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February 27, 2004

Use strong induction to show that $\forall n \in \mathbf{N}$ $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ where $f_n$ is the n$^{th}$ term of the Fibonacci sequence.

Claim: $\forall n \in \mathbf{N}$, $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Base Claim: $f_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$ and $f_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$

Proof: $f_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = \frac{\sqrt{5}}{\sqrt{5}} = 1 = f_1$

$$f_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{(2)^2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+2\sqrt{5}+5)-(1-2\sqrt{5}+5)}{4}\right] = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4\sqrt{5}}{4}\right) = \frac{\sqrt{5}}{\sqrt{5}} = 1 = f_2$$

Inductive Claim: For every $n \in \mathbf{N}$, if $f_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$ for every k = {1, 2, …, n}, then $f_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

Proof: Assume that $f_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$ for all $k \in \{1, 2, \ldots, n\}$. We must show that $f_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$. Note that by the definition of the Fibonacci sequence, $f_{n+1} = f_n + f_{n-1}$, $f_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$

So, we must show that

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Note that $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1+\left(\frac{1+\sqrt{5}}{2}\right)^{-1}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1+\left(\frac{1-\sqrt{5}}{2}\right)^{-1}\right)\right]$$

Note that:

$$1+\left(\frac{1+\sqrt{5}}{2}\right)^{-1} = 1+\frac{2}{1+\sqrt{5}} = \frac{1+\sqrt{5}}{1+\sqrt{5}} + \frac{2}{1+\sqrt{5}} = \frac{3+\sqrt{5}}{1+\sqrt{5}}$$

$$= \left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right) = \frac{(1+\sqrt{5})(3+\sqrt{5})}{6+2\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})(3+\sqrt{5})}{2(3+\sqrt{5})} = \frac{1+\sqrt{5}}{2}$$

and

$$1+\left(\frac{1-\sqrt{5}}{2}\right)^{-1} = 1+\frac{2}{1-\sqrt{5}} = \frac{1-\sqrt{5}}{1-\sqrt{5}} + \frac{2}{1-\sqrt{5}} = \frac{3-\sqrt{5}}{1-\sqrt{5}}$$

$$= \left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{1-\sqrt{5}}\right) = \frac{(1-\sqrt{5})(3-\sqrt{5})}{6-2\sqrt{5}}$$

$$= \frac{(1-\sqrt{5})(3-\sqrt{5})}{2(3-\sqrt{5})} = \frac{1-\sqrt{5}}{2}$$

Thus,

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(1+\left(\frac{1+\sqrt{5}}{2}\right)^{-1}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(1+\left(\frac{1-\sqrt{5}}{2}\right)^{-1}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

$$= f_{n+1}$$