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Calculus 173

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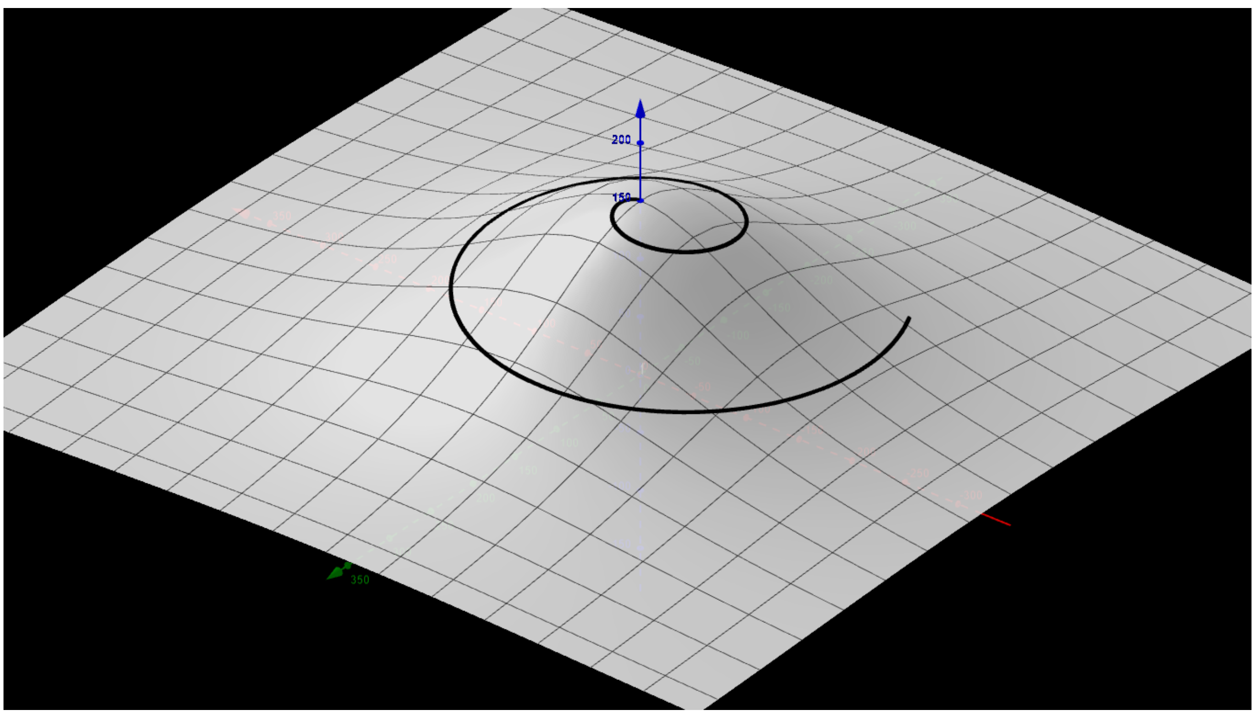
Sledding down a Parametric Mountain with a nonconstant friction force



Description

A person with a certain mass is sledding down a predetermined parametric curve that lies on a mountain-like surface. The main goal is to be able to find the work done by gravity and friction on our theoretical sledder using line integrals and vector fields. If I can find the work done by gravity and friction, I should theoretically be able to find exactly the time on the space curve where the sled starts and stops, as well as calculate the total distance traveled before friction brings the sled to a stop. **The real challenge will be figuring out how to do this over a non-constant angle of inclination, which makes finding the work due to friction much more complicated and should require use of a line integral.**

The Question

A person on a sled of mass $m = 65\ kg$ sits atop a 150 meter tall mountain given by the surface equation

$$f(x,y) = \frac{1}{\frac{1}{150}+\left(\frac{x}{2000}\right)^2+\left(\frac{y}{2000}\right)^2}\ .$$

The sled slides down the mountain on a spiral path represented by the 2d parametric curve

$$\boldsymbol{v}(t) = \langle 20tcos(t), 20tsint\rangle\ .$$

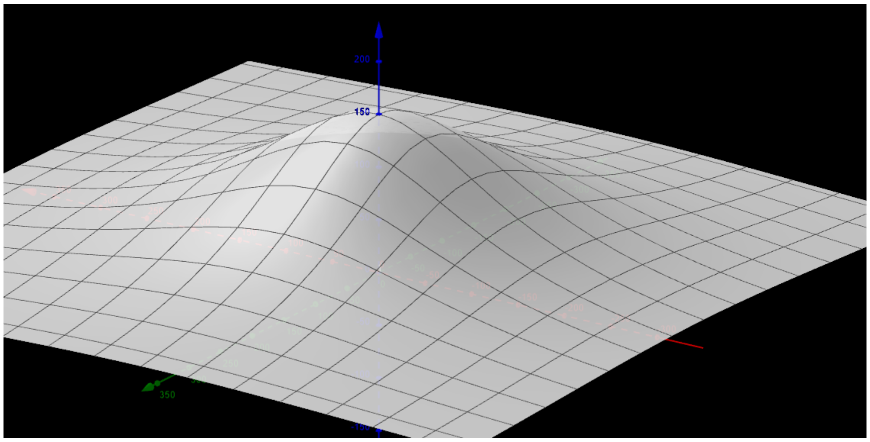
If the sled undergoes a friction force with a friction coefficient of μ_k =0.1, find

(a) The parametric space curve that lies on the mountain's surface
(b) The total work done by friction and gravity at any given time
(c) The maximum velocity
(d) The total distance traveled before the sled stops due to friction

Inventory of Calculus Concepts Applied

- 12.5 – Angle Between Vector and Plane
- 13.1 – Intersection of Surfaces and Parametric Curves
- 13.2 – Finding the Unit Tangent
- 13.3 – Arc Length
- 16.2 – Line Integral of a Parametric Curve through a Force Field

Data to Collect



For the data, I had to do a bit of trial and error to figure out what would be a good surface to use for the mountain. It didn't take long for me to find a nice-looking mountain surface from Stewart's book. The equation was

$$f(x,y) = \frac{1}{1+x^2+y^2}.$$

After messing around a bit with the parameters in Geogebra, I figured out how to scale it up to any size. Ideally, if a sled were to go down it, the friction would cause the sled to come to a complete stop. I also wanted to make sure that the sledder wouldn't be going hundreds of miles per hour. We can estimate the speed of the sled due to gravity at any height using conservation of energy.

$$KE = mgh_0 - mgh_f = 95\,550\,J - 0\,J = 54.2\frac{m}{s} \approx 120.8\,mph$$

120.8 mph seems like a lot, but if we estimate the opposing friction force at a 10 degree angle of inclination and a friction coefficient, it actually slows the sled down quite a lot. For the friction coefficient, I did some research and found [this](#) Cambridge study that stated the friction coefficient of skiing on snow ranged from $0.03 \leq \mu_k \leq 0.1$ depending on velocity.

For simplicity, I'll assume a constant coefficient of $\mu_k = 0.1$ since a sled has more surface area.

$$F_k = \mu_k mgcos(\theta) = 0.1(65\,kg)\left(9.8\frac{m}{s^2}\right)\cos(10°) = 62.7\,N$$

Then the friction force would be about 62.7 newtons compared to the a gravitational force of

$$F_g = mgsin(\theta) = (65\,kg)\left(9.8\frac{m}{s^2}\right)\cos(10°) = 110.6\,N.$$

Since friction is about half of the gravitational force, we can say that the total work done would be about half of the gravitational work done. Using the work-energy theorem, velocity of the sledder would be

$$KE = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{2\left(\frac{95550}{2}J\right)}{65\,kg}} \approx 84\,mph.$$

Still pretty fast. But if we consider the angle goes from 0 to 10 back to 0 quickly, the actual velocity will be significantly lower. In retrospect, I got pretty lucky with the surface I chose here just by eyeballing what the angle of inclination would look like in a spiral space curve. If I wanted to be sure, I could have found the gradient of the surface, but that would have required first finding the space curve on the surface.

Answering the Questions

Applying the 2-d space curve to the Mountain Surface

We're given a 2-d path that the skiier would want to take down the mountain.

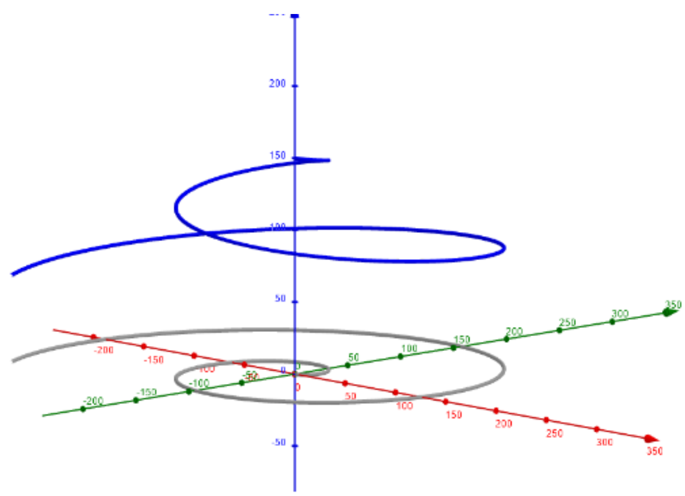
$$\boldsymbol{v}(t) = \langle 20t\cos(t), 20t\sin(t) \rangle$$

We need to find the path that intersects with the mountain. Since we already have the x and y components of the space curve, all that's left to do is reparametrize the z component.

$$\boldsymbol{r}(t) = \langle 20t\cos(t), 20t\sin(t), z \rangle$$

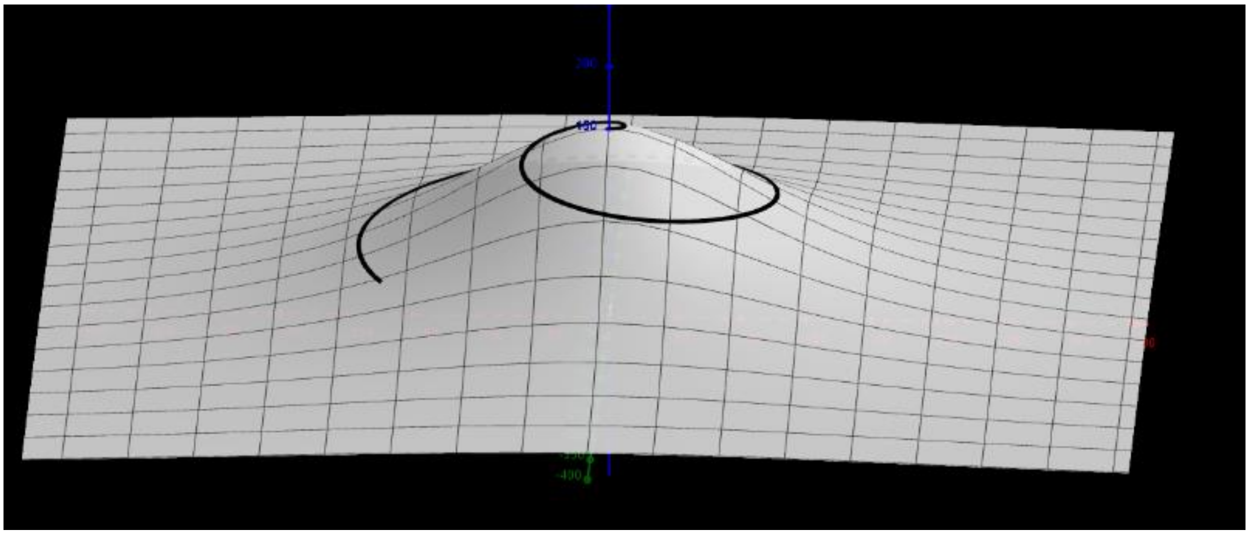
$$z = f(x,y) = \frac{1}{\frac{1}{150} + \left(\frac{x}{2000}\right)^2 + \left(\frac{y}{2000}\right)^2} = \frac{1}{\frac{1}{150} + \left(\frac{20t\cos(t)}{2000}\right)^2 + \left(\frac{20t\sin(t)}{2000}\right)^2} = \frac{30000}{200 + 3t^2}$$

$$\therefore \boldsymbol{r}(t) = \langle 20t\cos(t), 20t\sin(t), \tfrac{30000}{200+3t^2} \rangle$$



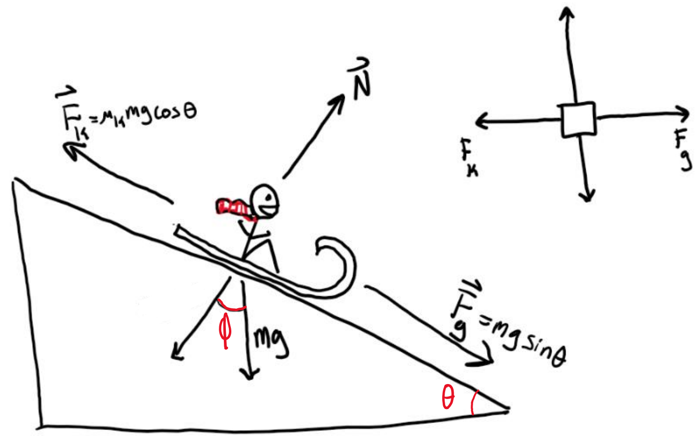
As you can see, Geogebra is showing that we have successfully parametrized the space curve with respect to the surface.

Ctrl+click the pictures to take you to the actual graph!



Finding the Work Done by Gravity and Friction

For finding the work done by either forces, we first need to be able to define the net force at any given time on the curve.



From Newtonian Mechanics we have that the force of gravity is $F_g = mgsin\theta$, and friction is $F_k = \mu_k mgcos\theta$, where θ is the angle of incline.

These two forces oppose each other to give a net force of

$\Sigma F = F_g - F_k$.

But there's one seriously big hurdle. The angle of incline is not constant. This means that as the sled moves down the space curve, the angle between the surface of the mountain and the flat ground is changing. At this point, I thought the project was kaput and we would need 2nd Order Differentials or something. But then, I had a lightbulb moment. Enter: The Unit Tangent Vector.

If we can find the unit tangent vector of our guy as he's hurling down the mountain, we can also find the angle between the slope of the mountain and the flat ground (the x-y plane) at any point in time. I harkened back to the days where we had to find the unit tangents and felt a sense of dread knowing what was to come, and it wasn't pretty. I will spare you the gruesome details of how I slayed the unit tangent monster, but here is what I came up with.

The Unit Tangent of the Sled as a function of t

$$\boldsymbol{T}(t) = \langle \frac{20(\cos t - \sin t)}{\sqrt{\left(\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}}, \frac{20(sint + tcost)}{\sqrt{\left(\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}}, -\frac{180\,000t}{(200+3t^2)^2\sqrt{\left(\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}} \rangle$$

To find the angle between the unit tangent and the x-y plane, we need the normal vector of the x-y plane, which is simply $\boldsymbol{n} = \langle 0,0,1 \rangle$.

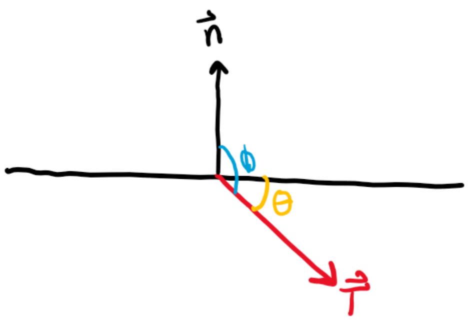
Then we have

$$\boldsymbol{T} \cdot \boldsymbol{n} = \left|\left|\boldsymbol{T}\right|\right|\left|\left|\boldsymbol{n}\right|\right| cos\phi$$

$$\boldsymbol{T} \cdot \langle 0,0,1 \rangle = \left|\left|\boldsymbol{T}\right|\right| cos\phi$$

$$\phi = \cos^{-1}\left(\frac{\boldsymbol{T} \cdot \langle 0,0,1 \rangle}{\left|\left|\boldsymbol{T}\right|\right|}\right)$$

$$\phi(t) = \cos^{-1}\left(-\frac{9000t}{(200+3t^2)^2\sqrt{t^2+1+\left(\frac{9000t}{(200+3t^2)^2}\right)^2}}\right)$$



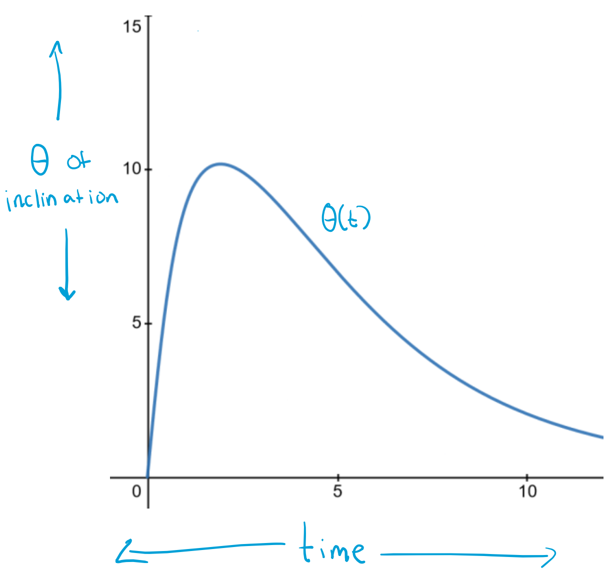
ϕ is the angle between but Unit Tangent and the normal vector of the x-y plane at any given time.

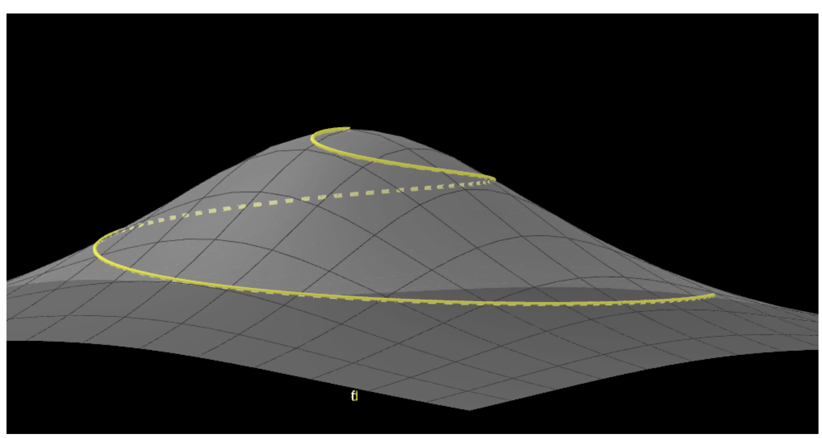
All we need to do now is subtract 90 degrees and flip a sign to get the angle of incline.

The angle of Incline as a function of t

$$\theta(t) = 90° - \cos^{-1}\left(\frac{9000t}{(200+3t^2)^2\sqrt{t^2+1+\left(\frac{9000t}{(200+3t^2)^2}\right)^2}}\right)$$

As you can see, θ starts off at zero, then increases quickly to a maximum angle just above 10 degrees and tapers off slower. Compare this with the path of the parametric curve and you can see how it matches up. Beautiful. (ctrl+click the image on the right to see for yourself)





https://www.geogebra.org/classic/jkngwyf7

Force as a Function of Time

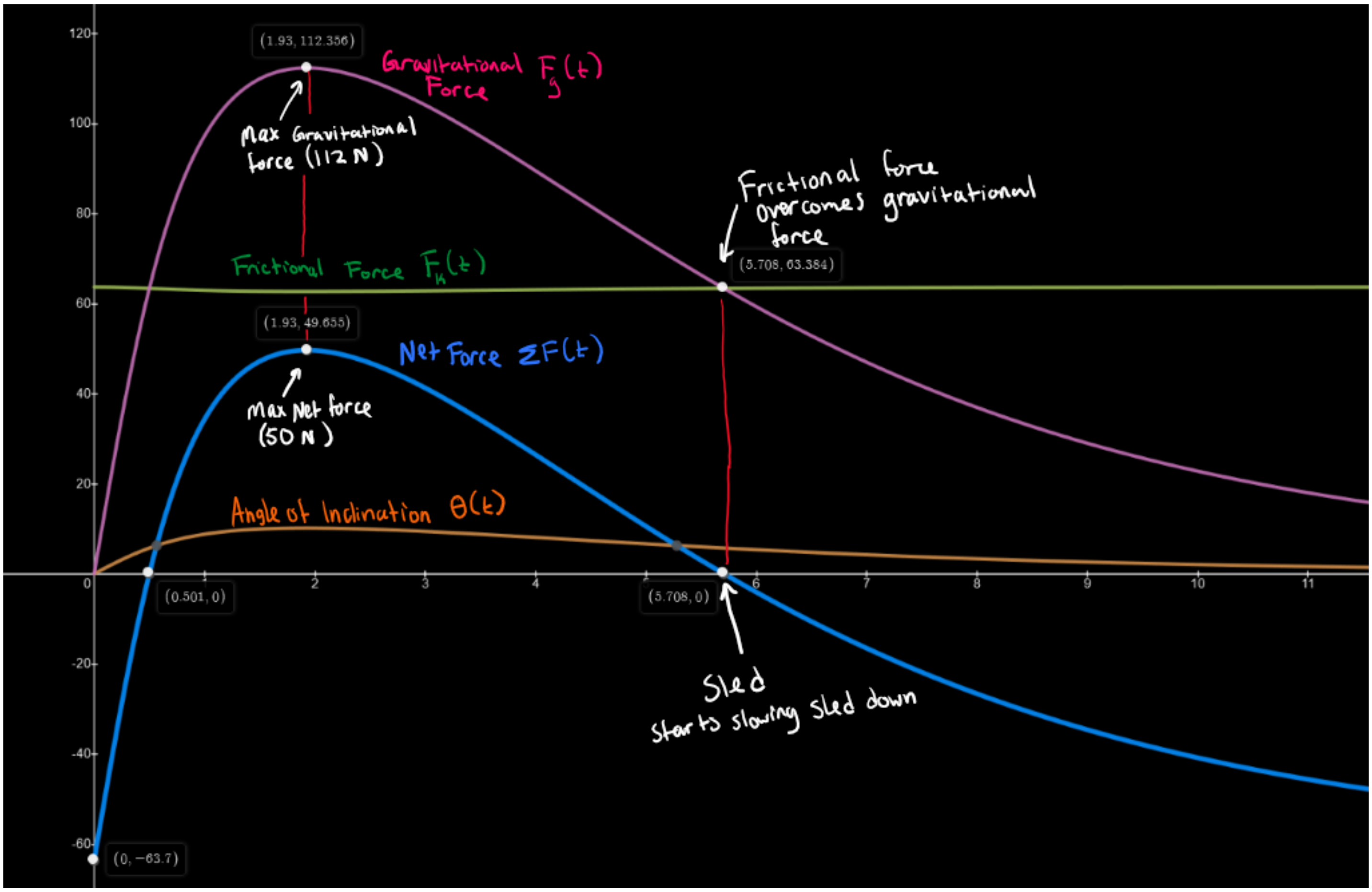
Now that we have the angle of incline at any time t, we should be able to calculate the net force at any time by plugging in our angle function.

$$F_g(t) = mgsin(\theta(t))$$

$$F_k(t) = \mu_k mgcos\big(\theta(t)\big)$$

$$\Sigma F(t) = F_g - F_k$$

Pictured below: Various Forces as t increases (ctrl+click the picture to go to graph)



https://www.desmos.com/calculator/haj6c2uxoy

Line Integrals over the Force Vectors to find Work Done

The final hurdle was me really trying to wrap my head around finding the total work done with this information. I needed to find the work done by gravity and subtract that from the work done by friction. Since gravity is a conservative force, and since we can easily infer the vector function of gravitational force, I started with that first. Doing this, I could also use The Law of Conservation of Energy to confirm that my angle functions are working as intended.

Setting up the Line Integral

A line integral is a force vector dotted with an infinitesimally small distance along a curve

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{s}$$

But we have a problem. The force function is not a vector so I need a way to get rid of the dot product.

Luckily, I figured it out.

$\boldsymbol{F} \cdot d\boldsymbol{s} = \boldsymbol{F} \cdot \boldsymbol{r}'(t)dt = \big||\boldsymbol{F}|\big|\big||\boldsymbol{r}'(t)|\big| cos\theta$dt

Note that for both forces, we have already parametrized the function so that the unit tangent is always parallel to the direction of force.

For gravity $\theta = 0°$

For kinetic friction $\theta = 180°$

Then

$$F_g = \left||\boldsymbol{F_g}|\right| \big||\boldsymbol{r}'(t)|\big| \cos(0°)dt = F_g(t)\big||\boldsymbol{r}'(t)|\big|dt$$

$$F_k = \big||\boldsymbol{F_k}|\big|\big||\boldsymbol{r}'(t)|\big| \cos(180°)dt = -F_k(t)\big||\boldsymbol{r}'(t)|\big|dt$$

Where our force functions equate to scalar values (magnitudes).

Thus, our line integrals become

$$\boxed{\begin{aligned} W_g &= \int_0^t F_g(t)||\boldsymbol{r}'(t)||dt \\ W_f &= -\int_0^t F_k(t)||\boldsymbol{r}'(t)||dt \end{aligned}}$$

And we already have $||\boldsymbol{r}'(t)||$ from finding $\boldsymbol{T}$ earlier.

$$||\boldsymbol{r}'(t)|| = \sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}$$

Work Done by Gravity

Starting with work done by gravity, let's take our new integrals for a spin.

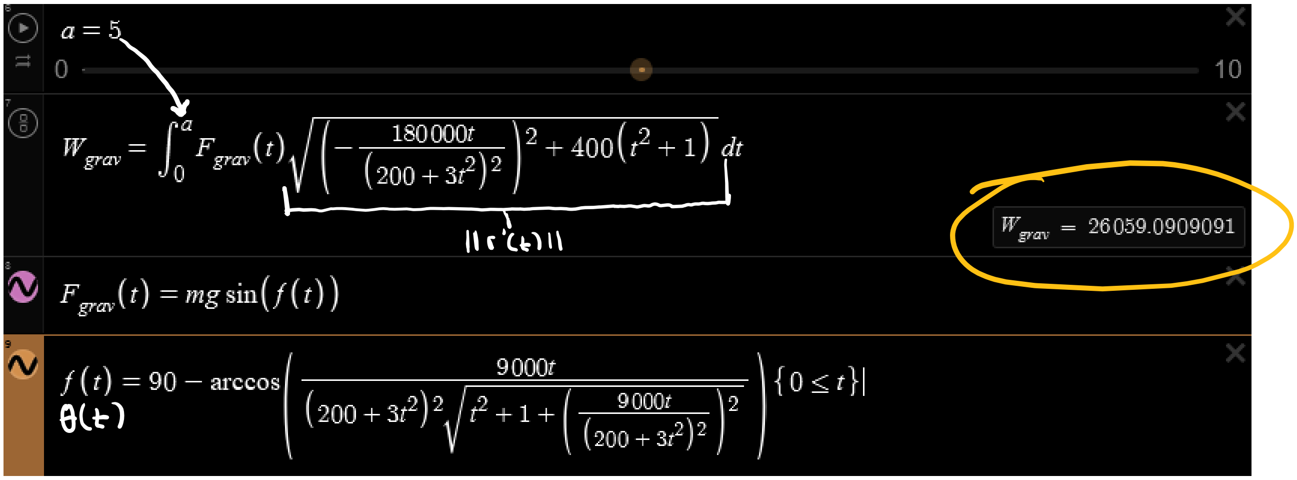
$$W_g = \int_0^t F_g(t)||\boldsymbol{r}'(t)||dt$$

This is what the gravity integral looks like fully expanded. Not very fun.

$$W_g = \int_0^t \left[mgsin\left(90 - \cos^{-1}\left(\frac{9000t}{(200+3t^2)^2 \sqrt{t^2+1+\left(\frac{9000t}{(200+3t^2)^2}\right)^2}} \right) \right) \cdot \sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)} \right] dt$$

Using Desmos, we can set up this integral and create a slider for the bounds of integration.

Let's see what we get for $0 \leq t \leq 5$.



So from $0 \leq t \leq 5$ there is about 26,000 J of work done by gravity. Let's compare this result with the Law of Conservation of Energy to see if it's working right.

Validifying the result

The Law of Conservation of Energy states that

$$\Delta KE = -\Delta PE$$

$$KE_f - KE_0 = PE_0 - PE_f$$

There is no initial kinetic energy so

$$KE_f = PE_0 - PE_f$$

$$KE = mgh_0 - mgh_f$$

For h_0 the sledder starts at the top of our surface at $f(0,0)$. As stated in the problem, our simplified surface equation is

$$f(x,y) = \frac{30000}{200+3t^2}$$

So then the height is

$$h_0 = f(0,0) = \frac{30\,000}{200} = 150\,m$$

For PE_f we will need to find where our sledder is at t = 5. From earlier, the position vector is

$$\boldsymbol{r}(t) = \langle 20tcos(t), 20tsin(t), \frac{30000}{200+3t^2} \rangle$$

So then

$$\boldsymbol{r}(5) = \langle 20(5)cos(5), 20(5)sin(5), \frac{30000}{200+3(5)^2} \rangle = \langle 99.6\,, 8.71\,, 109.1 \rangle\, m$$

For height, we really only need the z-component, which is 109.1 m.

$$h_f = 109.1\,m$$

Now, plugging our values into the energy equation we get

$$KE = (65\,kg\,)\left(9.8\frac{m}{s^2}\right)(150\,m) - (65\,kg)\left(9.8\frac{m}{s^2}\right)(109.1m)$$

$$\boldsymbol{KE = 26{,}053\,J}$$

Very close to the value we got from the vector function.

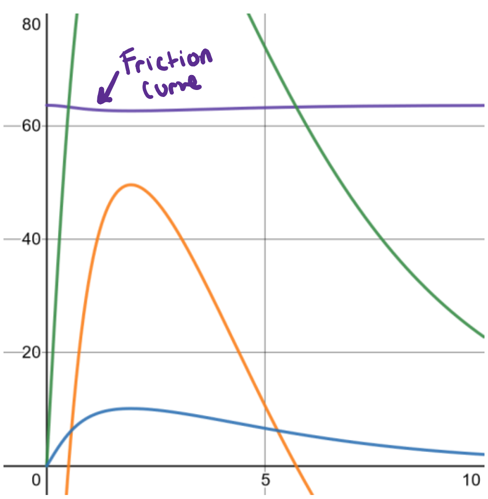
$$Work\ Energy{:}\ KE = 26053\,J$$

$$Line\ Integral{:}\ W_g = 26059\,J$$

Close enough. It works!!!

Work Done by Friction Force

The last big piece of the puzzle is calculating the work done by friction.



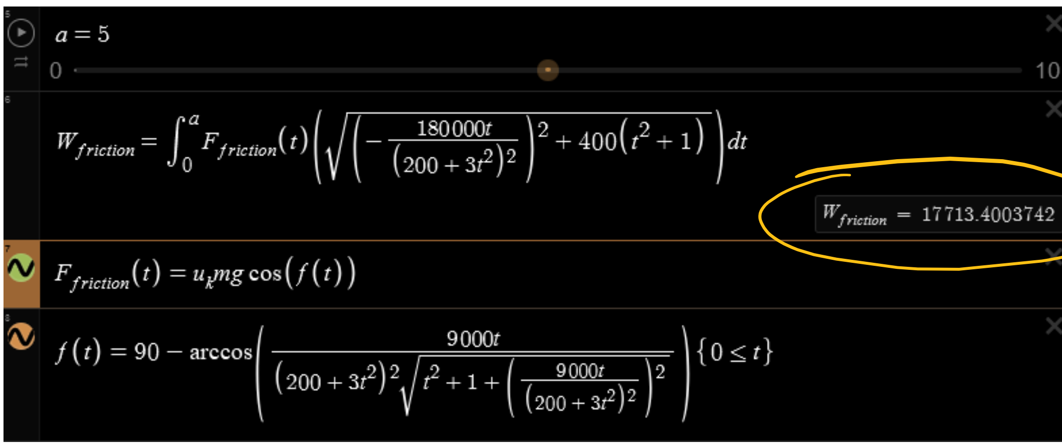
It's worth noting that because the angles of inclination are not very steep, friction force does not change very much.

That being said, to be as accurate as possible we can and should still perform a line integral to find the work done. As shown earlier we were able to deal with the dot product by realizing that by parametrizing our force function with the angle of inclination, they are already parallel.

$$W_f = \int_0^t F \cdot d\boldsymbol{s} = \int_0^t F_k(t)\left|\left|\boldsymbol{r}'(t)\right|\right|dt$$

I’ll use Desmos as the integral isn’t very pretty here either.

$$W_f = \int_0^5 \left[\mu_k mgcos\left(90 - \cos^{-1}\left(\frac{9000t}{(200+3t^2)^2\sqrt{t^2+1+\left(\frac{9000t}{(200+3t^2)^2}\right)^2}} \right) \right) \cdot \sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)} \right] dt$$



The Work done by friction is about 17,700 J.

Total Work and How to use the Desmos Program

Finally, I have an answer for work done by gravity and friction at any given time on the space curve. I have done all the plumbing in Desmos so that you can move the slider to output the total work output from 0 to any t.

Ctrl+click the picture below to see everything in action. Moving the "a" slider at top and watch the work values change below. Note that the velocity is undefined until the total work is above zero. This is because friction has done more total work than gravity and the sled is no longer moving. You can also change constants such as mass, friction coefficient, and gravitational acceleration**, but note that all my values in this paper are based on 65 kg mass with a friction coeff. Of 0.1 and gravitational acceleration 9.80 m/s^2.** Also note that when the green curve (gravitational force) crosses the purple curve (friction force), the velocity starts decreasing.

If the picture link is not working, visit this link https://www.desmos.com/calculator/efql8aoawe

$a = 5$

0 10

$$W_{total} = W_{grav} - W_{friction}$$

$W_{total} = 8345.69053491$

$$W_{grav} = \int_0^a F_{grav}(t)\sqrt{\left(-\frac{180000t}{\left(200+3t^2\right)^2}\right)^2 + 400\left(t^2+1\right)}\,dt$$

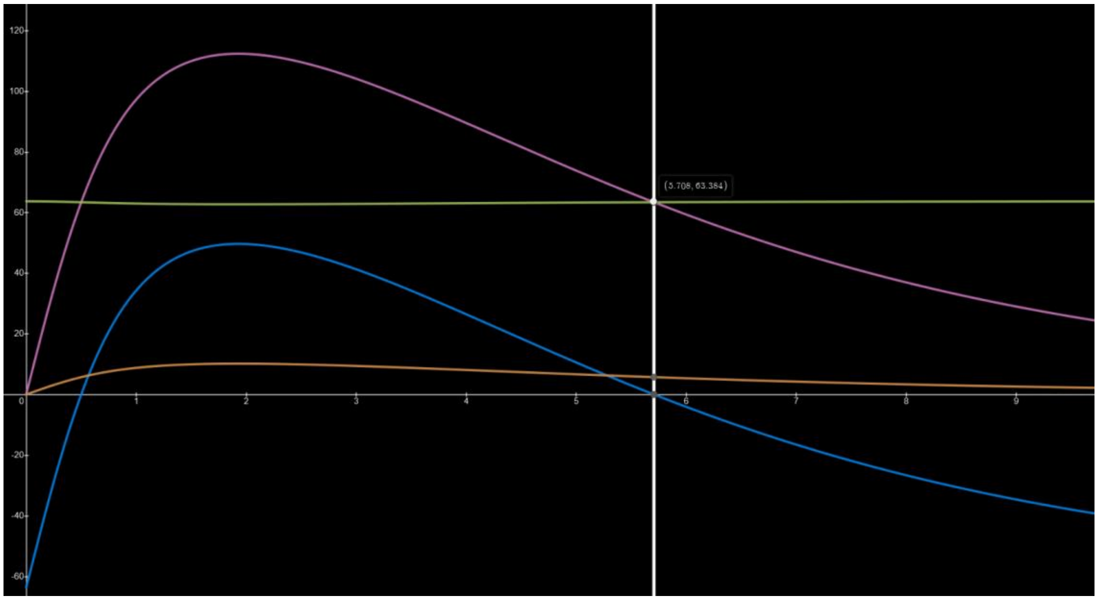
$W_{grav} = 26059.0909091$

$$W_{friction} = \int_0^a F_{friction}(t)\left(\sqrt{\left(-\frac{180000t}{\left(200+3t^2\right)^2}\right)^2 + 400\left(t^2+1\right)}\right)dt$$

$W_{friction} = 17713.4003742$

Maximum Velocity

Using the data from the Desmos graph, we can move the slider along t until we see that the maximum velocity occurs at a key point.



The maximum velocity occurs when the frictional force exceeds the gravitational force and then begins slowing the sled. This value occurs at $t = 5.708$ and the total Work done at that time is $\Sigma W = 8741\ J$.

Using the Work Energy Theorem, we know that the total work done on an object equals the object's kinetic energy, so

$$\Sigma W = KE = 8741\ J$$

And since

$$KE = \frac{1}{2}mv^2$$

We can calculate the velocity at that time.

$$v_{max} = \sqrt{\frac{2(8741\ J)}{65\ kg}} \approx 16.4\frac{m}{s} = \mathbf{36.6\ mph}$$

Total Distance Traveled until Sled Stops

To find the total distance traveled due to work, we can look at the Desmos graph to find a value of t where the work is negative. As you can see, at the beginning the sled is not moving. According to the graph, the net force has a deficit of -63.7 N at t = 0. This means that 63.7 Newtons are required to get it moving, or approximately 14.3 lbs of force. For this scenario, let's assume that the person kicks off with exactly 14.3 lbs of force to start moving, so that the work in the end will be the same as it says on the graph.

If you can scroll slowly through the graph, you'll see that the velocity reaches 0 at around t = 8.7.

I found a more precise value to be t = 8.77799356. Therefore, the sled stops moving at this value. Now we can take the arc length from 0 to that value to find the total distance traveled.

$$L = \int_0^{8.77799356} \left\| \boldsymbol{r}'(t) \right\| dt$$

We found $\left\|\boldsymbol{r}'(t)\right\|$ earlier.

$$||\boldsymbol{r}'(t)|| = \sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}$$

So then

$$Distance\ Traveled = \int_0^{8.77799356} \left[\sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}\right] dt \approx \mathbf{808.87}\ \boldsymbol{m}$$

∴ The total distance the sled travels before it comes to a stop is about 810 meters.

Conclusion

So, to answer the questions stated:

If the sled undergoes a friction force with a friction coefficient of μ_k =0.1, find

(a) The parametric space curve that lies on the mountain's surface

$$\boldsymbol{r}(t) = \langle \mathbf{20t}\boldsymbol{cos}(\boldsymbol{t}), \mathbf{20t}\boldsymbol{sin}(\boldsymbol{t}), \frac{\mathbf{30000}}{\mathbf{200+3t^2}}\rangle$$

(b) The total work done by friction and gravity at any given time

$$\Sigma W = \int_0^t F_g(t)\left\|\boldsymbol{r}'(t)\right\| dt - \int_0^t F_k(t)\left\|\boldsymbol{r}'(t)\right\| dt$$

Where $$F_g(t) = mgsin(\theta(t)),$$

$$F_k(t) = \mu_k mgcos(\theta(t)),$$

$$\theta(t) = 90° - \cos^{-1}\left(\frac{9000t}{(200+3t^2)^2\sqrt{t^2+1+\left(\frac{9000t}{(200+3t^2)^2}\right)^2}}\right),$$

And $$||\boldsymbol{r}'(t)|| = \sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)}\,.$$

(c) The maximum velocity

$$v_{max} = \sqrt{\frac{2(8741\,J)}{65\,kg}} \approx 16.4\frac{m}{s} = \mathbf{36.6\,mph}$$

(d) The total distance traveled before the sled stops due to friction

$$Distance\ Traveled = \int_0^{8.77799356} \left[\sqrt{\left(-\frac{180\,000t}{(3t^2+200)^2}\right)^2 + 400(t^2+1)} \right] dt \approx \mathbf{808.87\,m}$$

Starting with a sled, a surface, and a 2 dimensional parametric curve, I wound up with the exact distance the sled stopped due to friction on a **nonconstant** angle of inclination. Starting off, I found where the space curve would lay on the multivariate surface. Then, I found a way to get the angle between the unit tangent and the x-y plane for any point on the space curve. Using this, I found force functions for friction and gravity that depended on the changing angle of inclination as a function of time. From there, I was able to reconfigure the general line integral definition for work so the force functions would work with the arc length. From there, I used Desmos to create a slider for the bounds of integration so I could see the work done for any time interval. With everything set up I was easily able to find the max velocity and the point where the sled stopped. Finally, after finding where the sled stopped, I was able to find the arc length of the total distance traveled.

I am pretty happy with how this project turned out, and hopefully the logic is correct. I struggled a lot with trying to see how my force functions could be integrated. The results obtained seem consistent with my estimates.

As far as practical applications go, this could be the basis behind a physics engine for a simple physics-based video game. Everything seems fully functional. The limitations are that it only works with a predefined parametric curve. We could potentially follow the same procedure for any parametric space curve and maybe even try to see if an applied force could overcome gravity going uphill. A parametric rollercoaster would be fun.