Homework 2

Theoretical problems (no calculator/computer is needed):

T1. Consider the following finite difference scheme to solve the heat equation $\partial_t u = \partial_{xx} u$ on the real line:

$$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}.$$

Since a centered finite difference is used in time (this is called the leapfrog method), the local truncation error of this scheme is expected to be $O(\Delta t^2 + \Delta x^2)$. When is this scheme L^2 stable (i.e., stable in the discrete L^2 norm)?

T2. Consider the initial value problem for the convection-diffusion equation:

$$\partial_t u + b\partial_x u = a\partial_{xx} u, \quad -\infty < x < \infty,\ t > 0,$$

subject to some initial condition, where $a > 0$ and b are constants. A finite difference scheme for this equation can be given by

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + b\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x} = a\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{(\Delta x)^2},$$

where the implicit Euler is used for time stepping and centered finite difference is used for spatial derivatives. The local truncation error of this scheme is expected to be $O(\Delta t + \Delta x^2)$. When is this scheme L^2 stable?

Coding problems (attach the code you used to generate the results):

C1. Consider the 1D heat equation. From class, we know that if the initial condition is a delta function centered at $x = 2$, then the exact solution at later time is given by

$$v(t, x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \delta(y - 2)dy = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-2)^2}{4t}}.$$

Using this knowledge, we can consider solving the following initial-boundary value problem:

$$\begin{aligned} &\partial_t u = \partial_{xx} u, \quad x \in (-5, 5),\ t > 0, \\ &u(0, x) = v(1, x), \quad x \in [-5, 5], \quad u(t, -5) = v(t + 1, -5),\ u(t, 5) = v(t + 1, 5). \end{aligned}$$

And we know the exact solution will be $u(t, x) = v(t + 1, x)$.

Implement the explicit Euler, implicit Euler, and Crank-Nicolson finite difference schemes for the above initial-boundary value problem up to time $T = 3$. Choose a few different Δx, e.g., $N_x = 20, 40, 80, 160$.

- For explicit Euler, fix $\Delta t = 0.4\Delta x^2$. Demonstrate the error at $T = 3$ is $O(\Delta t + \Delta x^2) = O(\Delta x^2)$.

- For implicit Euler, fix $\Delta t = \Delta x$. Demonstrate the error at $T = 3$ is $O(\Delta t + \Delta x^2) = O(\Delta x)$.
- For Crank-Nicolson, fix $\Delta t = \Delta x$. Demonstrate the error at $T = 3$ is $O(\Delta t^2 + \Delta x^2) = O(\Delta x^2)$.