Summary of the course · 1MA170

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This document gives a summary of the entire course, with keywords highlighted, in colours indicating whether and how it might appear on the exam.

How this document works

The document goes through each lecture in order, noting what we covered in the lecture. In particular, it highlights the key topics of the course, and marks for each how it may appear on the exam.

In particular, the coding works as follows:

1. If it is highlighted like this, that indicates that you are expected to be familiar with the statement to the degree that you could use it to show some other statement or solve an exercise, if the highlighted statement is provided to you.

   You could also be asked to provide a precise statement given a prompt as to what it is about – so for example if you see Dirac's theorem in this document, an exam question might also be "What does Dirac's theorem about the existence of Hamiltonian paths say?". You are not expected to know the proof of the result.

2. If it is highlighted like this, you are expected to not only know the statement in the same sense as in the previous point, but also to have an idea of the proof of the theorem. So you might be asked to fill in a key step of the proof of the statement, write a precise proof given a prompt of what the general outline is, or write an outline of the idea of the proof.

   So if you see Dirac's theorem in this document, an exam question might give you the proof of the result with the step where the maximal length path is turned into a cycle, and you are asked to fill in that step. Or you could be asked to write a proof, given that the drawings for the proof from the lecture notes are given to you. Or you could be asked to draw those figures and explain the broad idea of taking a maximal length path and showing that it can be turned into a cycle, which must be a Hamilton cycle.

3. If it is highlighted like this, you are not expected to recall the exact statement without a prompt, but you are expected to be able to prove it without a reminder of the idea of the proof. So if you see Dirac's theorem, an exam question might look like "State and prove Dirac's theorem about the existence of Hamilton cycles".

4. If it is highlighted like this, you are expected to know the statement of the theorem without any prompt, but not expected to know the proof.

   So if you see Dirac's theorem in this document, an exam question might be "State Dirac's theorem", but you would not be asked about the proof.

5. If it is highlighted like this, you are expected to know the statement of the result, and additionally to have an idea of the proof. So this is the same as this and this together.

6. Finally, if it is highlighted like this, you are expected to know both the theorem and its proof. If you see Dirac's theorem, that means you could see an exam question just ask "State and prove Dirac's theorem".

For definitions, it of course makes no sense to refer to knowing a proof, so we simply highlight definitions like this if you are expected to know and be able to state the definitions, and like this if you are just expected to be able to use the definition and explain the idea of it if given it, but not to be able to state it.

L2: Eulerianity, simple graphs and subgraphs

In our first lecture of the course we started softly, giving the definitions of a multigraph, a walk, a trail, a path, a circuit, and a cycle.

Then we defined what it means for a graph to be connected, and what its connected components are.

Having made all these definitions, we defined an Eulerian trail to be a trail using every edge exactly once, and stated and proved Euler's theorem on Eulerian paths, which characterizes when a graph is Eulerian in terms of the degree of its vertices.

Then, we stated and proved the handshake lemma, which says that

$$2|E| = \sum_{v \in V} d_v.$$

Having done all this, we defined a simple graph[2], and what a graph morphism of simple graphs is, in terms of which we could then define isomorphism of graphs.

[2] Which is of course, for most of the course, the only notion of graph we referred to – so generally we end up just calling these *graphs*.

Once we knew what it meant for graphs to be isomorphic, we could define an unlabelled graph to be an isomorphism class of graphs.[3]

[3] We largely did not end up actually using this concept – other than in a few counting arguments, where we needed to be clear that we were *not* considering unlabelled but labelled graphs.

We ended the lecture by defining what a subgraph, an induced subgraph, an edge-induced subgraph, and a spanning subgraph is.

L3: Common graph families, trees, and Cayley's theorem

We started the lecture with giving definitions of a couple of commonly occuring graph families: the complete graphs, path graphs, cycle graphs, complete bipartite graphs, and complete multipartite graphs.

Then we moved on to the main topic of the lecture: Trees. We started by proving that any tree on n vertices has $n - 1$ edges. We then stated, but did not prove,[4] Cayley's formula on the number of labelled trees on n vertices.

Having done this, we proved a characterisation of trees in terms of three properties equivalent to being a tree. Then, we defined the notion of a spanning tree, and proved that all multigraphs have a spanning tree, assuming the axiom of choice.[5]

[4] The proof was deferred until the next lecture, when we were able to prove it as a corollary of a more general result.

[5] In fact, the two statements – existence of spanning trees for arbitrary graphs and the axiom of choice – are equivalent.

L4: Spectral graph theory and the matrix-tree theorem

We started by defining the adjacency matrix of a graph, then we defined what we mean by a directed graph, and used that to define what an incidence matrix of a graph is.

We then proved that the rank of the incidence matrix equals the number of vertices minus the number of connected components. We then defined our final matrix associated to a graph, the Laplacian Q of a graph, and we showed that the Laplacian satisfies $Q = DD^t$.

Then we did a bunch more linear algebra stuff in order to finally arrive at the Kirchhoff matrix-tree theorem. We then used this to give a proof of Cayley's formula, which we had stated in the previous lecture.

L6: Weights, distances, and minimum spanning trees

We started by defining Prim's algorithm. Then we stated and proved that removing an edge from a tree yields a forest of two trees. Then we proved that Prim's algorithm is correct.

Then, we defined Kruskal's algorithm, and proved that it is correct. After this, we defined the graph distance and thus the diameter of a graph. Having done this, we could define Dijkstra's algorithm.

L7: The max-flow min-cut and marriage theorems

We defined what a weighted directed graph is, and what a flow network is. Then we defined what a flow on these networks is, and its value.

We then defined a cut on a flow network and its capacity, and showed that the value of any flow is upper bounded by the capacity of any cut. We then defined the residual network of a flow, and defined an augmenting path in this network.

We stated that any augmenting path can be used to find a higher-value flow, and used this to show the Ford-Fulkerson theorem.

Then, we defined what a matching and a bipartite graph is, and stated and proved Hall's marriage theorem using the max-flow-min-cut duality we had just seen.

L8: Vertex covers, Hamilton cycles, independent sets

We continued on the themes of the previous lecture, using max-flow-min-cut to prove König's theorem relating vertex covers to the largest matching on a bipartite graph.

Then, we defined Hamilton cycles, and stated and proved Dirac's theorem.

The penultimate topic of the lecture was independent sets and the indepence number $\alpha(G)$ of a graph G. We defined the line graph of a graph, and noted that matchings are just independent sets in the line graph.

Then we stated and proved that the problem of determining whether a graph has an independent set of size k is NP-Complete. Nevertheless, we were able to prove the Caro-Wei theorem giving a lower bound on the independence number of a graph, which was our first example of a proof using the probabilistic method.

Finally, in a little section of only definitions and nearly no theorems, we defined vertex colourings of graphs, the chromatic number $\chi(G)$ of a graph, and the clique number of a graph.

L10: Connectivity

We started by defining what it means for a graph to be k-connected, and defined the connectivity of a graph. We then defined what it means for a set to separate two vertices, or two sets.

We then showed that the minimum size of a set that separates two non-adjacent vertices equals the connectivity for any graph. Then, we showed Menger's theorem, stating that the minimum size of a set separating two non-adjacent vertices is precisely the largest size of a set of independent paths between them – a fact we proved using the min-flow-max-cut duality yet again.

Having done this general work, we proceeded to study the structure of two-connected graphs, showing a characterisation of two-connected graphs.

Then, we moved on to generalizing the notion of dividing a disconnected graph into its connected components into dividing a connected graph into its two-connected components, which we called blocks. To do this, we also needed the notion of a cutvertex and a bridge.

The crucial lemma we needed to understand the structure of the blocks was that every cycle intersects exactly one block in more than one vertex. We then defined the block graph of a graph, and proved that the block graph is a tree.

L11: Planarity

Despite the lecture primarily being about planarity, we started by defining an edge contraction, and then stated a lemma about the structure of three-connected graphs – there is an edge in any three-connected graph[6] that can be contracted to yield another three-connected graph.

Having set up our lemma, we defined what it means for a graph to be planar, and what the planar dual of a graph is. We then used the notion of a planar dual to prove Euler's formula for planar graphs. As a corollary to this, we got an upper bound on the number of edges of a planar graph (or a triangle-free planar graph) in terms of the number of vertices.

Then we defined the notions of a topological minor and a minor of a graph, and stated and partially proved the forbidden minor characterisation of planar graphs due to Kuratowski and Wagner.

[6] On at least four vertices.

L12: Vertex colourings

In this lecture, we finally made use of our earlier definitions of a vertex colouring and the chromatic number of a graph.

We started by defining the greedy colouring algorithm, and used it to prove that $\chi(G) \leq \Delta + 1$ for all graphs G.

Then we defined breadth-first search in a graph, and used this to show that in fact $\chi(G) \leq \Delta$ if G is not regular. Then, we showed Brooks' theorem using a more complicated version of this idea.

We then proved that all planar graphs are five-colourable, using an argument with Kempe chains. Continuing our study of colourings of planar graphs, we defined the notion of an outerplanar graph, and proved that all outerplanar graphs are three-colourable, from which followed the art gallery theorem.

L14: The probabilistic method and the Erdős-Rényi random graph

In this lecture, we introduced one of the major techniques of modern combinatorics and graph theory, the *probabilistic method*. We started by showing a lower bound on the minimum bisection problem.

Then we defined the Erdős-Rényi model for a random graph, and proved a bound on the probability of it having high independence number using the union bound method.

As an application of Markov's inequality, and the first-moment method, we showed that a $G(n, p)$ a.a.s. has diameter 2 for fixed p. (We also defined what we mean by a property holding asymptotically almost surely/with high probability.)

Then, as an application of the second-moment method, we showed a result about when a $G(n, p)$ has a triangle. We then sketched a picture of the growth of an Erdős-Rényi graph, defined the girth of a graph, and gave a sketch of the proof of Erdős's result that there are graphs of arbitrarily high girth and chromatic number.

L16: Edge-colourings and Ramsey theory

We started by defining an edge colouring of a graph, and the edge-chromatic number of a graph. Then we stated and proved König's line-colouring theorem, using the idea of a Kempe change for edge-colourings as well.

Then we defined the Ramsey number $R(r, k)$, and proved their existence using a recursion for the Ramsey numbers, which also gave an upper bound on these numbers. We then proved a lower bound on the diagonal Ramsey numbers using the probabilistic method, with a proof due to Erdős.

L17: The Rado graph

We started the lecture by defining what the Rado graph is, namely as the unique homogeneous countable graph containing every finite graph as an induced subgraph.

We then defined the notion of a graph being k-saturated, and proved that a graph is the Rado graph if and only if it is countable and k-saturated for every k.[7]

Then we finally proved the existence of the Rado graph, by showing that an Erdős-Rényi graph on infinitely many vertices is almost surely isomorphic to the Rado graph.

We then used a consequence of this proof to see that removing any finite number of vertices or edges of the Rado graph leaves you with a graph isomorphic to the Rado graph. As a second statement about the fractal-ness of the Rado graph, we showed a result about partitioning the Rado graph into induced subgraphs.

We ended the lecture by giving two explicit constructions of the Rado graph, one of which wasn't actually explicit at all, and one of which was.

[7] The proof of this was divided into several lemmas – if this appears on the exam, it won't be one monster question, it'll be a proof of one of the constituent lemmas.

L18: Extremal graphs and Szemerédi's regularity lemma

We started by defining the extremal function of a graph, the Turán graphs, and stating Turán's theorem. We then gave two proofs of this – one using the Caro-Wei theorem and Cauchy-Schwarz inequality, and one directly proving that the Turán graphs are extremal.

Then we proved a beautiful stability result about nearly-extremal graphs.

Then, moving on to the second topic of the lecture, we defined the notion of the density of a pair of vertex sets, and the notion of such a pair being ε-regular. Having defined this, we stated the Szemerédi regularity lemma,[8] and then we sketched how one can use this to prove the triangle removal lemma.

Finally, we stated and gave a mostly complete proof of Roth's theorem.

[8] Obviously, this highlighting does not (as it could in general) indicate that you might be asked to give a precise statement of the lemma – but a question that gives you the statement and asks you about the idea of it relating to random graphs could appear.