Reflection Coefficient

In this section, we will derive the equation for the reflection coefficient. The reflection coefficient relates the forward-going voltage with reflected voltage.

Reflection coefficient at the load

Equations 1-2 represent the voltage and current on a lossless transmission line shown in Figure 1.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} \tag{1}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta z} \tag{2}$$

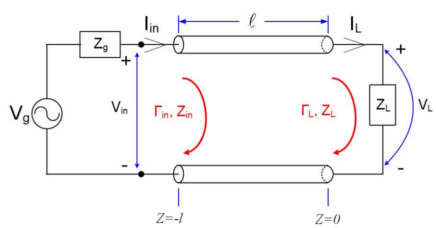

Figure 1: Transmission Line connects generator and the load.

We set up the z-axis so that the $z = 0$ is at the load, and the generator is at $z = -l$. At $z = 0$, the load impedance is connected. The definition of impedance is $Z = V/I$, therefore at the z=0 end of the transmission line, the voltage and current on the transmission line at that point have to obey boundary condition that the load impedance imposes.

$$Z_L = \frac{V(0)}{I(0)}$$

Substituting z=0, the boundary condition, in Equations 1-2, we get Equations 3-4.

Learning outcomes: Derive and calculate the transmission line impedance and reflection coefficient. Relate reflection coefficient to impedance.

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$$\tilde{V}(0) = \tilde{V}_0^+ e^{-j\beta 0} + \tilde{V}_0^- e^{j\beta 0} = \tilde{V}_0^+ + \tilde{V}_0^- \tag{3}$$

$$I(0) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta 0} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta 0} = \frac{\tilde{V}_0^+}{Z_0} - \frac{\tilde{V}_0^-}{Z_0} \tag{4}$$

Dividing the two above equations, we get the impedance at the load.

$$Z_L = Z_0 \frac{\tilde{V}_0^+ + \tilde{V}_0^-}{\tilde{V}_0^+ - \tilde{V}_0^-} \tag{5}$$

We can now solve the above equation for $\tilde{V}_0^-$

$$\frac{Z_L}{Z_0}(\tilde{V}_0^+ - \tilde{V}_0^-) = \tilde{V}_0^+ + \tilde{V}_0^-$$

$$(\frac{Z_L}{Z_0} - 1)\tilde{V}_0^+ = (\frac{Z_L}{Z_0} + 1)\tilde{V}_0^-$$

$$\frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{z_L - 1}{z_L + 1}$$

$$\Gamma_L = \frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{6}$$

Lowercase z_L is called the "normalized load impedance". It is the actual impedance divided by the transmission line impedance $z_L = \dfrac{Z_L}{Z_0}$. For example, if the load impedance is $Z_L = 100\Omega$, and the transmission-line impedance is $Z) = 50\Omega$, then the normalized impedance is $z_L = \dfrac{100\Omega}{50\Omega} = 2$. Normalized impedance is a unitless quantity.

Definition 1. $\Gamma_L = \dfrac{\tilde{V}_0^-}{\tilde{V}_0^+} = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ *is the voltage reflection coefficient at the load.* Γ_L *relates the reflected and incident voltage phasor and the load* Z_L *and transmission line impedance* Z_0*. The voltage reflection coefficient at the load is, in general, a complex number, it has a magnitude and a phase* $\Gamma_L = |\Gamma_L| e^{j\angle\Gamma_L}$.

Example

(a) $100\,\Omega$ transmission line is terminated in a series connection of a $50\,\Omega$ resistor and $10\,\text{pF}$ capacitor. The frequency of operation is $100\,\text{MHz}$. Find the voltage reflection coefficient.

(b) For purely reactive load $Z_L = j50\Omega$, find the reflection coefficient.

Voltage and Current on a transmission line

Now that we related forward and reflected voltage on a transmission line with the reflection coefficient at the load, we can re-write the equations for the current and voltage on a transmission line as:

$$\tilde{V}(z) = \tilde{V}_0^+(e^{-j\beta z} + \Gamma_L e^{j\beta z}) \tag{7}$$
$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \tag{8}$$

We see that if we know the length of the line, line type, the load impedance, and the transmission line impedance, we can calculate all variables above, except for $\tilde{V}_0^+$. In the following chapters, we will derive the equation for the forward going voltage at the load, but first, we will look at little more at the various reflection coefficients on a transmission line.

Reflection coefficient anywhere on the line

Equations 1-2 can be concisely written as

$$\tilde{V}(z) = \tilde{V}(z)^+ + \tilde{V}(z)^- \tag{9}$$
$$\tilde{I}(z) = \tilde{I}(z)^+ + \tilde{I}(z)^- \tag{10}$$

Where $\tilde{V}(z)^+$ is the forward voltage anywhere on the line, $\tilde{V}(z)^-$ is reflected voltage anywhere on the line, $\tilde{I}(z)^+$ is the forward current anywhere on the line, and $\tilde{I}(z)^-$ is the reflected current anywhere on the line.

We can then define a reflection coefficient anywhere on the line as

Definition 2. $\Gamma(z) = \dfrac{\tilde{V}(z)^-}{\tilde{V}(z)^+} = \dfrac{\tilde{V}_0^- e^{j\beta z}}{\tilde{V}_0^+ e^{-j\beta z}} = \dfrac{\tilde{V}_0^-}{\tilde{V}_0^+} e^{2j\beta z}$ *is a voltage reflection coefficient anywhere on the line.* $\Gamma(z)$ *relates the reflected and incident voltage phasor at any* z.

Since we already defined $\Gamma_L = \dfrac{\tilde{V}_0^-}{\tilde{V}_0^+}$ *as the reflection coefficient at the load, we can now simplify the general reflection coefficient as*

$$\Gamma(z) = \Gamma_L e^{2j\beta z} \tag{11}$$

It is important to remember that we defined points between the generator and the load as the negative z-axis. If the line length is, for example, l *m long, the* *generator is then at z=-l m, and the load at z=0. To find the reflection coefficient at some distance $l/2\,m$ away from the load, at $z = -l/2\,m$, the equation for the reflection coefficient will be*

$$\Gamma(z = -l/2) = \Gamma_L e^{-2j\beta l/2} \tag{12}$$

Since we already defined the reflection coefficient at the load, the reflection at any point on the line $z = -l$ is

$$\Gamma(z = -l) = \Gamma_L e^{-2j\beta l} \tag{13}$$
$$\Gamma(z = -l) = |\Gamma_L| e^{j(\angle\Gamma_L - 2\beta l)} \tag{14}$$

Reflection coefficient at the input of the transmission line

Using the reasoning above, the reflection coefficient at the input of the line whose length is l is

$$\Gamma(z = -l) = \Gamma_{in} = \Gamma_L e^{-2j\beta l} \tag{15}$$

Example 1. *The reflection coefficient at the load is $\Gamma_L = 0.5e^{j60^0}$. Find the input reflection coefficient if the electrical length of the line is $\beta l = 45^0$.*

Explanation. *The reflection coefficient at the input of the line is $\Gamma(z = -l) = \Gamma_{in} = |\Gamma_L| e^{j(\angle\Gamma_L - 2\beta l)}$.*

We substitute the expression for $\Gamma_L = 0.5e^{j60^0}$ and $\beta l = 90^0$, we get the reflection coefficient at the input of the line $\Gamma_{in} = 0.5e^{-j30}$.