Hartshorne Exercises

Jaehwan Kim

Chapter 1

Varieties

Chapter 2

Schemes

2.1 Sheaves

Exercise 1. Let A an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \emptyset$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated with this presheaf.

Proof. Let A' be the constant presheaf associated with A. Define $A' \to \mathcal{A}$ by

$$\begin{aligned} A'(U) &\to \mathcal{A}(U) \\ a &\to a \end{aligned}$$

maps an element $a \in A'(U)$ to the constant function $a \in \mathcal{A}(U)$. This map is injective. For a continuous map $f : U \to A$, the map restricted to $f^{-1}(f(a))$ is a constant function. Therefore the map is surjective on each stalk. Thus this map induces an isomorphism of each stalk, and the sheaf associated with A' is $\mathcal{A}$ by the universal property. Check this. I'm not sure this is correct when X is not locally connected.

□

Exercise 2. (a) For any morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, show that for each point P, $(\ker \phi)_P = \ker(\phi_P)$ and $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$.

(b) Show that ϕ is injective(respectively, surjective) if and only if the induced map on the stalks ϕ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence $\cdots \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \cdots$ of sheaves and morphisms is exact if and only if for each $P \in X$, the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof. (a) Two groups are subgroups of $\mathcal{F}_P$. Since $\phi_P(s_P) = \phi(s)_P = 0$ for any section $s \in \mathcal{F}(U)$, $\ker(\phi_P) \subset (\ker \phi)_P$. For $s_P \in \ker(\phi_P)$, choose a representative $s \in \mathcal{F}(U)$. Then $\phi(s)_P = \phi(s_P) = 0$ implies, there exists V containing P such that $\phi(s)|_V = 0$. Thus $s|_V \in \ker(\phi(V))$, and $s_P = 0$ in $\ker(\mathcal{F})_P$, and the map is surjective.

For the image, two groups are subgroups of $\mathcal{G}_P$ and the result is more straightforward than the kernel side.

(b) We can use the result from (a) directly.

$$\begin{aligned} \ker(\phi) = 0 &\iff \ker(\phi)_P = 0 \text{ for all } P \\ &\iff \ker(\phi_P) = 0 \text{ for all } P \\ &\iff \phi_P \text{ is injective for all } P \end{aligned}$$

Also, if we think of a natural inclusion $\mathrm{im}(\phi) \hookrightarrow \mathcal{G}$ and induced map on stalks,

$$\begin{aligned} \mathrm{im}(\phi) = \mathcal{G} &\iff \mathrm{im}(\phi)_P = \mathcal{G}_P \text{ for all } P \\ &\iff \mathrm{im}(\phi_P) = \mathcal{G}_P \text{ for all } P \\ &\iff \phi_P \text{ is surjective for all } P \end{aligned}$$

(c) Subsheaves are isomorphic if and only if the stalks are isomorphic.

$$\begin{aligned} \mathrm{im}(\phi^{i-1}) = \ker(\phi^i) &\iff \mathrm{im}(\phi^{i-1})_P = \ker(\phi^i)_P \text{ for all } P \\ &\iff \mathrm{im}(\phi_P^{i-1}) = \ker(\phi_P^i) \text{ for all } P \\ &\iff \text{ the sequence of stalks is exact as a sequence of abelian groups.} \end{aligned}$$

□

Exercise 3. (a) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Show that ϕ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathcal{F}(U_i)$, such that $\phi(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, and an open set U such that the induced map $\mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective.

Proof. (a) ($\Rightarrow$) Let $U \subseteq X$ be an open set and s be a section of $\mathcal{G}$ on U. Since ϕ is surjective, for each $P \in U$, there exists an $t_P \in \mathcal{F}_P$ such that $\phi_P(t_P) = s_P$. Then the set of representatives (t_{U_P}, U_P) of t_P for each $P \in U$.

($\Leftarrow$) For any $P \in X$, $s_P \in \mathcal{G}_P$, take a representative $s \in \mathcal{F}(U)$. By the assumption, there exists an open set V containing P and $t_V \in \mathcal{F}(V)$ such that $\phi(t_V) = s_V$. Then $s_P = \phi(t_V)_P = \phi_P(t_V) = 0$. Thus ϕ_P is surjective for all P.

(b) Let $X = S^1$ and $\mathcal{F}$ be a sheaf of differentiable functions that has a derivative 1. Then there is a trivial map from $\mathcal{F}$ to the constant sheaf $\mathcal{A}$ of the trivial group. The induced map on stalks is trivially surjective. However, there is no global section of $\mathcal{F}$, thus $\mathcal{F}(S^1) \to \mathcal{A}(S^1)$ is not surjective.

□

Exercise 4. (a) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves such that $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for each U. Show that the induced map $\phi^+ : \mathcal{F}^+ \to \mathcal{G}^+$ of associated sheaves is injective.

(b) Use part (a) to show that if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\mathrm{im}\,\phi$ can be naturally identified with a subsheaf of $\mathcal{G}$, as mentioned in the text.

Proof. (a) By the universal property of a sheafification, we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d, "\eta_{\mathcal{F}}"] & \mathcal{G} \arrow[d, "\eta_{\mathcal{G}}"] \\ \mathcal{F}^+ \arrow[r, "\phi^+"] & \mathcal{G}^+ \end{tikzcd}$$

We have the following commutative diagram for each $P \in X$.

$$\begin{tikzcd} \mathcal{F}_P \arrow[r, "\phi_P"] \arrow[d, "\eta_{\mathcal{F}_P}"] & \mathcal{G}_P \arrow[d, "\eta_{\mathcal{G}_P}"] \\ \mathcal{F}_P^+ \arrow[r, "\phi_P^+"] & \mathcal{G}_P^+ \end{tikzcd}$$

Since $\eta_{\mathcal{F}_P}$ and $\eta_{\mathcal{G}_P}$ are isomorphisms, injectivity of ϕ_P implies that of ϕ_P^+.

(b) We have an injective map $\mathrm{im}\,\phi^{\mathrm{pre}} \to \mathcal{G}$. By the part (a), this induces a natural injective morphism of sheaves $\mathrm{im}\,\phi \to \mathcal{G}$. Thus we can identify $\mathrm{im}\,\phi$ with a subsheaf of $\mathcal{G}$.

□

Exercise 5. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof. ($\Rightarrow$) is straightforward. ($\Leftarrow$) We can think of it at the level of stalks with the induced morphism. Surjectivity and injectivity of the induced map on stalks tell the stalk map is an isomorphism. $\square$

Exercise 6. (a) Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map of $\mathcal{F}$ to the quetient sheaf $\mathcal{F}/\mathcal{F}'$ is surjective, and has kernel $\mathcal{F}'$. Thus there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

(b) Conversely, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and that $\mathcal{F}''$ is isomorphic to the quotient sheaf $\mathcal{F}$ by this subsheaf.

Proof. (a) In the level of presheaf, the given sequence is exact in all open $U \subseteq X$ by definition. Thus it induces an exact sequence of stalks as follows.

$$0 \to \mathcal{F}'_P \to \mathcal{F}_P \to \mathcal{F}_P/\mathcal{F}'_P \to 0$$

(b) If a morphism of sheaves is injective, a section map is also injective. Also, $\mathcal{F} \to \mathcal{F}''$ factors through the presheaf $\mathcal{F}/\mathcal{F}'$. By the universal property of sheafification, we have a morphism $\mathcal{F}/\mathcal{F}' \to \mathcal{F}''$. Since its stalk map commutes with $\mathcal{F}'_P \to \mathcal{F}_P \to$, the morphism is an isomorphism. Thus we have $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$. $\square$

Exercise 7. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

(a) Show that $\operatorname{im}\phi \cong \mathcal{F}/\ker\phi$.

(b) Show that $\operatorname{coker}\phi \cong \mathcal{G}/\operatorname{im}\phi$.

Proof. (a) Since image sheaf is a subsheaf of $\mathcal{G}$, ϕ factors through the image sheaf $\operatorname{im}\phi$. Thus we have an exact sequence

$$0 \to \ker\phi \to \mathcal{F} \to \operatorname{im}\phi \to 0$$

By the previous exercise, $\operatorname{im}\phi \cong \mathcal{F}/\ker\phi$.

(b) By the definition of a cokernel, we have a short exact sequence

$$0 \to \operatorname{im}\phi^{pre} \to \mathcal{G} \to \operatorname{coker}\phi^{pre} \to 0.$$

Using the universal property and sheafification map, we have a short exact sequence

$$0 \to \operatorname{im}\phi \to \mathcal{G} \to \operatorname{coker}\phi \to 0.$$

$\square$

Exercise 8. For any open subset $U \subseteq X$, show that the functor $\Gamma(U, -)$ from sheaves on X to abelian groups is a let exact functor. i.e. if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$ is an exact sequence of abelian groups. The functor $\Gamma(U, -)$ need not be exact: see (Ex. 1.21) below

Proof. We denote the maps in the exact sequence of sheaves by ϕ and ψ. Suppose $\psi(s) = 0$ for a section $s \in \Gamma(U, \mathcal{F})$. For all point $P \in X$, $\psi(s_P) = 0$ implies $s_P = 0$ by injectivity of ϕ. Thus $s = 0$ and $\phi(U)$ is injective. It is straightforward that $\psi(U) \circ \phi(U) = 0$ using a similar argument via stalks. If $\psi(t) = 0$ for some $t \in \Gamma(U, \mathcal{F})$, then there is a covering $\{U_i\}$ and corresponding sections $\{s_i\}$ of $\mathcal{F}$ such that $\phi(s_i) = t|_{U_i}$ for all i. Also, injectivity implies that the sections $\{s_i\}$ admit a global section s of $\mathcal{F}$ such that $\phi(s) = t$. Thus $\Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$ is exact. $\square$

Exercise 9. *Direct Sum.* Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the direct sum of $\mathcal{F}$ and $\mathcal{G}$, and is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of direct sum and direct product in the category of sheaves of abelian groups on X.

Proof. It is clear that the direct sum of sheaves is a sheaf. Let's check the universal property of direct sum. Consider a morphism of sheaves $f : \mathcal{F} \to \mathcal{H}$, $g : \mathcal{G} \to \mathcal{H}$.

$$\begin{tikzcd}
\mathcal{F} \arrow[dr, "\iota_{\mathcal{F}}"] \arrow[ddr, "f"'] & & \mathcal{G} \arrow[dl, "\iota_{\mathcal{G}}"'] \arrow[ddl, "g"] \\
& \mathcal{F} \oplus \mathcal{G} \arrow[d, dashed] & \\
& \mathcal{H} &
\end{tikzcd}$$

Let $h : \mathcal{F} \oplus \mathcal{G} \to \mathcal{H}$ be the morphism of sheaves induced by the universal property of direct sum. $h \circ \iota_{\mathcal{F}} = f$, $h \circ \iota_{\mathcal{G}} = g$ implies $h(s \oplus t) = f(s) + g(t)$. Thus h is uniquely determined. $\square$

Exercise 10. *Direct Limit.* Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the *direct limit* of the system $\{\mathcal{F}_i\}$, denoted $\varinjlim \mathcal{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, *i.e.*, that it has the following universal property: given a sheaf $\mathcal{G}$, and a collection of morphisms $\mathcal{F}_i \to \mathcal{G}$ compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathcal{F}_i \to \mathcal{G}$ suth chat for each i, the original map $\mathcal{F}_i \to \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \to \mathcal{G}$.

Proof. The restriction map defined through the universal property of direct limit makes $U \to \varinjlim \mathcal{F}_i(U)$ a presheaf. Let $\mathcal{F}$ be a sheaf associated with this presheaf. From the universal property of a direct limit, there is a unique map $\varinjlim \mathcal{F}_i(U) \to \mathcal{G}(U)$ and this map is compatible with the restriction map we defined. Thus there is a unique sheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$ commute with the sheafification map by the universal property. $\square$

Exercise 11. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case, show that the presheaf $U \to \varinjlim \mathcal{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i)$.

Proof. Let $\iota_i : \mathcal{F}_i \to \mathcal{F}$ be the direct limit presheaf of $\{\mathcal{F}_i\}$. Suppose $s_i \in \mathcal{F}(U_i)$ be the sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists a section $s_i' \in \mathcal{F}_{k(i)}(U_i)$ for some $k(i) \in I$ such that $s_i = \iota_{k(i)}(s_i')$ for all i. Since X is noetherian, there exists a covering $\{U_i \mid i \in I'\}$ of X with finite I'. We can choose sufficiently large k such that $k(i) \le k$ and any two sections s_i, s_j are equal if we restrict the domain to $U_i \cap U_j$. Since $\mathcal{F}(U_k)$ is a sheaf, we get the unique section $s' \in \mathcal{F}(U_k)$ such that $\iota_k(s_k)|_{U_i} = s_i$ for $i \in I'$. We define $s = \iota_k(s') \in \Gamma(U, \mathcal{F})$. For arbitrary $i \in I$, repeating the above argument with $I' \cup \{i\}$ gives the same section s. Thus $U \to \varinjlim \mathcal{F}_i(U)$ is a sheaf. $\square$

Exercise 12. *Inverse Limit.* Let $\{\mathcal{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \to \varprojlim \mathcal{F}_i(U)$ is already a sheaf. It is called the *inverse limit* of the system $\{\mathcal{F}_i\}$ and is denoted by $\varprojlim \mathcal{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Proof. Recall that the inverse limit is given by

$$\varprojlim \mathcal{F}_i(U) = \{(s_i) \in \prod \mathcal{F}_i(U) \mid \phi_{ij}(s_j) = s_i\}$$

Let $\{U^x\}$ be the open covering of U and $s^x \in \varprojlim \mathcal{F}_i(U^x)$ be the sections with gluing condition. The gluing condition implies the gluing condition for each $s_i^x \in \mathcal{F}_i(U^x)$. Thus there exists a unique section $s_i \in \mathcal{F}_i(U)$ such that $s_i|_{U^x} = s_i^x$. (s_i) is the unique section satisfying the gluing condition, thus $U \to \varprojlim \mathcal{F}_i(U)$ is a sheaf. $\square$

Exercise 13. *Espace Étalé of a Presheaf.* Given a presheaf $\mathcal{F}$ on X, we define a topological space $\text{Spé}(\mathcal{F})$, called the *espace étalé* of $\mathcal{F}$, as follows. As a set, $\text{Spé}(\mathcal{F}) = \bigcup_{P \in X} \mathcal{F}_P$. We define a projection map $\pi : \text{Spé}(\mathcal{F}) \to X$ by sending $s \in \mathcal{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathcal{F}(U)$, we obtain a map $\bar{s} : U \to \text{Spé}(\mathcal{F})$ by sending $P \mapsto s_P$ its germ at P. This map has the property that $\pi \circ \bar{s} = \text{id}_U$, in other words, it is a section of π over U. We now make $\text{Spé}(\mathcal{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \to \text{Spé}(\mathcal{F})$ are continuous for all U and all $s \in \mathcal{F}(U)$, are continuous. Now show that the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathcal{F}^+(U)$ is the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U. In particular, the original presheaf $\mathcal{F}$ was a sheaf if and only if for each U, $\mathcal{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathcal{F})$ over U.

Proof. It can be directly checked from the construction of a sheafification.

$$\mathcal{F}^+(U) = \{(s_P)_{P \in U} \mid s_P \in \mathcal{F}_P, s_Q = (s^P)_Q \text{ for all } Q \in U_P \text{ for some section } s^P \in \mathcal{F}(U_P)\}$$

Let $(s_P) \in \mathcal{F}(U)$ be a section. Then s^P is open in $\text{Spé}(\mathcal{F})$. Thus $\bar{s}$ is a continuous section. Also, a continuous section is a section of $\mathcal{F}^+$. □

Exercise 14. *Support.* Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section over an open set U. The *support* of s, denoted $\text{Supp}(s)$, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathcal{F}_P$. Show that $\text{Supp}x$ is a closed subset of U. We define the *support* of $\mathcal{F}$, $\text{Supp}\mathcal{F}$, to be $\{P \in X \mid \mathcal{F}_P \neq 0\}$. It need not be a closed subset.

Proof. Let $P \notin \text{Supp}(s)$. Then $s_P = 0$ and there exists a neighborhood V of P such that $s|_V = 0$. Also, $s_Q = 0$ for all $Q \in V$, thus $X \setminus \text{Supp}(s)$ is open. □

Exercise 15. *Sheaf $\mathscr{H}om$.* Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$, sheaf hom for short, and is denoted $\mathscr{H}om(\mathcal{F}, \mathcal{G})$.

Proof. It is clear that $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a presheaf. Suppose $\phi_i \in \text{Hom}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ be the sections such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ where $\{U_i\}$ is an open covering of U. Then for any open set $V \subseteq U$, we can glue the morphisms $\phi_i|_{V \cap U_i}$ to get $\phi_V \in \text{Hom}(\mathcal{F}(V), \mathcal{G}(V))$. These morphisms form a unique glued morphism, thus $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf. □

Exercise 16. *Flasque Sheaves.* A sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ is flasque, show that for every open set U, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is also exact.

(c) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.

(d) if $f : X \to Y$ is a continuous map, and if $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is a flasque sheaf on Y.

(e) Let $\mathcal{F}$ be any sheaf on X. We define a new sheaf $\mathcal{G}$, called the sheaf of *discontinuous sections* of $\mathcal{F}$ as follows. For each open set $U \subseteq X$, $\mathcal{G}(U)$ is the set of maps $s : U \to \cup_{P \in U} \mathcal{F}_P$ such that for each $P \in U$, $s(P) \in \mathcal{F}_P$. Show that $\mathcal{G}$ is a flasque sheaf and that there is a natural injective morphism of $\mathcal{F}$ to $\mathcal{G}$.

Proof. (a) Consider the following constant presheaf:

$$U \mapsto \begin{cases} A & \text{if } U \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

If X is irreducible, all nonempty open subsets have nonempty intersection. Thus the constant presheaf automatically satisfies the gluing axiom, and it is a sheaf. Then all the restriction maps are surjective and the sheaf is flasque.

(b) Global section functor is left exact in general. Let U be the given open set and ϕ, ψ be the map of sheaves. Suppose s'' is any section of $\mathcal{F}''(U)$. Since $\mathcal{F} \to \mathcal{F}''$ is surjection, there is a sections $s_i \in \mathcal{F}(U_i)$ where $\{U_i\}$ is an open covering of U such that $s''|_{U_i} = \phi(s_i)$. Let s_i and s_j be two sections of $\mathcal{F}(U_i \cap U_j)$. There exists a section $s'_{ij} \in \mathcal{F}'(U)$ such that $\phi(s'_{ij}) = s_i - s_j$ since $\psi(s_i - s_j) = 0$. The flasqueness of $\mathcal{F}'$ implies that there exists a section $s' \in \mathcal{F}'(U_i)$ such that $s'|_{U_i \cup U_j} = s'_{ij}$. Then we can glue two sections and corresponding open sets by replacing s_i with $s_i - \phi(s')$. Applying this argument with Zorn's lemma gives a global section s of $\mathcal{F}$ such that $\psi(s) = s''$. zorn's lemma argument is not correct.

(c) Let $U \subseteq V$ be two open sets of X. For any section $s''_U \in \mathcal{F}''(U)$, there exists $s_U \in \mathcal{F}(U)$ such that $\psi(s_U) = s''_U$ since $\mathcal{F}'$ is flasque. Also there exists $s''_V \in \mathcal{F}''(U)$ such that $\text{res}_{VU}(s_V) = s_U$ since $\mathcal{F}$ is flasque. $\text{res}_{VU}(\psi(s)) = s''_U$ implies $\mathcal{F}''$ is flasque.

(d) Results comes from the definition of a direct image.

(e) The sheaf of discontinuous sections is very fat. For a section $s_U \in \mathcal{F}(U)$, we can easily extend a section s_V of $\mathcal{F}(V)$ by

$$s_V(P) = \begin{cases} s_U(P) & \text{if } P \in U \\ 0 & \text{otherwise} \end{cases}.$$

Also, there is a natural injection $\mathcal{F} \hookrightarrow \mathcal{G}; s \mapsto (s_P)_P$.

□

Exercise 17. *Skyscraper Sheaves.* Let X be a topological space. Let P be a point, and A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U)$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf'.' Show that this sheaf could also be described as $i_*(A)$ where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i : \{P\}^- \to X$ is the inclusion.

Proof. The stalk at Q is described as a direct limit $i_P(A)_Q = \varinjlim i_P(U)$. If $Q \in \{P\}^-$, we have $i_P(U) = A$ for all open set U containing Q. Thus $i_P(A)_Q = A$. Otherwise, for any open set U, there exists $V \subseteq U$ such that $Q \notin V$. Thus we have $i_P(A)_Q = 0$.

$\{P\}^-$ is irreducible. If $U \cap \{P\}^- \neq \emptyset$ if and only if U contains P. Thus $i_*(A)(U) = 0$ if $P \notin U$. On the other hand, $i_*A(U) = A$ where $P \in U$ by Exercise 16 (a).

Of course, $i_*(A_P) = i_*(A)$ also holds. □

Exercise 18. *Adjoint Property of f^{-1}.* Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y,

$$\text{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \text{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

Hence we say that f^{-1} is a *left adjoint* of f_*, and that f_* is a *right adjoint* of f^{-1}.

Proof.

- Let U be an open set of X and V be an open set of Y containing $f(U)$. We have a natural restriction map $\phi_V : f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ also it commutes with the restriction map $\text{res}_{V'V}$ for any $V \subseteq V'$. Thus $(V, \phi_V)_{V \subseteq f(U)}$ forms a direct system. Then we have a natural presheaf morphism ${f^{-1}}^{\text{pre}} f_*\mathcal{F}(U) = \varinjlim f_*\mathcal{F}(V) \to \mathcal{F}(U)$. By the universal property of the sheafification, we have a natural sheaf morphism $\epsilon : f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

- Let V be an open set of Y and W be an open set of containing $f(f^{-1}(V))$. Since $f(f^{-1}(V)) \subseteq V$, we have a natural sheaf morphism $\eta : \mathcal{G}(V) \to f^{-1}\mathcal{G}(f^{-1}(V)) = f_*f^{-1}\mathcal{G}(V)$.

Similarly, ϵ and η are natural transforms. We have a natural isomorphism $f^{-1}f_*f^{-1}\mathcal{G} \cong f^{-1}\mathcal{G}$ since the stalks are isomorphic through ϵ and η.

$$\begin{aligned}(f^{-1}f_*f^{-1}\mathcal{G})_P &= (f_*f^{-1}\mathcal{G})_{f(P)} \\ &= \varinjlim_{f(P)\in V} f^{-1}\mathcal{G}(f^{-1}(V)) \\ &= \varinjlim_{f(P)\in V} \varinjlim_{f(f^{-1}(V))\subseteq W} \mathcal{G}(W) \\ &= \varinjlim_{P\in V} \mathcal{G}(V) = f^{-1}\mathcal{G}_P\end{aligned}$$

Also, we can show that $f_*f^{-1}f_*\mathcal{F} \cong f_*\mathcal{F}$ is a natural isomorphism through ϵ and η. Thus, we show that ϵ and η are the counit and the unit of the adjunction respectively. □

Exercise 19. *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X \setminus Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(a) Let $\mathcal{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathcal{F})_P$ of the direct image sheaf on X is $\mathcal{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathcal{F}$ the sheaf obtained by extending $\mathcal{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathcal{F}$ instead of $i_*\mathcal{F}$, and say "consider $\mathcal{F}$ as a sheaf on X," when we mean "consider $i_*\mathcal{F}$."

(b) Now let $\mathcal{F}$ be a sheaf on U. Let $j_!\mathcal{F}$ be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!\mathcal{F})_P$ is equal to $\mathcal{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathcal{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathcal{F}$. We call $j_*\mathcal{F}$ the sheaf obtained by *extending $\mathcal{F}$ by zero outside U.*

(c) Now let $\mathcal{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

Proof. (a) $(i_*\mathcal{F})_P = 0$ if $P \notin Z$. Otherwise, if we think of the direct limit, we have

$$\begin{aligned}(i_*\mathcal{F})_P &= \varinjlim_{P\in U} i_*\mathcal{F}(U) = \varinjlim_{P\in U} \mathcal{F}(U \cap Z) \\ &= \varinjlim_{P\in V} \mathcal{F}(V) = \mathcal{F}_P.\end{aligned}$$

(b) The former results directly follow from the definition of $j_!(\mathcal{F})$. Let $\mathcal{G}$ and $\mathcal{H}$ be two sheaves on X such that $\mathcal{G}|_U = \mathcal{H}|_U = \mathcal{F}$ and $\mathcal{G}_P = \mathcal{H}_P = 0$ for $P \notin U$. Let $\phi : \mathcal{G}|_U \xrightarrow{\cong} \mathcal{F} \xrightarrow{\cong} \mathcal{H}|_U$ be the isomorphism. Define ϕ_P as the stalk map induced from ϕ. Let V be an open set and $(s_P)_{P\in V} \in V$ be a section of $\mathcal{G}$ and P be any point of V. If $P \in U$, there exists an open subset W of V containing P and $(\phi_Q(s_Q))_{Q\in W}$ is a section of $\mathcal{H}$ on W. If $P \notin U$, there is an open subset W of V containing P such that $(s)_{Q\in W} = 0$ since $\mathcal{F}_P = 0$. This implies $(\phi_Q(s_Q))_{Q\in W}$ is a zero section of $\mathcal{H}$ on W. Thus ϕ_P is indeed an isomorphism of sheaves $\mathcal{G}$ and $\mathcal{H}$.

Although $j_!\mathcal{F}|_U = j_*\mathcal{F}|_U = \mathcal{F}|_U$ holds, $j_!\mathcal{F}$ and $j_*\mathcal{F}$ are different sheaves. Let $\mathcal{F}$ be a sheaf of smooth sections of U where $j : U = \mathbb{C} \setminus \{0\} \to \mathbb{C}$. Then the global section of a direct image is simply $j_*\mathcal{F}(\mathbb{C}) = \mathcal{F}(U)$. However, $j_!\mathcal{F}(\mathbb{C})$ is a section of $\mathcal{F}$ vanishing at a neighborhood of 0. The distinction comes from the stalk at the boundary of U.

(c) The induced diagram on each stalk is exact.

□

Exercise 20. *Subsheaf with Supports.* Let Z be a closed subset of X, and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ consisting of all sections whose support (Ex 14) is contained in Z.

(a) Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathcal{F}$ with supports in Z, and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$.

(b) Let $U = X \setminus Z$, and let $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X.

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_* j^{-1}\mathcal{F}$$

Furthermore, if $\mathcal{F}$ is flasque, the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective.

Proof. (a) A support of glued sections is a union of supports of each section. Thus the given presheaf is a subsheaf of $\mathcal{F}$.

(b) Let's calculate a stalk of a sheaf $jP_* j^{-1}\mathcal{F}$ at a point $P \in X$.

$$\begin{aligned}(j_* j^{-1}\mathcal{F})_P &= \varinjlim_{P \in V} j^{-1}\mathcal{F}(j^{-1}(V)) \\ &= \varinjlim_{P \in V} j^{-1}\mathcal{F}(V \setminus Z) \\ &= \varinjlim_{P \in V} \mathcal{F}(V \setminus Z)\end{aligned}$$

The kernel of $\mathcal{F} \to j_* j^{-1}\mathcal{F}$ is a sheaf of sections vanishing on $X \setminus Z$ which is $\mathcal{H}^0_Z(\mathcal{F})$. If $\mathcal{F}$ is flasque, $\mathcal{F}(V \setminus Z) \to \mathcal{F}(V)$ is surjective for any open set V. Thus the stalk map is surjective.

□

Exercise 21. *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. 1. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.

(a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanishes at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathcal{I}_Y$ *of* Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

(b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

(c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathcal{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{F} \to 0.$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{F})$ is not surjective. This shows that the global section functor $\Gamma(X, -)$ is not exact(cf. (Ex 1.8) which shows that it is left exact).

(d) Again let $X = \mathbb{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathcal{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathcal{K}$. Show that the quotient sheaf $\mathcal{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf(Ex 17) given by I_P at the point P.

(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables.)

Proof. (a) Since vanishes at all points of $Y \cap U$ is a local property, thus the given presheaf is a subsheaf of $\mathcal{O}_X$. Also, it is an $\mathcal{O}_X$-module.

(b) For a subvariety Y, regular function on $\mathcal{O}_Y$ is a restriction of a regular function on X to Y.

$$\begin{aligned}i_*(\mathcal{O}_Y)(U) &= \mathcal{O}_Y(U \cap Y) \\ &= \{g|_Y \mid \text{regular } g \text{ on } U\} \\ &= \{g : \text{regular}\}/\{g : \text{vanishes on } Y\} \\ &= \mathcal{O}_X(U)/\mathcal{I}_Y(U)\end{aligned}$$

Thus $i_*(\mathcal{O}_Y) \cong \mathcal{O}_X/\mathcal{I}_Y$.

(c) $\Gamma(X, \mathcal{O}_X) = k$ and $\Gamma(X, \mathcal{F}) = k \oplus k$. The map $f \mapsto (f(P), f(Q))$ is not surjective.

(d) $\mathcal{O}(U) \to K = K(U); f \mapsto (f, U)$ is injection since any open set is dense in X. The following holds.

$$I_P = K/\mathcal{O}_P \cong \mathcal{F}(X \setminus P)/k = k[(x-P)^{-1}]/k$$

Any element of K is uniquely represented as a finite sum of $k[(x-P)^{-1}]/k$ for some points $P \in X$ and a constant $c \in k$. Also, $\mathcal{F}(U) = \bigoplus_{P \notin U} I_P \oplus k$ and $\mathcal{K} = \bigoplus_P I_P \oplus k$. Thus $\mathcal{K}(U)/\mathcal{O}(U) \cong \bigoplus_{P \in X} I_P = (\sum_{P \in X} i_P(I_P))(U)$. Check please.

(e) Clear from (d).

$\square$

Exercise 22. *Gluing Sheaves.* Let X be a topological space, let $\mathcal{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j and isomorphism $\phi_{ij} : F_i|_{U_i \cap U_j} \xrightarrow{\cong} \mathcal{F}_j|_{U_i \cap U_j}$, such that (1) for each i, $\phi_{ii} = \mathrm{id}$, and (2) for each i, j, k, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\cong} \mathcal{F}_i$ such that for each i, j, $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by *gluing* the sheaves $\mathcal{F}_i$ via the isomorphisms ϕ_{ij}.

Proof. The following defines a presheaf $\mathcal{G}$.

$$\begin{aligned} \mathcal{G}(U) &= \prod_{U \in U_i} \mathcal{F}_i(U)/\{s_i \sim \phi_{ij}(s_i)\} \\ &\cong \mathcal{F}_i(U) \text{ for } U \in U_i \end{aligned}$$

The sheaf $\mathcal{G}$ associated to $\mathcal{G}$ is the desired sheaf with the canonical isomorphisms ψ_i. $\square$