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AMATH 586

# Homework #3

1.) a.) Let $u(t,x)$ be the true solution to $\partial_t u + a\partial_x u = 0$, $a>0$.

$T_j^{n+1} = u(t+\Delta t, x) - u_j^{n+1}$ (LTE).

Taylor expanding, we see that

$u(t, x-\Delta x) = u - \Delta x\,\partial_x u + \frac{\Delta x^2}{2}\partial_{xx}u - \frac{\Delta x^3}{6}\partial_{xxx}u + \mathcal{O}(\Delta x^4)$

$u(t, x-2\Delta x) = u - 2\Delta x\,\partial_x u + 2\Delta x^2\partial_{xx}u - \frac{4\Delta x^3}{3}\partial_{xxx}u + \mathcal{O}(\Delta x^4)$

$u(t+\Delta t, x) = u + \Delta t\,\partial_t u + \frac{\Delta t^2}{2}\partial_{tt}u + \mathcal{O}(\Delta t^3)$

Since $\partial_t u = -a\partial_x u$ and $\partial_{tt}u = a^2\partial_{xx}u$, we see that

$u(t+\Delta t, x) = u - a\Delta t\,\partial_x u + \frac{1}{2}a^2\Delta t^2\partial_{xx}u + \mathcal{O}(\Delta t^3)$

We see that if $u = u_j^n$, then

$$3u_j^n - 4u_{j-1}^n + u_{j-2}^n = 3u - 4u + 4\Delta x\,\partial_x u - 2\Delta x^2\partial_{xx}u + \frac{2\Delta x^3}{3}\partial_{xxx}u + \mathcal{O}(\Delta x^4)$$
$$+ u - 2\Delta x\,\partial_x u + 2\Delta x^2\partial_{xx}u - \frac{4}{3}\Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)$$

$$= 2\Delta x\,\partial_x u - \frac{2}{3}\Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)$$

$$u - 2u_{j-1}^n + u_{j-2}^n = u - 2u + 2\Delta x\,\partial_x u - \Delta x^2\partial_{xx}u + \frac{1}{3}\Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)$$
$$+ u + 2\Delta x\,\partial_x u + 2\Delta x^2\partial_{xx}u - \frac{4}{3}\Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)$$

$$= \Delta x^2\partial_{xx}u - \Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)$$

$$\Rightarrow u_j^{n+1} = u - \frac{\Delta t}{2\Delta x}a[2\Delta x\,\partial_x u - \frac{2}{3}\Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)] + \frac{\Delta t^2}{2\Delta x^2}a^2[\Delta x^2\partial_{xx}u - \Delta x^3\partial_{xxx}u + \mathcal{O}(\Delta x^4)]$$

$$= u - a\Delta t[\partial_x u - \frac{1}{3}\Delta x^2\partial_{xxx}u + \mathcal{O}(\Delta x^3)] + a^2\Delta t^2[\frac{1}{2}\partial_{xx}u - \frac{1}{2}\Delta x\,\partial_{xxx}u + \mathcal{O}(\Delta x^2)]$$

$$= u - a\Delta t\,\partial_x u + \frac{1}{2}a^2\Delta t^2\partial_{xx}u + \frac{1}{3}a\Delta t\Delta x^2\partial_{xxx}u - \frac{1}{2}a^2\Delta t^2\Delta x\,\partial_{xxx}u + \mathcal{O}(\Delta x$$

$$= u - a\Delta t\,\partial_x u + \frac{1}{2}a^2\Delta t^2\partial_{xx}u - \frac{1}{2}a^2\Delta t^2\Delta x\,\partial_{xxx}u + \mathcal{O}(\Delta x^2)$$

$\Rightarrow T_j^{n+1} = u(t+$

1.) a) continued)

Plugging in $u(t+\Delta t, x)$ & $u_j^{n+1}$ into the LTE formula gives

$$T_j^{n+1} = \left(u - a\Delta t \partial_x u + \tfrac{1}{2}a^2\Delta t^2 \partial_{xx}u + \mathcal{O}(\Delta t^3)\right)$$
$$- \left(u - a\Delta t \partial_x u + \tfrac{1}{2}a^2\Delta t^2\partial_{xx}u - \tfrac{1}{2}a^2\Delta t^2 \Delta x\, \partial_{xxx}u + \mathcal{O}(\Delta x^2)\right)$$
$$= \tfrac{1}{2}a^2\Delta t^2 \Delta x\, \partial_{xxx}u + \mathcal{O}(\Delta t^3 + \Delta x^2) = \mathcal{O}(\Delta t^2 + \Delta x^2).$$

Thus the LTE of this scheme is $\mathcal{O}(\Delta t^2 + \Delta x^2)$.

b) First, we will rewrite the scheme and let $v = \frac{\Delta t}{\Delta x}a$.

$$U_j^{n+1} = U_j^n - \tfrac{1}{2}v\left(3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right) + \tfrac{1}{2}v^2\left(U_j^n - 2U_{j-1}^n + U_{j-2}^n\right)$$
$$= \left(1 - \tfrac{3}{2}v + \tfrac{1}{2}v^2\right)U_j^n + \left(2v - v^2\right)U_{j-1}^n + \left(-\tfrac{1}{2}v + \tfrac{1}{2}v^2\right)U_{j-2}^n$$
$$= \tfrac{1}{2}(v-1)(v-2)U_j^n + v(2-v)U_{j-1}^n + \tfrac{1}{2}v(v-1)U_{j-2}^n$$

Now let $U_j^n = g(\xi)^n e^{i\xi j\Delta x}$. Then

$$g(\xi) = \tfrac{1}{2}(v-1)(v-2) + v(2-v)e^{-i\xi\Delta x} + \tfrac{1}{2}v(v-1)e^{-2i\xi\Delta x}$$
$$= e^{-i\xi\Delta x}\left[\tfrac{1}{2}(v-1)(v-2)e^{i\xi\Delta x} + v(2-v) + \tfrac{1}{2}v(v-1)e^{-i\xi\Delta x}\right]$$
$$= e^{-i\xi\Delta x}\left[\tfrac{1}{2}v(v-1)e^{i\xi\Delta x} - (v-1)e^{i\xi\Delta x} + v(2-v) + \tfrac{1}{2}v(v-1)e^{-i\xi\Delta x}\right]$$
$$= e^{-i\xi\Delta x}\left[v(v-1)\cos\xi\Delta x + v(2-v) - (v-1)e^{i\xi\Delta x}\right]$$
$$= e^{-i\xi\Delta x}\left[v(v-1)\cos\xi\Delta x + v(2-v) - (v-1)\cos\xi\Delta x - i(v-1)\sin\xi\Delta x\right]$$
$$= e^{-i\xi\Delta x}\left[(v-1)^2\cos\xi\Delta x + v(2-v) - i(v-1)\sin\xi\Delta x\right]$$

$$|g(\xi)|^2 = \left|e^{-i\xi\Delta x}\left[(v-1)^2\cos\xi\Delta x + v(2-v) - i(v-1)\sin\xi\Delta x\right]\right|^2$$
$$= |e^{-i\xi\Delta x}|^2\,|(v-1)^2\cos\xi\Delta x + v(2-v) - i(v-1)\sin\xi\Delta x|^2$$
$$= \left[(v-1)^2\cos\xi\Delta x + v(2-v)\right]^2 + (v-1)^2\sin^2\xi\Delta x$$
$$= (v-1)^4\cos^2\xi\Delta x + 2v(2-v)(v-1)^2\cos\xi\Delta x + v^2(2-v)^2 + (v-1)^2\sin^2\xi\Delta x$$

1.) b.) continued.)

$$|g(\xi)| = (\nu-1)^2\left[(\nu-1)^2\cos^2\xi\Delta x + \sin^2\xi\Delta x\right] + 2\nu(2-\nu)(\nu-1)^2\cos\xi\Delta x + \nu^2(2-\nu)^2$$

$$= (\nu-1)^2\left[\nu^2\cos^2\xi\Delta x - 2\nu\cos^2\xi\Delta x + 1\right] + 2\nu(2-\nu)(\nu-1)^2\cos\xi\Delta x + \nu^2(2-\nu)^2$$

$$= (\nu-1)^2\left[-\nu(2-\nu)\cos^2\xi\Delta x + 1\right] + 2\nu(2-\nu)(\nu-1)^2\cos\xi\Delta x + \nu^2(2-\nu)^2$$

$$= (\nu-1)^2 + \nu^2(2-\nu)^2 + 2\nu(2-\nu)(\nu-1)^2\cos\xi\Delta x - \nu(2-\nu)\cos^2\xi\Delta x$$

*Note that

$$(\nu-1)^2 + \nu^2(2-\nu)^2 = \nu^2 - 2\nu + 1 + \nu^2(\nu^2 - 4\nu + 4)$$

$$= \nu^4 - 4\nu^3 + 5\nu^2 - 2\nu + 1 = 1 + \nu(\nu^3 - 4\nu^2 + 5\nu - 2)$$

also we see that

$$(2-\nu)(\nu-1)^2 = (2-\nu)(\nu^2 - 2\nu + 1) = -\nu^3 + 4\nu^2 - 5\nu + 2 = -(\nu^3 - 4\nu^2 + 5\nu - 2)$$

This means we have

$$(\nu-1)^2 + \nu^2(2-\nu)^2 = 1 + \nu(\nu^3 - 4\nu^2 + 5\nu - 2) = 1 - \nu(2-\nu)(\nu-1)^2$$

$$\Rightarrow |g(\xi)| = 1 - \nu(2-\nu)(\nu-1)^2 + 2\nu(2-\nu)(\nu-1)^2\cos\xi\Delta x - \nu(2-\nu)\cos^2\xi\Delta x$$

$$= 1 - \nu(2-\nu)(\nu-1)^2(1 - 2\cos\xi\Delta x + \cos^2\xi\Delta x)$$

$$= 1 - \nu(2-\nu)(\nu-1)^2(1-\cos\xi\Delta x)^2.$$

We require $|g(\xi)| \leq 1$ for $L^2$ stability, so

$$1 - \nu(2-\nu)(\nu-1)^2(1-\cos\xi\Delta x)^2 \leq 1$$

$$\Rightarrow -\nu(2-\nu)(\nu-1)^2(1-\cos\xi\Delta x)^2 \leq 0$$

Since $\nu = a\frac{\Delta t}{\Delta x} \geq 0$, $(\nu-1)^2 \geq 0$, $(1-\cos\xi\Delta x)^2 \geq 0$, this means we need $(2-\nu) \geq 0$ to satisfy this condition. Thus $\nu \leq 2$, so $a\frac{\Delta t}{\Delta x} \leq 2$ is the $L^2$ stability condition for this scheme.

2.) Let us replace $U_j^n$ by a continuous function $v(t,x)$. Then

$$\frac{v(t+\Delta t,x)-v(t,x)}{\Delta t} + a\frac{v(t,x+\Delta x)-v(t,x-\Delta x)}{2\Delta x} = 0$$

Taylor expanding, we see that

$$v(t+\Delta t,x) = v(t,x) + \Delta t\,\partial_t v(t,x) + \frac{\Delta t^2}{2}\partial_{tt}v(t,x) + \mathcal{O}(\Delta t^3)$$
$$v(t,x+\Delta x) = v(t,x) + \Delta x\,\partial_x v(t,x) + \frac{\Delta x^2}{2}\partial_{xx}v(t,x) + \mathcal{O}(\Delta x^3)$$
$$v(t,x-\Delta x) = v(t,x) - \Delta x\,\partial_x v(t,x) + \frac{\Delta x^2}{2}\partial_{xx}v(t,x) + \mathcal{O}(\Delta x^3)$$

Plugging into above and assuming $\frac{\Delta t}{\Delta x}$ is fixed gives

$$\partial_t v + \frac{\Delta t}{2}\partial_{tt}v + a\partial_x v + \mathcal{O}(\Delta x^2) = 0$$

$$\Rightarrow \partial_t v + a\partial_x v = -\frac{\Delta t}{2}\partial_{tt}v + \mathcal{O}(\Delta x^2)$$

We see that we can differentiate w.r.t. $t$ to get

$$\partial_{tt}v + a\partial_{tx}v = \mathcal{O}(\Delta x)$$

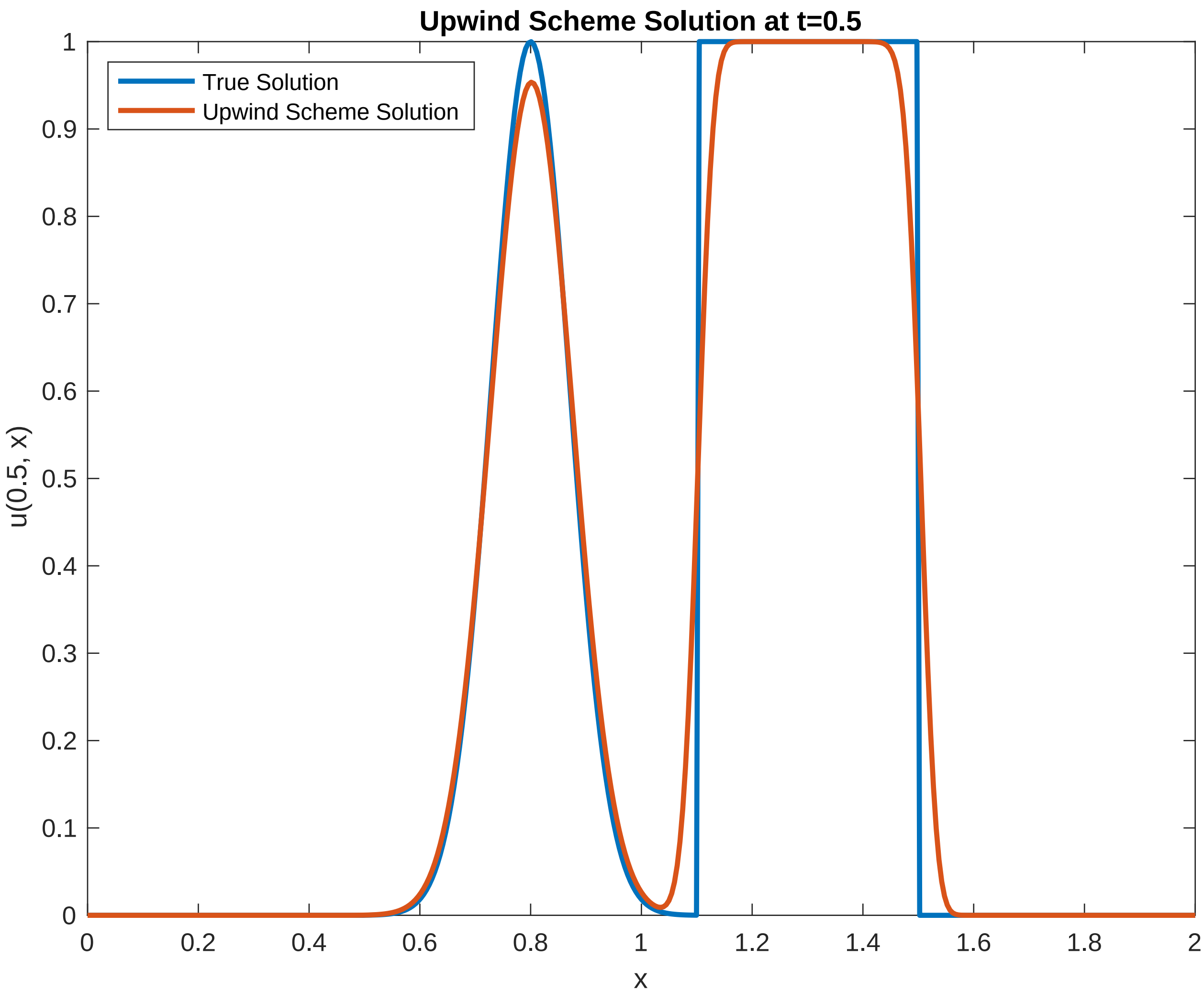
$$\Rightarrow \partial_{tt}v = -a\partial_x(\partial_t v) + \mathcal{O}(\Delta x)$$

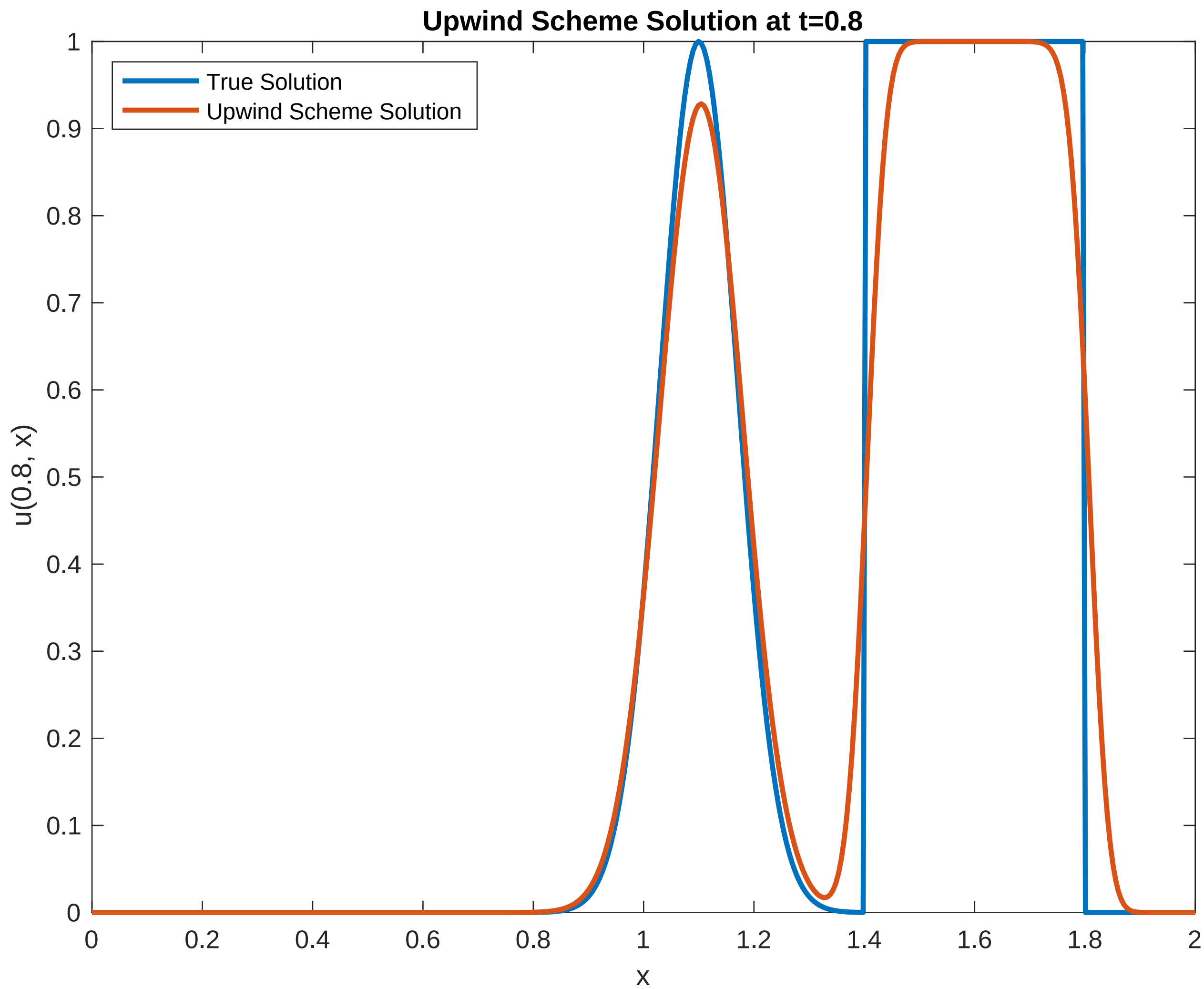
$$\Rightarrow \partial_{tt}v = -a\partial_x(-a\partial_x v) + \mathcal{O}(\Delta x) = a^2\partial_{xx}v + \mathcal{O}(\Delta x)$$

Plugging this in for $\partial_{tt}v$ gives

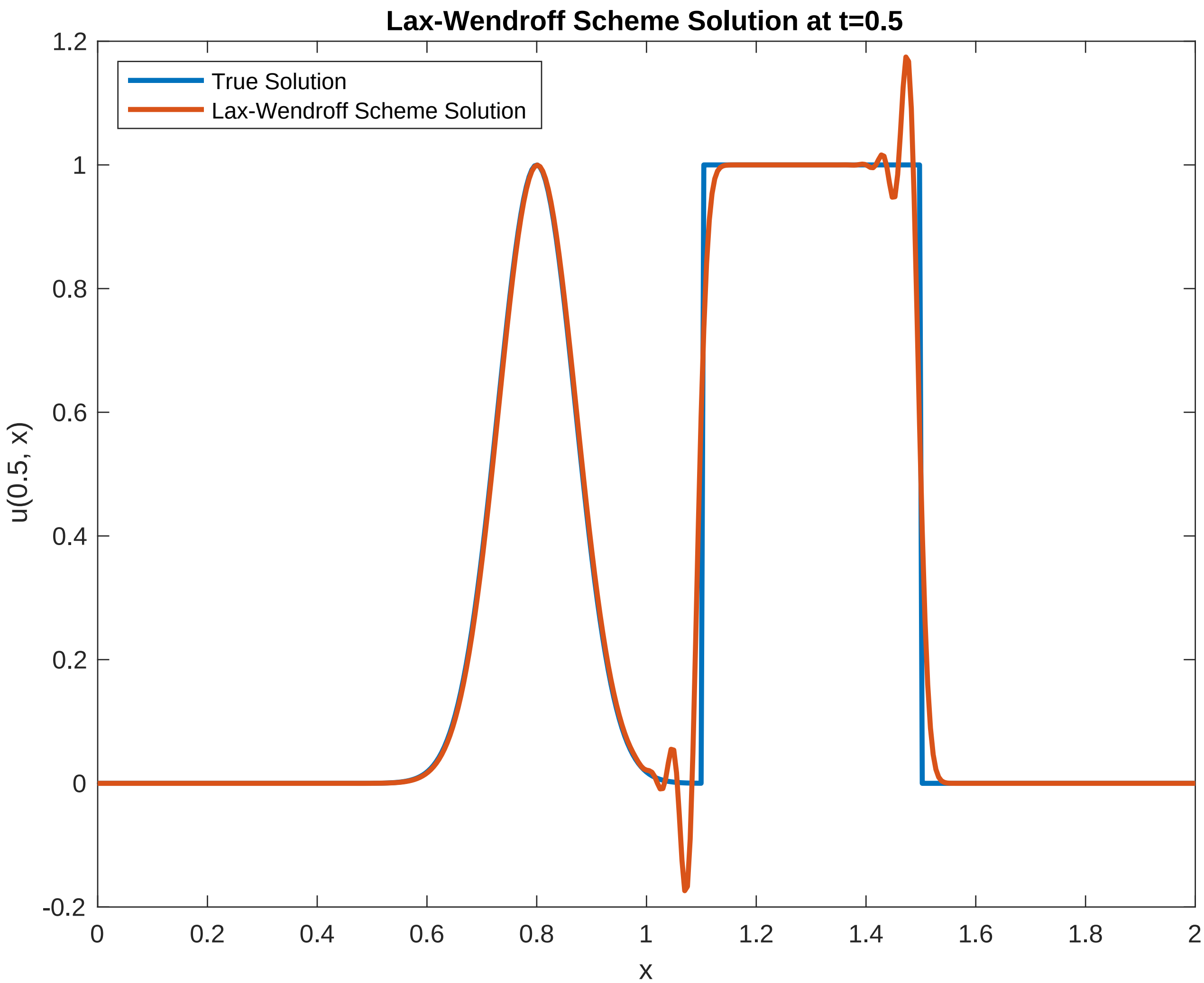
$$\partial_t v + a\partial_x v = -\frac{\Delta t}{2}\left[a^2\partial_{xx}v + \mathcal{O}(\Delta x)\right] + \mathcal{O}(\Delta x^2)$$
$$= -\frac{\Delta t}{2}a^2\partial_{xx}v + \mathcal{O}(\Delta x^2)$$

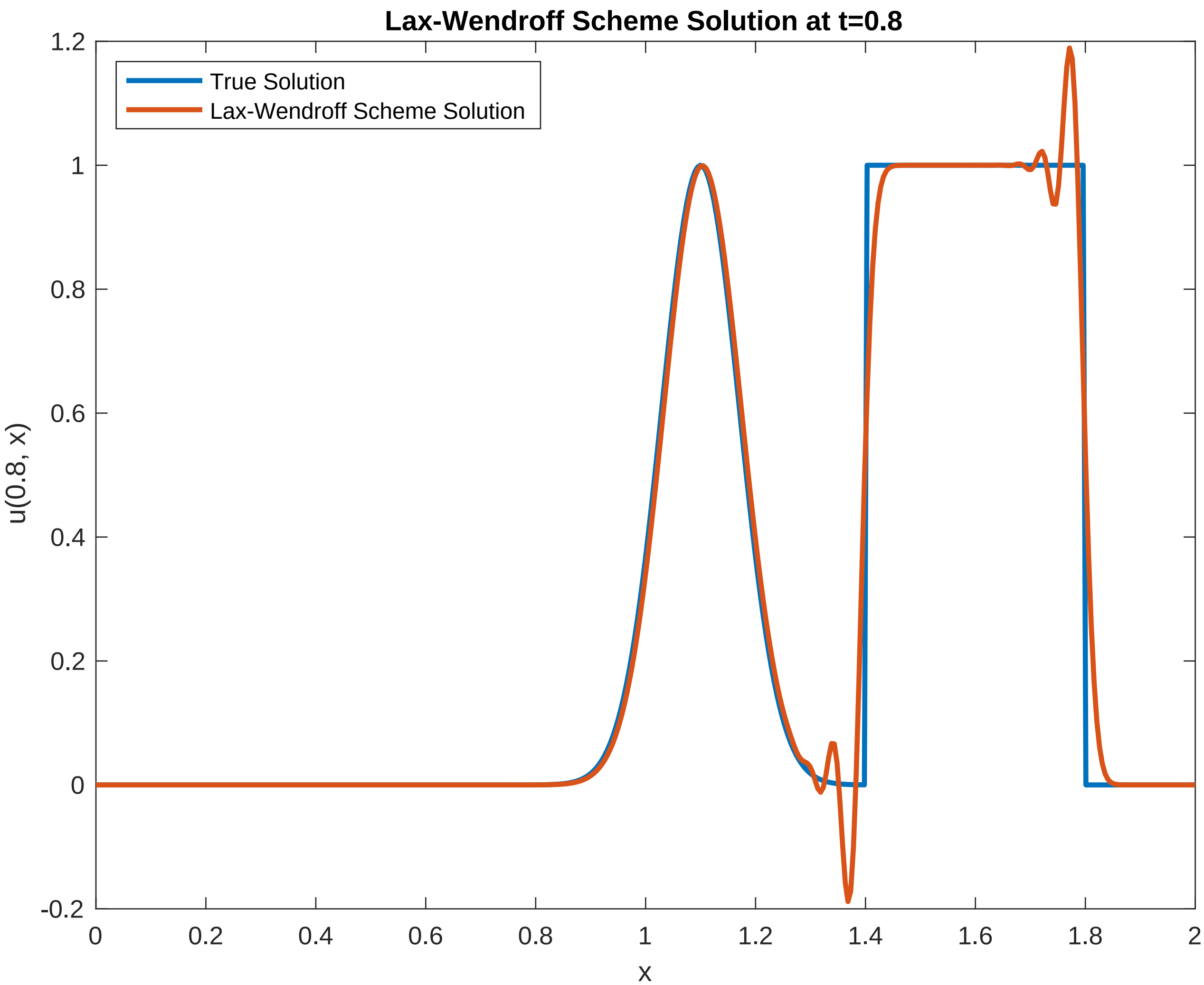
Since $-\frac{\Delta t}{2}a^2 < 0$ always, this diffusion term can never be positive. Negative diffusion produces instability, so this implies this scheme is always unstable for all $\frac{\Delta t}{\Delta x}$.


Upwind Scheme Solution at t=0.5
True Solution
Upwind Scheme Solution
u(0.5, x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

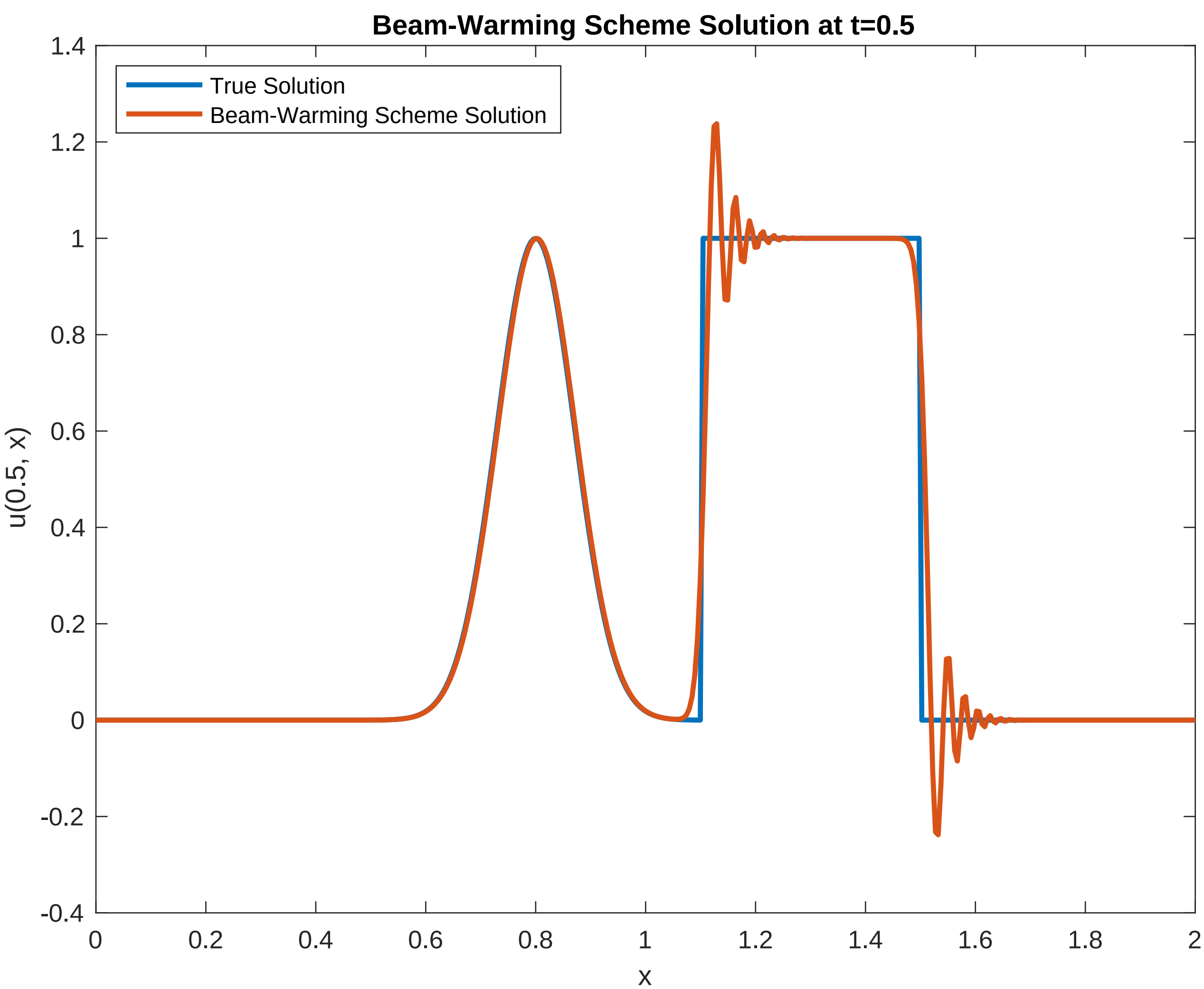


Upwind Scheme Solution at t=0.8
True Solution
Upwind Scheme Solution
u(0.8, x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

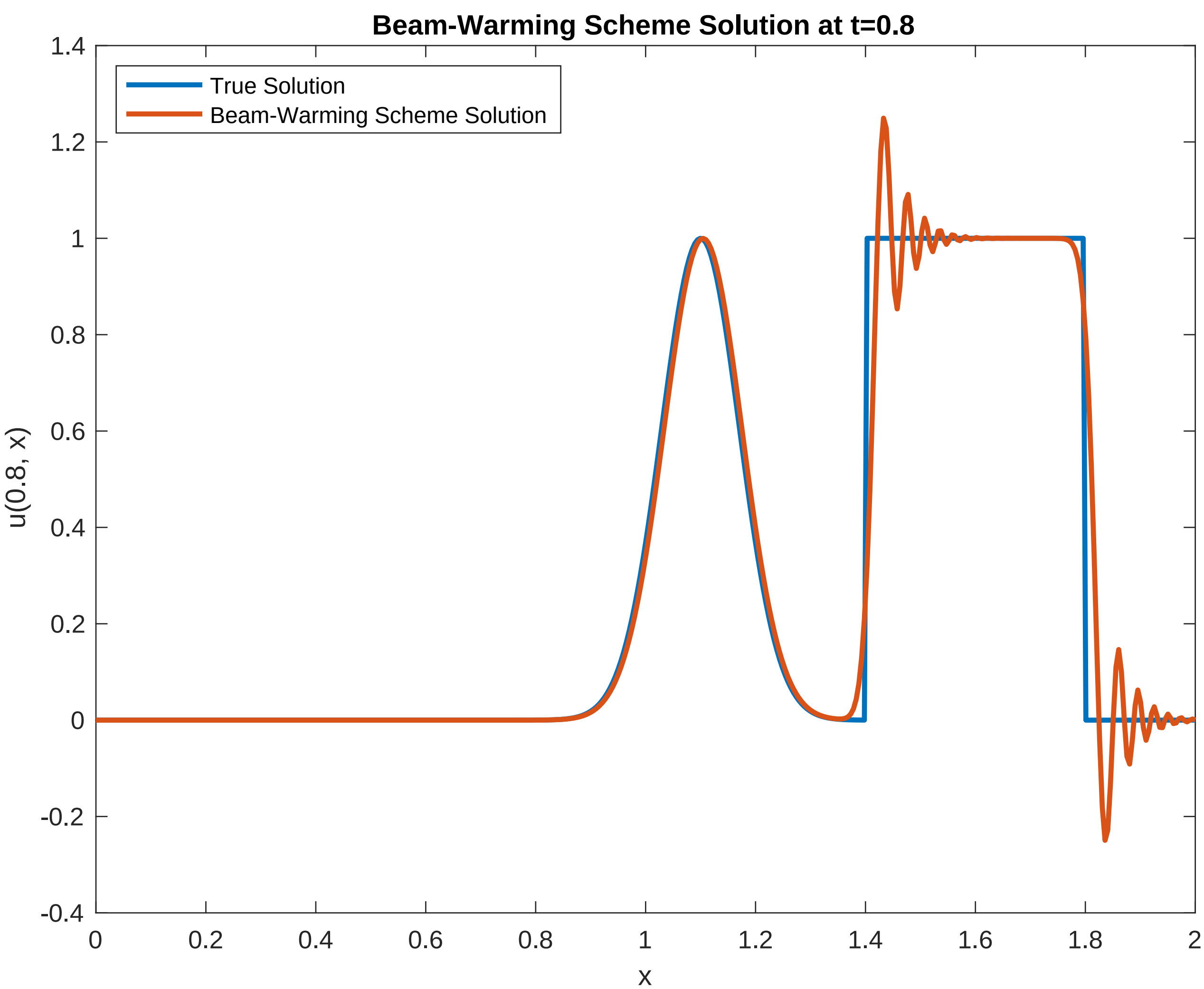

Lax-Wendroff Scheme Solution at t=0.5
True Solution
Lax-Wendroff Scheme Solution
u(0.5, x)
x
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



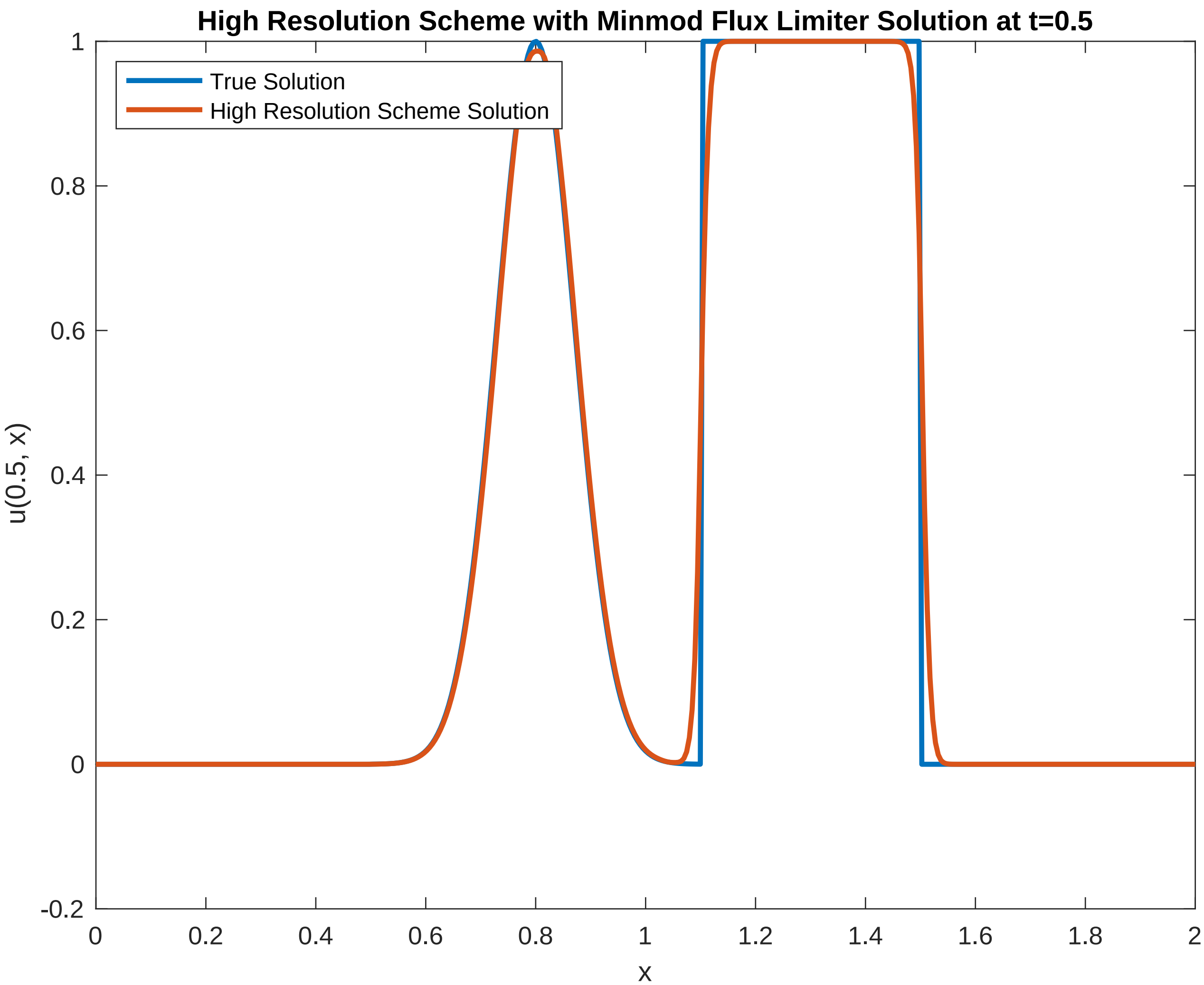
Lax-Wendroff Scheme Solution at t=0.8
True Solution
Lax-Wendroff Scheme Solution
u(0.8, x)
x
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

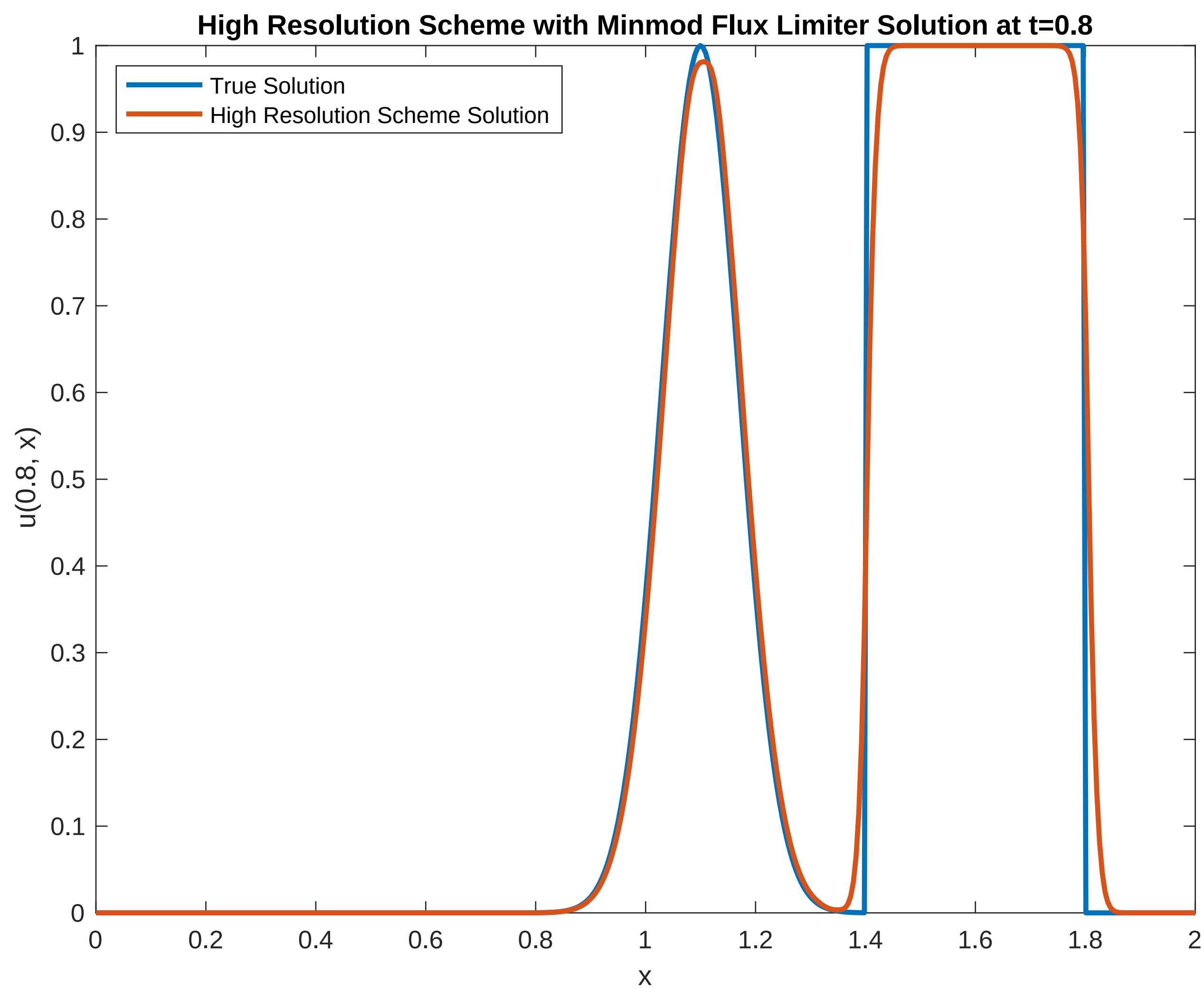


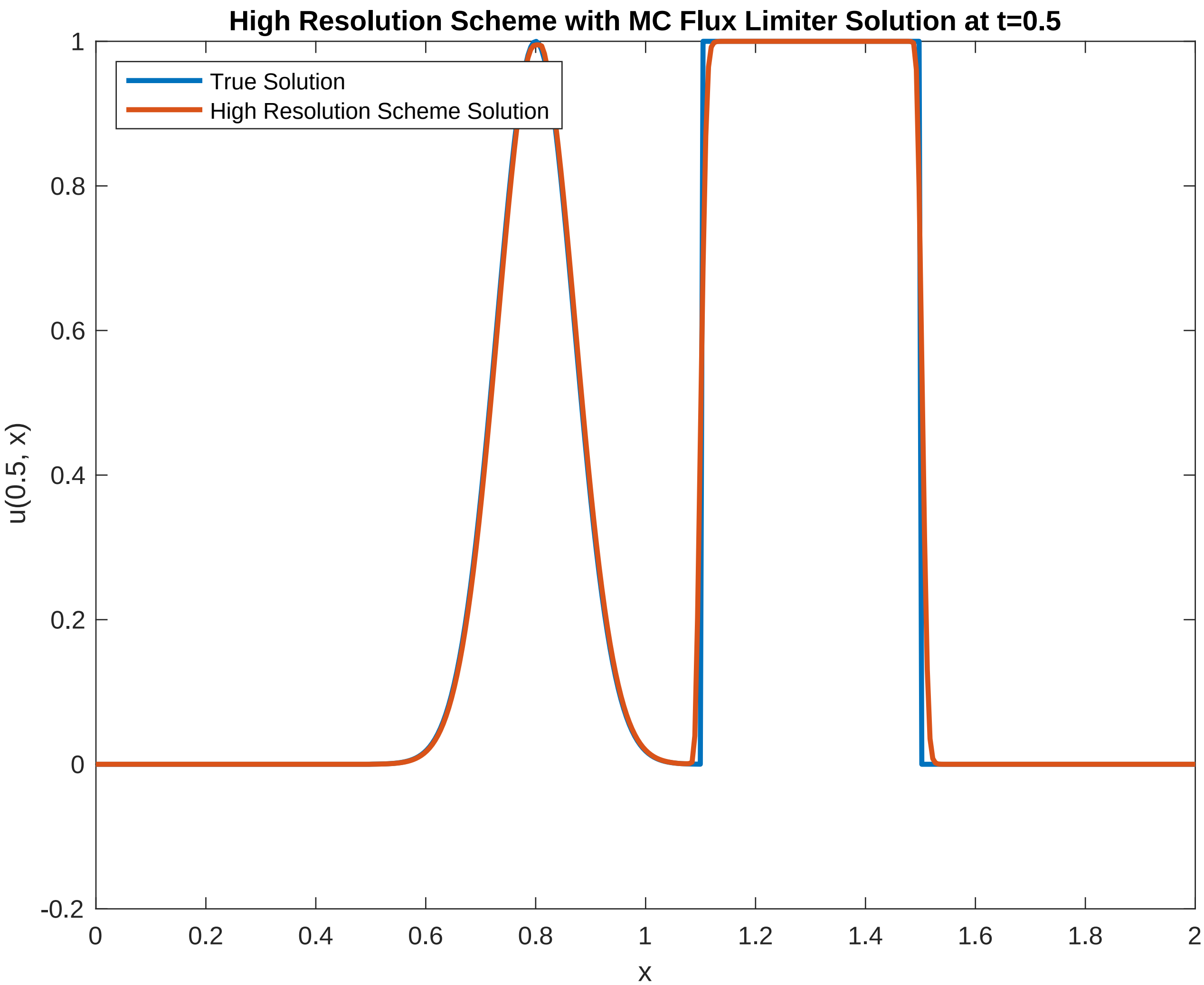
Beam-Warming Scheme Solution at t=0.5
True Solution
Beam-Warming Scheme Solution
u(0.5, x)
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



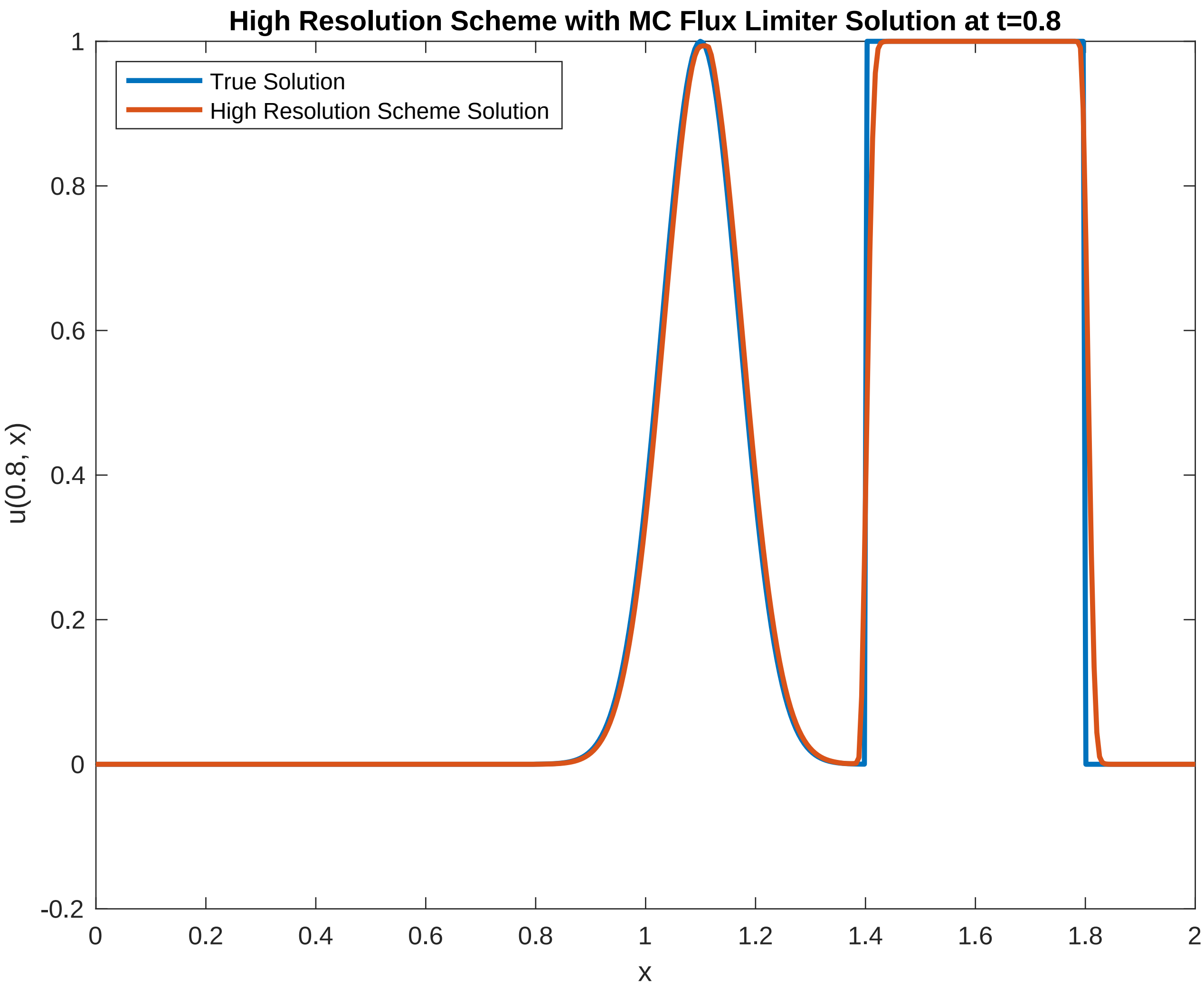
Beam-Warming Scheme Solution at t=0.8
True Solution
Beam-Warming Scheme Solution
u(0.8, x)
x
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


High Resolution Scheme with Minmod Flux Limiter Solution at t=0.5
True Solution
High Resolution Scheme Solution
u(0.5, x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


High Resolution Scheme with Minmod Flux Limiter Solution at t=0.8
True Solution
High Resolution Scheme Solution
u(0.8, x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



High Resolution Scheme with MC Flux Limiter Solution at t=0.5
True Solution
High Resolution Scheme Solution
u(0.5, x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


High Resolution Scheme with MC Flux Limiter Solution at t=0.8
True Solution
High Resolution Scheme Solution
u(0.8, x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2