Homework 3

Theoretical problems (no calculator/computer is needed):

T1. Consider the linear advection equation $\partial_t u + a\partial_x u = 0$ with $a > 0$ a constant. A one-sided finite difference scheme can be given by

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x}a(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{\Delta t^2}{2\Delta x^2}a^2(U_j^n - 2U_{j-1}^n + U_{j-2}^n).$$

(a) Find the local truncation error of this scheme.

(b) Find the stability condition of this scheme.

T2. Derive the modified equation for the scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0,$$

which approximates the equation $\partial_t u + a\partial_x u = 0$, with a being a constant. We already know from Von Neumann analysis this is an unstable scheme. Using the modified equation, explain why this scheme might be expected to be unstable for all $\Delta t/\Delta x$.

Coding problems (attach the code you used to generate the results):

C1. Consider the linear advection equation subject to periodic boundary condition:

$$\partial_t u + \partial_x u = 0, \quad x \in [0, 2].$$

The initial condition is given by

$$u(0, x) = u_0(x) = \begin{cases} \exp(-100(x - 0.3)^2), & 0 \le x \le 0.6, \\ 1, & 0.8 \le x \le 1. \end{cases}$$

Solve this problem using 1) Upwind scheme; 2) Lax-Wendroff scheme; 3) Beam-Warming scheme; 4) high resolution scheme with minmod flux limiter; 5) high resolution scheme with MC flux limiter. For these schemes, use CFL number $\frac{\Delta t}{\Delta x} = 0.8$ and 200 spatial grid points. Plot your results at time $t = 0.5$ and $t = 0.8$. For each scheme, generate two plots, one for time $t = 0.5$ and one for $t = 0.8$. For each plot, overlay your numerical solution on the exact solution (which is given by $u_0(x - t)$) using different legends.