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AMATH 586

Homework #1

1.) $-(\rho(x)u'(x))' + q(x)u(x) = f(x)$, $a < x < b$, $u(a) = \alpha$, $u(b) = \beta$.

a) Let $u \in C^4[a,b]$ be the exact solution to the BVP. Then the consistency error T_i is

$$T_i = -(\rho(x_i)u'(x_i))' - \left[\frac{1}{h^2}\left[\rho(x_{i+\frac12})u(x_{i+1}) - [\rho(x_{i+\frac12}) + \rho(x_{i-\frac12})]u(x_i) + \rho(x_{i-\frac12})u(x_{i-1})\right]\right].$$

We will first find the Taylor expansion of $-(\rho(x_i)u'(x_i))'$.

$$u(x_{i+1}) = u(x_i) + u'(x_i)h + \frac{u''(x_i)}{2}h^2 + \frac{u'''(x_i)}{6}h^3 + \frac{u^{(4)}(\xi_1)}{24}h^4, \quad x_i \le \xi_1 \le x_{i+1}$$

$$u(x_{i-1}) = u(x_i) - u'(x_i)h + \frac{u''(x_i)}{2}h^2 - \frac{u'''(x_i)}{6}h^3 + \frac{u^{(4)}(\xi_2)}{24}h^4, \quad x_{i-1} \le \xi_2 \le x_i$$

$$\Rightarrow \rho(x_{i+\frac12})u(x_{i+1}) = \rho(x_{i+\frac12})u(x_i) + \rho(x_{i+\frac12})u'(x_i)h + \rho(x_{i+\frac12})\frac{u''(x_i)}{2}h^2 + \rho(x_{i+\frac12})\frac{u'''(x_i)}{6}h^3 + \rho(x_{i+\frac12})\frac{u^{(4)}(\xi_1)}{24}h^4$$

$$\rho(x_{i-\frac12})u(x_{i-1}) = \rho(x_{i-\frac12})u(x_i) - \rho(x_{i-\frac12})u'(x_i)h + \rho(x_{i-\frac12})\frac{u''(x_i)}{2}h^2 - \rho(x_{i-\frac12})\frac{u'''(x_i)}{6}h^3 + \rho(x_{i+\frac12})\frac{u^{(4)}(\xi_2)}{24}h^4$$

$$\Rightarrow \rho(x_{i+\frac12})u(x_{i+1}) + \rho(x_{i-\frac12})u(x_{i-1}) = [\rho(x_{i+\frac12}) + \rho(x_{i-\frac12})]u(x_i) + [\rho(x_{i+\frac12}) - \rho(x_{i-\frac12})]u'(x_i)h + [\rho(x_{i+\frac12}) - \rho(x_{i-\frac12})]\frac{u''(x_i)}{2}h^2 + [\rho(x_{i+\frac12}) - \rho(x_{i-\frac12})]\frac{u'''(x_i)}{6}h^3 + [\rho(x_{i+\frac12}) + \rho(x_{i-\frac12})]\frac{u^{(4)}(\xi)}{24}h^4, \quad x_{i-1} \le \xi \le x_{i+1}$$

$$\Rightarrow \rho(x_{i+\frac12})u(x_{i+1}) - [\rho(x_{i+\frac12}) + \rho(x_{i-\frac12})]u(x_i) + \rho(x_{i-\frac12})u(x_{i-1}) = [\rho(x_{i+\frac12}) - \rho(x_{i-\frac12})]u'(x_i)h + [\rho(x_{i+\frac12}) + \rho(x_{i-\frac12})]\frac{u''(x_i)}{2}h^2 + [\rho(x_{i+\frac12}) - \rho(x_{i-\frac12})]\frac{u'''(x_i)}{6}h^3 + [\rho(x_{i+\frac12}) + \rho(x_{i-\frac12})]\frac{u^{(4)}(\xi)}{24}h^4$$

1.) a) continued.)

$$\Rightarrow -\frac{1}{h^2}\left[\rho(x_{i+\frac{1}{2}})u(x_{i+1}) - \{\rho(x_{i+\frac{1}{2}}) + \rho(x_{i-\frac{1}{2}})\}u(x_i) + \rho(x_{i-\frac{1}{2}})u(x_{i-1})\right] =$$

$$-\left[\frac{\rho(x_{i+\frac{1}{2}}) - \rho(x_{i-\frac{1}{2}})}{h}\right]u'(x_i) - \left[\rho(x_{i+\frac{1}{2}}) - \rho(x_{i-\frac{1}{2}})\right]\frac{u''(x_i)}{2}$$

$$-\left[\frac{\rho(x_{i+\frac{1}{2}}) - \rho(x_{i-\frac{1}{2}})}{h}\right]\frac{u'''(x_i)}{6}h^2 - \left[\rho(x_{i+\frac{1}{2}}) + \rho(x_{i-\frac{1}{2}})\right]\frac{u^{(4)}(\xi)}{24}h^2$$

Observe that we have found the centered difference formula for the first derivative with grid size $h/2$, so

$$\frac{\rho(x_{i+\frac{1}{2}}) - \rho(x_{i-\frac{1}{2}})}{h} = \rho'(x_i) + \mathcal{O}(h^2).$$

For the $\rho(x_{i+\frac{1}{2}}) + \rho(x_{i-\frac{1}{2}})$ term, we will Taylor expand:

$$\rho(x_{i+\frac{1}{2}}) = \rho(x_i) + \rho'(x_i)\left(\tfrac{h}{2}\right) + \frac{\rho''(x_i)}{2}\left(\tfrac{h}{2}\right)^2 + \frac{\rho'''(x_i)}{6}\left(\tfrac{h}{2}\right)^3 + \frac{\rho^{(4)}(\xi_1)}{24}\left(\tfrac{h}{2}\right)^4$$

$x_i \leq \xi_1 \leq x_{i+\frac{1}{2}}$

$$\rho(x_{i-\frac{1}{2}}) = \rho(x_i) - \rho'(x)\left(\tfrac{h}{2}\right) + \frac{\rho''(x_i)}{2}\left(\tfrac{h}{2}\right)^2 - \frac{\rho'''(x_i)}{6}\left(\tfrac{h}{2}\right)^3 + \frac{\rho^{(4)}(\xi_2)}{24}\left(\tfrac{h}{2}\right)^4$$

$x_{i-\frac{1}{2}} \leq \xi_2 \leq x_i$

$$\Rightarrow \rho(x_{i+\frac{1}{2}}) + \rho(x_{i-\frac{1}{2}}) = 2\rho(x_i) + \rho''(x_i)\left(\tfrac{h}{2}\right)^2 + \frac{\rho^{(4)}(\xi)}{12}\left(\tfrac{h}{2}\right)^4, \quad x_{i-\frac{1}{2}} \leq \xi \leq x_{i+\frac{1}{2}}$$

$$= 2\rho(x_i) + \mathcal{O}(h^2)$$

Plugging these into our above Taylor expansion gives

$$-\frac{1}{h^2}\left[\rho(x_{i+\frac{1}{2}})u(x_{i+1}) - \{\rho(x_{i+\frac{1}{2}}) + \rho(x_{i-\frac{1}{2}})\}u(x_i) + \rho(x_{i-\frac{1}{2}})u(x_{i-1})\right] =$$

$$-\left[\rho'(x_i) + \mathcal{O}(h^2)\right]u'(x_i) - \left[2\rho(x_i) + \mathcal{O}(h^2)\right]\frac{u''(x_i)}{2}$$

$$-\left[\rho'(x_i) + \mathcal{O}(h^2)\right]\frac{u'''(x_i)}{6}h^2 - \left[\rho'(x_i) + \mathcal{O}(h^2)\right]\frac{u^{(4)}(\xi)}{24}h^2$$

$$= -\rho'(x_i)u'(x_i) - \rho(x_i)u''(x_i) + \mathcal{O}(h^2) = -(\rho(x_i)u'(x_i))' + \mathcal{O}(h^2).$$

Thus

$$T_i = -(\rho(x_i)u'(x_i))' - \left[-(\rho(x_i)u'(x_i))' + \mathcal{O}(h^2)\right] = \mathcal{O}(h^2).$$

Therefore the consistency error must be $\mathcal{O}(h^2)$ as $h \to 0$, provided $u \in C^4[a,b]$. $\boxtimes$

1.) b.) We may rewrite the BVP as

$$\left[-\frac{p(x_{i+1/2})}{h^2}\right]U_{i+1} + \left[q(x_i) + \frac{p(x_{i+1/2})}{h^2} + \frac{p(x_{i-1/2})}{h^2}\right]U_i + \left[-\frac{p(x_{i-1/2})}{h^2}\right]U_{i-1} = f(x_i).$$

Define the following:

$$U = [U_1, U_2, \ldots, U_N]^T$$

We know $U_0 = \alpha$ & $U_{N+1} = \beta$, so this means

$$\left[-\frac{p(x_{3/2})}{h^2}\right]U_2 + \left[q(x_1) + \frac{p(x_{3/2})}{h^2} + \frac{p(x_{1/2})}{h^2}\right]U_1 + \left[-\frac{p(x_{1/2})}{h^2}\right]\alpha = f(x_1)$$

$$\Rightarrow \left[-\frac{p(x_{3/2})}{h^2}\right]U_2 + \left[q(x_1) + \frac{p(x_{3/2})}{h^2} + \frac{p(x_{1/2})}{h^2}\right]U_1 = f(x_1) + \frac{\alpha \cdot p(x_{1/2})}{h^2}$$

$$\left[-\frac{p(x_{N+1/2})}{h^2}\right]\beta + \left[q(x_N) + \frac{p(x_{N+1/2})}{h^2} + \frac{p(x_{N-1/2})}{h^2}\right]U_N + \left[-\frac{p(x_{N-1/2})}{h^2}\right]U_{N-1} = f(x_N)$$

$$\Rightarrow \left[q(x_N) + \frac{p(x_{N+1/2})}{h^2} + \frac{p(x_{N-1/2})}{h^2}\right]U_N + \left[-\frac{p(x_{N-1/2})}{h^2}\right]U_{N-1} = f(x_N) + \frac{\beta \cdot p(x_{N+1/2})}{h^2}$$

Thus we may define the coefficient matrix

$$A = \begin{bmatrix} q(x_1) + \frac{p(x_{3/2})}{h^2} + \frac{p(x_{1/2})}{h^2} & -\frac{p(x_{3/2})}{h^2} & & 0 \\ -\frac{p(x_{3/2})}{h^2} & q(x_2) + \frac{p(x_{5/2})}{h^2} + \frac{p(x_{3/2})}{h^2} & -\frac{p(x_{5/2})}{h^2} & \ddots \\ & \ddots & \ddots & \ddots \\ 0 & & -\frac{p(x_{N-1/2})}{h^2} & q(x_N) + \frac{p(x_{N+1/2})}{h^2} + \frac{p(x_{N-1/2})}{h^2} \end{bmatrix}.$$

Therefore, if $F = [f(x_1) + \frac{\alpha p(x_{1/2})}{h^2}, f(x_2), \ldots, f(x_N) + \frac{\beta p(x_{N+1/2})}{h^2}]^T$, then our BVP can be rewritten as $AU = F$, where A is a tridiagonal, symmetric matrix. To show that this system has a unique solution, we will consider

$$(AU, U)_h = (-D_x(p(x)D_xU) + q(x)U, U)_h$$

$$= (-D_x(p(x)D_xU), U)_h + \underbrace{(q(x)U, U)_h}_{\geq 0}$$

$$\geq (-D_x(p(x)D_xU), U)_h \overset{\text{lemma 1}}{=} (p(x)D_xU, D_xU)_h$$

$$\geq p(x)\sum_{i=1}^{N+1} h|D_xU_i|.$$

If $AU = 0$, then $0 \geq p(x)\sum_{i=1}^{N+1} h|D_xU_i| \Rightarrow D_xU_i = 0 \ \forall\, i = 1, 2, \ldots$

1.) b.) continued.)

Since we are considering the homogeneous version of the BVP when $AU=0$, $D_x U_i = 0$ for all $i=0,1,2,\dots,N$ implies that $U_0 = U_1 = U_2 = \dots = U_N = 0$ (since $U_0 = U_N = 0$). Thus $AU=0$ implies $U=0$, so A must be nonsingular. Therefore the linear system $AU=F$ must have a unique solution.

2) $-p(x)u''(x) - p'(x)u'(x) + q(x)u(x) = f(x), \quad a < x < b, \quad u(a) = \alpha, \; u(b) = \beta.$

a) $$-p(x_i)\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - p'(x_i)\frac{U_{i+1} - U_{i-1}}{2h} + q(x_i)U_i = f(x_i).$$

We may rewrite this finite difference scheme as

$$\left[-\frac{p(x_i)}{h^2} - \frac{p'(x_i)}{2h}\right]U_{i+1} + \left[\frac{2p(x_i)}{h^2} + q(x_i)\right]U_i + \left[-\frac{p(x_i)}{h^2} + \frac{p'(x_i)}{2h}\right]U_{i-1} = f(x_i)$$

Since $U_0 = \alpha$ & $U_{N+1} = \beta$, we know

$$\left[-\frac{p(x_1)}{h^2} - \frac{p'(x_1)}{2h}\right]U_2 + \left[\frac{2p(x_1)}{h^2} + q(x_1)\right]U_1 = f(x_1) + \alpha\frac{p(x_1)}{h^2} - \alpha\frac{p'(x_1)}{2h}$$

$$\left[\frac{2p(x_N)}{h^2} + q(x_N)\right]U_N + \left[-\frac{p(x_N)}{h^2} + \frac{p'(x_N)}{2h}\right]U_{N-1} = f(x_N) + \beta\frac{p(x_N)}{h^2} + \beta\frac{p'(x_N)}{2h}.$$

Define the following:

$$U = [U_1, U_2, \dots, U_N]^T$$

$$F = \left[f(x_1) + \alpha\frac{p(x_1)}{h^2} - \alpha\frac{p'(x_1)}{2h}, f(x_2), \dots, f(x_N) + \beta\frac{p(x_N)}{h^2} + \beta\frac{p'(x_N)}{2h}\right]^T$$

$$A = \begin{bmatrix} \frac{2p(x_1)}{h^2} + q(x_1) & -\frac{p(x_1)}{h^2} - \frac{p'(x_1)}{2h} & & & 0 \\ -\frac{p'(x_2)}{h^2} + \frac{p'(x_2)}{2h} & \frac{2p(x_2)}{h^2} + q(x_2) & -\frac{p(x_2)}{h^2} - \frac{p'(x_2)}{2h} & & \\ & -\frac{p(x_3)}{h^2} + \frac{p'(x_3)}{2h} & \frac{2p(x_3)}{h^2} + q(x_3) & \ddots & \\ & & \ddots & \ddots & -\frac{p(x_{N-1})}{h^2} - \frac{p'(x_{N-1})}{2h} \\ 0 & & & -\frac{p(x_N)}{h^2} + \frac{p'(x_N)}{2h} & \frac{2p(x_N)}{h^2} + q(x_N) \end{bmatrix}.$$

We see that our system can then be written as $AU = F$, where A is a tridiagonal matrix. (A is not symmetric).

b) $-u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad a < x < b, \quad u(a) = \alpha, \; u(b) = \beta$

A finite difference scheme for this BVP is

$$-\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - a(x_i)\frac{U_{i+1} - U_{i-1}}{2h} + b(x_i)U_i = f(x_i), \quad i = 1, 2, \dots, N.$$

$$\Rightarrow \left[-\frac{1}{h^2} - \frac{a(x_i)}{2h}\right]U_{i+1} + \left[\frac{2}{h^2} + b(x_i)\right]U_i + \left[-\frac{1}{h^2} - \frac{a(x_i)}{2h}\right]U_{i-1} = f(x_i)$$

If we define $F = \left[f(x_1) + \alpha\left(\frac{1}{h^2} + \frac{a(x_1)}{2h}\right), f(x_2), \dots, f(x_N) + \beta\left(\frac{1}{h^2} + \frac{a(x_N)}{2h}\right)\right]^T$ and $U = [U_1, U_2, \dots, U_N]$, then the coefficient matrix A is a symmetric, tridiagonal matrix.

2.) b.) continued.)

If A is strictly diagonally dominant, then A is nonsingular.* This would imply that the system $AU=F$ has a unique solution. In order for A to be strictly diagonally dominant, the following must be satisfied:

$$\left|-\frac{1}{h^2}-\frac{a(x_i)}{2h}\right|+\left|-\frac{1}{h^2}-\frac{a(x_i)}{2h}\right|<\left|\frac{2}{h^2}+b(x_i)\right| \quad \text{for all } i=1,2,\dots,N$$

$$\Rightarrow 2\left|-\frac{1}{h^2}-\frac{a(x_i)}{2h}\right|<\left|\frac{2}{h^2}+b(x_i)\right| \Rightarrow \left|-\frac{2}{h^2}-\frac{a(x_i)}{2h}\right|<\left|\frac{2}{h^2}+b(x_i)\right|.$$

Thus if $\left|-\frac{2}{h^2}-\frac{a(x_i)}{2h}\right|<\left|\frac{2}{h^2}+b(x_i)\right|$ for all $i=1,2,\dots,N$, then A is nonsingular, so the system $AU=F$ must have a unique solution.

*Gershgorin's theorem

3.) $-u'' = f(x)$, $x \in (0,1)$, $u'(0)=\alpha$, $u'(1)=\beta$.

a) Integrating both sides gives

$$-\int_0^1 u''(x)\,dx = \int_0^1 f(x)\,dx$$

$$\Rightarrow -[u'(x)]_0^1 = \int_0^1 f(x)\,dx$$

$$\Rightarrow \int_0^1 f(x)\,dx = -[u'(1)-u'(0)] = -[\beta-\alpha] = \alpha-\beta$$

Thus $\int_0^1 f(x)\,dx = \alpha-\beta$ in order for $u(x)$ to solve $-u''=f(x)$ on $[0,1]$ with $u'(0)=\alpha$ & $u'(1)=\beta$.

The solution is not unique since the boundary conditions only specify the derivative on the boundary. This means that for any solution $u(x)$ of the BVP, $u(x)+c$ is also a solution for any $c\in\mathbb{R}$.

b) We may use the second order centered difference approximation for u''. This gives

$$-\frac{U_{i-1}-2U_i+U_{i+1}}{h^2} = f(x_i), \quad h=\frac{1}{N+1}, \quad x_i = i\cdot h, \quad i=0,1,\dots,N+1$$

$$\Rightarrow \frac{1}{h^2}[-U_{i-1}+2U_i-U_{i+1}] = f(x_i)$$

We will introduce two "ghost points" U_{-1} & U_{N+2} to apply the BCs. Using the second order centered difference for u', we see that

$$u'(0) \approx \frac{U_1-U_{-1}}{2h} = \alpha \Rightarrow U_1-U_{-1} = 2h\alpha \Rightarrow U_{-1} = U_1 - 2h\alpha$$

$$\Rightarrow \frac{1}{h^2}[-U_{-1}+2U_0-U_1] = f(x_0) \Rightarrow \frac{1}{h^2}[2U_0-2U_1+2h\alpha] = f(x_0)$$

$$\Rightarrow \frac{1}{h^2}[2U_0-2U_1] = f(x_0) - \frac{2\alpha}{h}$$

$$u'(1) \approx \frac{U_{N+2}-U_N}{2h} = \beta \Rightarrow U_{N+2}-U_N = 2h\beta \Rightarrow U_{N+2} = U_N + 2h\beta$$

$$\Rightarrow \frac{1}{h^2}[-U_N+2U_{N+1}-U_{N+2}] = f(x_{N+1}) \Rightarrow \frac{1}{h^2}[-2U_N+2U_{N+1}-2h\beta] = f(x_{N+1})$$

$$\Rightarrow \frac{1}{h^2}[-2U_N+2U_{N+1}] = f(x_{N+1}) + \frac{2\beta}{h}$$

3.) b.) continued.)

This means we now have the following system:

$$\frac{2}{h^2}U_0 - \frac{2}{h^2}U_1 = f(x_0) - \frac{2\alpha}{h}$$

$$-\frac{1}{h^2}U_{i-1} + \frac{2}{h^2}U_i - \frac{1}{h^2}U_{i+1} = f(x_i), \quad i = 1, 2, \dots, N$$

$$-\frac{2}{h^2}U_N + \frac{2}{h^2}U_{N+1} = f(x_{N+1}) + \frac{2\beta}{h}$$

Define the following:

$$U = [U_0, U_1, \dots, U_{N+1}]^T$$

$$F = [f(x_0) - \frac{2\alpha}{h}, f(x_1), \dots, f(x_N), f(x_{N+1}) + \frac{2\beta}{h}]^T$$

$$A = \begin{bmatrix} 2/h^2 & -2/h^2 & & & & & \\ -1/h^2 & 2/h^2 & -1/h^2 & & & 0 & \\ & -1/h^2 & 2/h^2 & -1/h^2 & & & \\ & & -1/h^2 & 2/h^2 & \ddots & & \\ & & & \ddots & 2/h^2 & -1/h^2 & \\ & 0 & & & -1/h^2 & 2/h^2 & -1/h^2 \\ & & & & & -2/h^2 & 2/h^2 \end{bmatrix}$$

Then we can rewrite our system as $AU = F$. We see by inspection that the sum of the rows of A are all zero. This means if we define $v = [1, 1, \dots, 1]^T$, then $Av = 0$ so A must be singular since $v \in Ker(A)$ & $v \neq 0$. This means that all constant functions are in the kernel of A. If we change the boundary condition $u'(0) = \alpha$ to $u(0) = \alpha$, then we obtain a nonsingular system since the PDE now has a unique solution due to the fact that we are now specifying a value that the function $u(x)$ must satisfy.

4,) a)

$$\Phi_{i,j} = \frac{1}{4}\left[(x_i - \tfrac{1}{2})^2 + (y_j - \tfrac{1}{2})^2\right] = \frac{1}{4}\left[x_i^2 - x_i + \tfrac{1}{4} + y_j^2 - y_j + \tfrac{1}{4}\right]$$

$$= \frac{1}{4}x_i^2 + \frac{1}{4}y_j^2 - \frac{1}{4}x_i - \frac{1}{4}y_j + \frac{1}{8}$$

$$L_h\Phi_{i,j} = \frac{1}{h^2}\left[4\Phi_{i,j} - \Phi_{i-1,j} - \Phi_{i+1,j} - \Phi_{i,j-1} - \Phi_{i,j+1}\right]$$

$$= \frac{1}{h^2}\Big[x_i^2 + y_j^2 - x_i - y_j + \tfrac{1}{2} - \tfrac{1}{2}x_i^2 - \tfrac{1}{2}y_j^2 + \tfrac{1}{2}x_i + \tfrac{1}{2}y_j - \tfrac{1}{2}$$

$$- \tfrac{1}{4}x_{i-1}^2 - \tfrac{1}{4}x_{i+1}^2 - \tfrac{1}{4}y_{j-1}^2 - \tfrac{1}{4}y_{j+1}^2 + \tfrac{1}{4}x_{i-1} + \tfrac{1}{4}x_{i+1} + \tfrac{1}{4}y_{j-1} + \tfrac{1}{4}y_{j+1}\Big]$$

$$= \frac{1}{h^2}\Big[\tfrac{1}{2}x_i^2 + \tfrac{1}{2}y_j^2 - \tfrac{1}{2}x_i - \tfrac{1}{2}y_j - \tfrac{1}{4}\left[x_{i-1}^2 + x_{i+1}^2 + y_{j-1}^2 + y_{j+1}^2\right]$$

$$+ \tfrac{1}{4}\left[x_{i-1} + x_{i+1} + y_{j-1} + y_{j+1}\right]\Big]$$

$$= \frac{1}{h^2}\Big[-\tfrac{1}{4}\left[(x_{i-1}^2 - x_i^2) + (x_{i+1}^2 - x_i^2) + (y_{j-1}^2 - y_j^2) + (y_{j+1}^2 - y_j)\right]$$

$$+ \tfrac{1}{4}\left[(x_{i-1} - x_i) + (x_{i+1} - x_i) + (y_{j-1} - y_j) + (y_{j+1} - y_j)\right]\Big]$$

$$= \frac{1}{h^2}\Big[-\tfrac{1}{4}\big[(x_{i-1} - x_i)(x_{i-1} + x_i) + (x_{i+1} - x_i)(x_{i+1} + x_i) + (y_{j-1} - y_j)(y_{j-1} + y_j)$$

$$+ (y_{j+1} - y_j)(y_{j+1} + y_j)\big] + \tfrac{1}{4}\left[-h + h - h + h\right]\Big]$$

$$= -\frac{1}{4h^2}\left[-h(x_{i-1} + x_i) + h(x_{i+1} + x_i) - h(y_{j-1} + y_j) + h(y_{j+1} + y_j)\right]$$

$$= -\frac{1}{4h}\left[(x_{i+1} - x_{i-1}) + (y_{j+1} - y_{j-1})\right] = -\frac{1}{4h}\left[2h + 2h\right] = -\frac{4h}{4h} = -1$$

$$\Rightarrow L_h\Phi_{i,j} = -1.$$

Now, let $(i,j) \in \Omega_h$. Then

$$L_h\phi_{ij} = L_h(e_{ij} + T\Phi_{ij}) = L_h e_{ij} + T L_h \Phi_{ij} = T_{ij} - T$$

$$\le T - T = 0.$$

Thus $L_h\phi_{ij} \le 0$ for all $(i,j) \in \Omega_h$.

4) a) continued)

Since $L_h \phi_{ij} \leq 0$, we know by the discrete maximum principle that $\max_{\bar{\Omega}_h} \phi_{ij} = \max_{\partial\Omega_h} \phi_{ij}$. This means

$$\max_{\bar{\Omega}_h} \phi_{ij} = \max_{\partial\Omega_h} \phi_{ij} = \max_{\partial\Omega_h} (e_{ij} + T\Phi_{ij}).$$

Since we know $e_{ij} = 0$ on $\partial\Omega_h$, this becomes

$$\max_{\bar{\Omega}_h} \phi_{ij} = \max_{\partial\Omega_h} T\Phi_{ij} = T \cdot \max_{\partial\Omega_h} \Phi_{ij}.$$

We see that for $x_i, y_j \in [0,1]$, the maximum value of Φ_{ij} is $\frac{1}{8}$ and it is attained at $(0,0)$ & $(1,1)$. This means

$$\max_{\bar{\Omega}_h} \phi_{ij} = T \cdot \max_{\partial\Omega_h} \Phi_{ij} = \frac{1}{8}T$$

$$\Rightarrow \phi_{ij} \leq \max_{\bar{\Omega}_h} \phi_{ij} = \frac{1}{8}T$$

$$\Rightarrow e_{ij} + T\Phi_{ij} \leq \frac{1}{8}T$$

Since $T \geq 0$ & $\Phi_{ij} \geq 0$, $T\Phi_{ij} \geq 0$. This means

$$e_{ij} \leq e_{ij} + T\Phi_{ij} \leq \frac{1}{8}T \Rightarrow \boxed{e_{ij} \leq \frac{1}{8}T} \boxtimes$$

b) Let $(i,j) \in \Omega_h$. Then

$$L_h \bar{\Phi}_{ij} = L_h(e_{ij} - T\Phi_{ij}) = L_h e_{ij} - TL_h \Phi_{ij} = T_{ij} + T \geq 0$$

Thus $L_h \bar{\Phi}_{ij} \geq 0$ for all $(i,j) \in \Omega_h$.

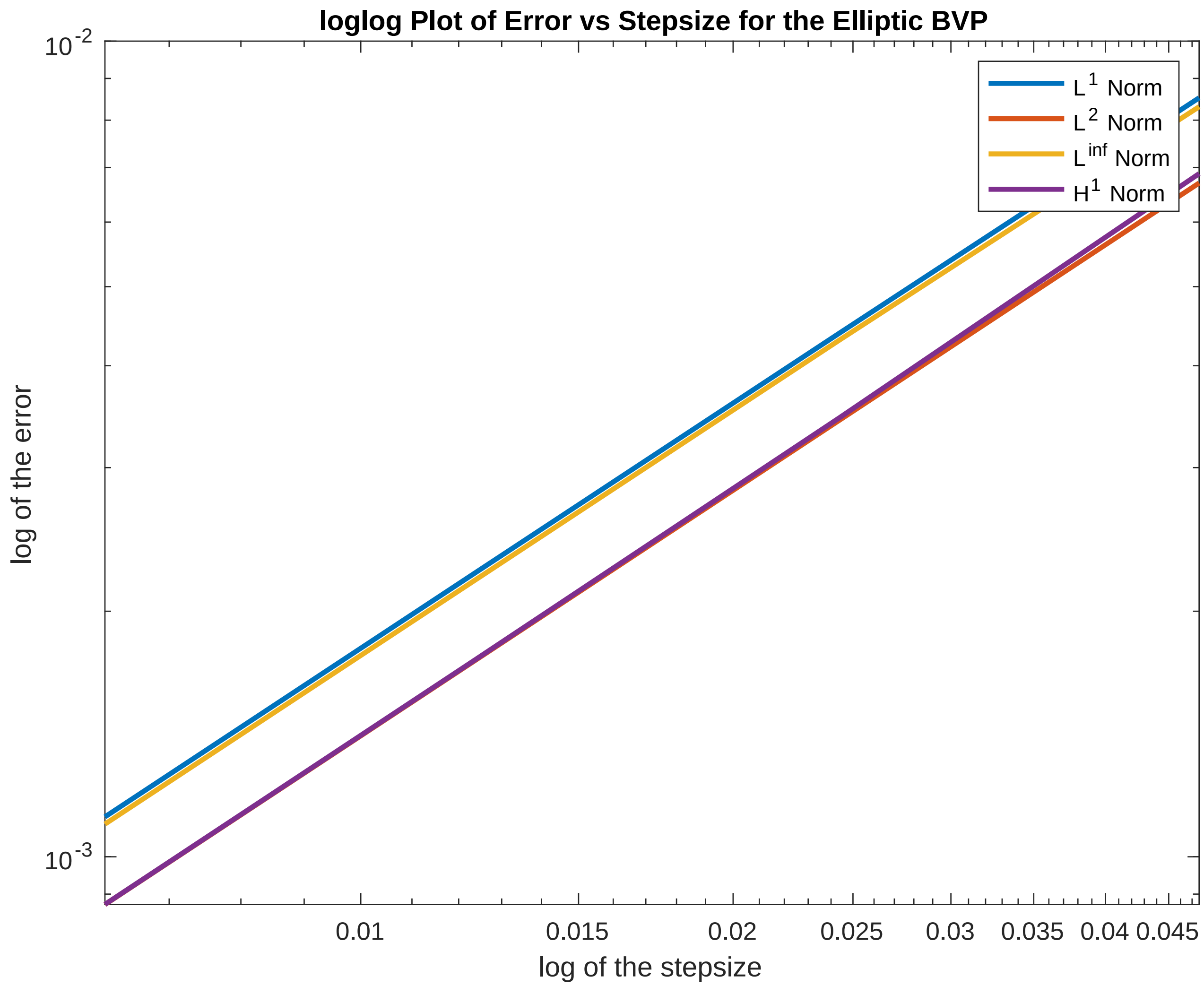
Since $L_h \bar{\Phi}_{ij} \geq 0$, we know by the discrete minimum principle that $\min_{\bar{\Omega}_h} \bar{\Phi}_{ij} = \min_{\partial\Omega_h} \bar{\Phi}_{ij}$. This means

$$\min_{\bar{\Omega}_h} \bar{\Phi}_{ij} = \min_{\partial\Omega_h} \bar{\Phi}_{ij} = \min_{\partial\Omega_h} (e_{ij} - T\Phi_{ij}) = \min_{\partial\Omega_h} (-T\Phi_{ij})$$

$$= T \min_{\partial\Omega_h} (-\Phi_{ij}) = T \cdot \left(-\frac{1}{8}\right) = -\frac{1}{8}T$$

$$\Rightarrow e_{ij} \geq e_{ij} - \underbrace{T\Phi_{ij}}_{\leq 0} \geq \min_{\bar{\Omega}_h} \bar{\Phi}_{ij} = -\frac{1}{8}T \Rightarrow \boxed{e_{ij} \geq -\frac{1}{8}T} \boxtimes$$

Scanned with CamScanner



loglog Plot of Error vs Stepsize for the Elliptic BVP
L^1 Norm
L^2 Norm
L^{inf} Norm
H^1 Norm
10^{-2}
10^{-3}
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045
log of the stepsize
log of the error

```
    U = fliplr(reshape(Uvec, N, N)).';
    U = [zeros(N, 1) U zeros(N, 1)];
    U = [-sin(pi*x); U; sin(pi*x);];

    Utrue = uexact(plotX, plotY);

    error = Utrue - U;

    L1errors(j) = max(sum(abs(error)));
    LInferrors(j) = norm(error, "inf");
    L2errors(j) = norm(error, 2);

    dxErrors = [diff(error, 1, 2) / h, (error(:, end) - error(:, end-1)) / h];
    dyErrors = [diff(error, 1, 1) / h; (error(end, :) - error(end-1, :)) / h];
    gradErrors = dxErrors.^2 + dyErrors.^2;

    H1errors(j) = L2errors(j) + norm(gradErrors, 2);

end

L1errors
L2errors
LInferrors
H1errors

hvals = (Nvals + 1).^-1

loglog(hvals, L1errors, "LineWidth", 2)
hold on;
loglog(hvals, L2errors, "LineWidth", 2)
hold on;
loglog(hvals, LInferrors, "LineWidth", 2)
hold on;
loglog(hvals, H1errors, "LineWidth", 2)
xlabel("log of the stepsize")
ylabel("log of the error")
title("loglog Plot of Error vs Stepsize for the Elliptic BVP")
legend("L^{1} Norm", "L^{2} Norm", "L^{inf} Norm", "H^{1} Norm")
```

Problem5.m test.m

```matlab
Nvals = [20 40 80 160];
L1errors = [0 0 0 0];
LInferrors = [0 0 0 0];
L2errors = [0 0 0 0];
H1errors = [0 0 0 0];

for j = 1:4
    N = Nvals(j);
    h = 1/(N+1);

    f = @(x, y) 2*pi*pi*sin(pi*x).*cos(pi*y);
    uexact = @(x, y) sin(pi*x).*cos(pi*y);

    x = h*(0:N+1);
    y = h*(N+1:-1:0);
    [plotX, plotY] = meshgrid(x, y);

    xvals = h*(1:N);
    yvals = h*(N:-1:1);
    [X, Y] = meshgrid(xvals, yvals);

    tempF = h*h*f(X, Y);
    tempF(1,:) = tempF(1,:) - sin(pi*xvals);
    tempF(N,:) = tempF(N,:) + sin(pi*xvals);

    F = reshape(fliplr(tempF.'), N*N, 1);

    maindiag = 4*ones(N*N, 1);
    subdiag1 = -1*ones((N*N)-1, 1);
    for k = 1:N-1
        subdiag1(k*N) = subdiag1(k*N) + 1;
    end
    subdiag2 = -1*ones(N*(N-1), 1);

    A = diag(maindiag) + diag(subdiag1, -1) + diag(subdiag1, 1) + diag(subdiag2, N) + diag(subdiag2, -N);

    Uvec = A \ F;

    U = fliplr(reshape(Uvec, N, N)).';
    U = [zeros(N, 1) U zeros(N, 1)];
```