
1.96
1.94
1.92
1.9
1.88
1.86
1.84
1.82
1.8
u2(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t



Classical Fourth-Order Runge-Kutta Solution vs True Solution (h = 0.0025)
Runge-Kutta
True Solution
$u_1(t)$
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Classical Fourth-Order Runge-Kutta Solution vs True Solution (h = 0.0025)
Runge-Kutta
True Solution
$u_2(t)$
t
2
1.98
1.96
1.94
1.92
1.9
1.88
1.86
1.84
1.82
1.8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1