The Riemann-Hilbert Correspondence (Part 2)

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Abstract

Consider M as a Riemann surface. The Riemann-Hilbert correspondence establishes an equivalence between the category of flat connections on meromorphic bundles over M with regular singularities and the category of locally constant sheaves of vector spaces on M.

In the previous talk, we introduced the local solution functor. Initially, we shall explore a trivial meromorphic bundle over a disc D to comprehensively understand the notion of a regular singularity. After that, we will briefly review the essential surjectivity of a functor and prove the full-faithfulness. As a corollary of the R-H correspondence, we will explore an algebraization of a Deligne lattice and an algebraization of the meromorphic bundle on $\mathbb{P}^1$ with some conditions.

In this article, M is a smooth complex manifold of dimension $n = 1$, and a local coordinate of M is always z.

1 Riemann-Hilbert Correspondence

Before beginning, we will explore the necessity of the regular singularity condition through the following example.

Example. (Locally constant sheaves on $\mathbb{C} \setminus \{0\}$ with rank 1.) For a given topological space X, there is a correspondence between locally constant sheaves on X and finite dimensional $\mathbb{C}$-representations of $\pi_1(X)$. In particular, locally constant sheaves on $\mathbb{C} \setminus \{0\}$ with rank 1 are classified by a monodromy matrix $T \in \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$.

$$\begin{aligned} \pi_1(\mathbb{C} \setminus \{0\}) \cong \langle\gamma\rangle &\to \mathrm{Aut}(\mathbb{C}) \cong \mathbb{C}^\times \\ \gamma &\mapsto T_\gamma = T \end{aligned}$$

Then, we can construct a locally constant sheaf from a given monodromy analytically.

Let ∇ be a meromorphic connection on a rank-1 trivial bundle E over $\mathbb{C}$ with a pole along $\{0\}$ and Ω be the connection matrix, i.e. $\nabla e_i = \Omega^i_j \otimes e_j$ for a basis $\{e_i\}$ of the trivial bundle where Ω is a meromorphic 1-form on $\mathbb{C}$. Then, the solution sheaf E^∇ is a locally constant sheaf on $\mathbb{C} \setminus \{0\}$ with rank 1 by the Cauchy-Kowalevski theorem. Explicitly, for each section $s \in \Gamma(U, E^\nabla)$ on a simply connected open set $U \subset \mathbb{C} \setminus \{0\}$, s is a holomorphic section of E on U and satisfies $\nabla s = 0$.

$$\begin{aligned} 0 = \nabla s = \nabla s^1 e_1 &= \frac{\partial s^1}{\partial z} dz \otimes e_1 + s^1 \Omega^1_1 dz \otimes e_1 \\ \implies \frac{\partial s^1}{\partial z} &= -s^1 \Omega^1_1 \end{aligned}$$

For this example, we will ignore the index 1 for simplicity. The local solution is given by $s(z) = c \exp(-\int_{z_0}^z \Omega)$ for $z_0, z \in U$ where U is the simply connected region, and automatically E^∇ becomes a locally constant sheaf. Using this, monodromy of E^∇ is given by $\exp(-\int_\gamma \Omega)$ for $\gamma \in \pi_1(U)$. And it is natural to think of the form $\Omega = R/z$ where $R = -\log(T)/2\pi i \in \mathbb{C}$ with the monodromy T.

Note. What kind of bundles and connections are needed for the equivalence of categories?
(a) If we restrict Ω to be holomorphic, E^∇ becomes a constant sheaf since $\int_\gamma \Omega$ vanishes. From this, we notice that at least a simple pole of a connection ∇ is necessary to construct a non-trivial locally constant sheaf.

(b) Trivial bundle with connection of the form $\Omega' = -\frac{3}{z} dz$ has trivial monodromy. We want this bundle and

a trivial bundle with a trivial connection to be isomorphic. Then the basis change map $P(z)$ should takes the meromorphic entries $P(z) = \dfrac{1}{z^3}$ after solving ode $\Omega' = P^{-1}\Omega P + PdP$. Thus, it is natural to think of the category of meromorphic bundles with connection rather than a holomorphic bundle with meromorphic connection
(c) Let's consider a connection matrix $\Omega' = R'/z^2$ with the trivial solution sheaf $E^{\nabla'}$. Like (b), we also wish two bundles (E, d) and (E, ∇') to be isomorphic. But in this case, the transition matrix should be $P(z) = e^{-\frac{1}{z}}$, which has an irregular singularity.

In sum, we need to consider the category of meromorphic bundles with connection ∇ having regular singularities along its poles.

Now, we can state the main theorem through the local solutions functor.

Definition 1.1. (Local solutions functor) Let $\mathscr{M}$ be a meromorphic bundle with connection ∇ having regular singularities along its poles. Then the local solutions functor *Sol* is defined as follows:

$$Sol : (\mathscr{M}, \nabla) \mapsto \mathscr{M}^{\nabla}$$

Where $\mathscr{M}^{\nabla}$ is a locally constant sheaf on $M \setminus \Sigma$. And for a morphism $f : (\mathscr{M}, \nabla) \to (\mathscr{N}, \nabla)$, $Sol(f)$ is a restriction of f to $\mathscr{M}^{\nabla}$.

Theorem 1.2. (Riemann-Hilbert correspondence) The local solutions functor *Sol* is an equivalence from the category of meromorphic bundles with connection ∇ having regular singularities along its poles and the category of locally constant sheaves on $M \setminus \Sigma$.

We will first prove the essential surjectivity part.

Theorem 1.3. (Logarithmic correspondence) Given a locally constant sheaf $\mathscr{F}$ on $M \setminus \Sigma$, there exists a holomorphic bundle E on M and a (flat) logarithmic connection with poles along Σ such that $\mathscr{F} \cong E^{\nabla}$.

Proof. Let $\mathscr{E}_{M\setminus\Sigma} = \mathscr{O}_{M\setminus\Sigma} \otimes_{\mathbb{C}} \mathscr{F}$ be a holomorphic bundle with connection $\nabla(f \otimes s) = df \otimes s$. Then $(\mathscr{E}_{M\setminus\Sigma})^{\nabla} = \mathscr{F}$, so we only need to fill the poles Σ. Take a small neighborhoods U_m for all $m \in \Sigma$ such that $U_m \cap U_n = \emptyset$ for $m \neq n$. If we identify U to a disc, we have locally free sheaf $\mathscr{F}|_{D^*}$ on D^*. Let T_m be a monodromy matrix of $\mathscr{F}|^*_{U_m}$. Similar to the previous example, we can construct a trivial bundle $\mathscr{E}_{U_m}$ over D^* with meromorphic connection $\nabla = \dfrac{R}{z}dz$, where $e^{-2\pi i R} = T_m$. Two bundles $\mathscr{E}|_{U_m}$ and $\mathscr{F}|^*_{U_m}$ are isomorphic since $\mathscr{F}|_{U_m\setminus\{m\}} \cong \mathscr{E}^{\nabla}{}_{U_m\setminus\{m\}}$, and the fact that the connections are built from these local solutions, Then, we can construct a global holomorphic bundle $\mathscr{E}$ on M by gluing $\mathscr{E}_{U_m}$ and $\mathscr{E}_{M\setminus\Sigma}$. $\square$

The immediate result of this theorem shows the essential surjectivity of the local solutions functor.

Corollary 1.4. (Meromorphic correspondence) Given a locally constant sheaf $\mathscr{F}$ on $M \setminus \Sigma$, there exists a meromorphic bundle with connection $(\mathscr{M}, \nabla)$ on M with poles along Σ having regular singularity.

Proof. Let $\mathscr{E}$ be the holomorphic bundle constructed from the previous theorem. Then $\mathscr{M} = \mathscr{E} \otimes \mathscr{O}(*\Sigma)$ is a desired meromorphic bundle. $\square$

The remaining part is the full-faithfulness. The following lemma is useful.

Lemma. Let $(\mathscr{E}, \nabla)$ and $(\mathscr{E}', \nabla')$ be holomorphic bundles with (holomorphic) connection. Then,
(a) $\mathscr{H}om_{\mathscr{O}_M}(\mathscr{E}, \mathscr{E}')|^{\nabla''}_{M\setminus\Sigma} \cong \mathscr{H}om_{\mathbb{C}}(\mathscr{E}^{\nabla}, \mathscr{E}'^{\nabla'})$ where ∇'' is a connection of a hom sheaf.
(b) If ∇ and ∇' are meromorphic connection having logarithmic poles along Σ, then ∇'' also has a logarithmic poles along Σ.

Proof. (a) There is a natural $\mathbb{C}$-linear map $Sol : \mathscr{H}om_{\mathscr{O}_M}(\mathscr{E}, \mathscr{E}')|^{\nabla''}_{M\setminus\Sigma} \to \mathscr{H}om_{\mathbb{C}}(\mathscr{E}^{\nabla}, \mathscr{E}'^{\nabla'})$. *Sol* is injective since a morphism between bundles is determined by where its basis maps to. Also, if $\phi : \mathscr{E}|^{\nabla}_U \to \mathscr{E}'|^{\nabla'}_U$ for and open set $U \in M \setminus \Sigma$ be a morphism of locally constant sheaves, induced morphism $\phi : \mathscr{E}|_U \cong \mathscr{E}|^{\nabla}_U \otimes \mathscr{O}|_U \to$

$\mathscr{E}'|_U^{\nabla} \otimes \mathscr{O}|_U \cong \mathscr{E}'|_U$, $s \otimes f \mapsto \phi^{\nabla''}(s) \otimes f$ is a section of $\mathscr{H}om_{\mathscr{O}_M}(\mathscr{E}, \mathscr{E}')|_{M\setminus\Sigma}^{\nabla''}$. Thus, two hom sheaves are isomorphic.
(b) From $\nabla''(\phi)(s) = \nabla'(\phi(s)) - \phi(\nabla(s))$, the connection matrix $\Omega''(A) = \Omega' A - A\Omega$ has at most a simple pole. □

Let us show that *Sol* is fully faithful. Let $\phi^{\nabla''} : \mathscr{M}|_{M\setminus\Sigma}^{\nabla} \to \mathscr{M}'|_{M\setminus\Sigma}^{\nabla}$ be a morphism. It is a matter of showing that there exists a unique morphism $\phi : \mathscr{M} \to \mathscr{M}'$ which induces it. Let us regard $\phi^{\nabla''}$ as a horizontal section $\mathscr{H}om_{\mathscr{O}_M}(\mathscr{M}, \mathscr{M}')^{\nabla''}$ on $M \setminus \Sigma$. By the above lemma, $\phi|_{M\setminus\Sigma}$ is a section of a sheaf $\mathscr{H}om_{\mathscr{O}_M}(\mathscr{M}, \mathscr{M}')$. It suffices to show that a horizontal section $s|_{M\setminus\Sigma}$ of a meromorphic bundle $\mathscr{M}$ with poles along Σ having regular singularity has the unique extension to M.

We can restrict the problem to the neighborhood $U_m \cong D$ of a pole $m \in \Sigma$. Let $(\mathscr{O}_D(*0)^d, \nabla)$ be a trivialized restricted bundle with a logarithmic pole at 0. We will show that the horizontal section s of a punctured disc D^* can be extended to D, i.e. s has a moderate growth with the following steps.

Step 1: Find a meromorphic base change $P(z) \in \mathrm{GL}_d(\mathscr{O}_U(*0))$ on some open set $U \subseteq D$ containing 0 s.t. transformed matrix has the form $\dfrac{B_0}{z}dz$ for some constant matrix B_0.

Step 2: Transform $\dfrac{B_0}{z}dz$ to $\dfrac{B_0'}{z}dz$ where B_0' is a Jordan normal form of B_0.

Step 3: Find the condition of Jordan normal form for the existence of a global horizontal section in D^* and show that the horizontal section can be uniquely extended to D.

In step 1, we will take a converging formal series $P(z)$ with a radius of convergence possibly smaller than 1. Thus, the domain could be shrunk to a smaller disc U. Instead of dealing with radius-varying discs, we will consider the germ of the meromorphic bundle in Step 1 & Step 2.

If we denote $\mathbb{C}\{z\}$ be the ring of converging formal series, the stalk $\mathscr{O}_{M,m}(*\Sigma)$ at a pole $m \in \Sigma$ is the field $k := \mathbb{C}\{z\}[1/z]$. A stalk of a meromorphic bundle $\mathscr{M}$ at m is a k-vector space with a meromorphic connection ∇. If we choose a basis, a connection matrix is given by a matrix $A(z) \in \mathrm{Mat}_d(k)$. Concepts in meromorphic bundles with connection can be defined on the (k, ∇)-vector spaces in the same way.

Definition 1.5. Let $\mathcal{M}$ be a (k, ∇)-vector space of rank d.
(a) $\mathbb{C}\{z\}$-free submodule $\mathcal{E}$ of $\mathcal{M}$ with rank d is called a lattice.
(b) If there exists a lattice $\mathcal{E}$ s.t. the connection matrix has a logarithmic pole, we say that $(\mathcal{M}, \nabla)$ has a regular singularity.

Considering the germs of a bundle has some advantages:

Note. Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a (k, ∇)-homomorphism.
(a) Then, $\mathrm{im}(\phi)$ is also a (k, ∇')-vector space. (This is not true for the case of meromorphic bundles with connection.)
(b) If $(\mathcal{E}, \nabla)$ is a lattice of $(\mathcal{M}, \nabla)$, then $\phi(\mathcal{E})$ is a lattice of $(\mathcal{M}', \nabla')$.

Before proceeding with Step 1, we will quickly prove the straightforward Steps 2 and 3. For a connection matrix $\Omega = \dfrac{B_0}{z}dz$, there is a basis change matrix $P(z) \in \mathrm{GL}_d(\mathbb{C})$ s.t. $B_0' = P^{-1}B_0P$ where B_0' is a Jordan normal form of B_0. The connection matrix in the basis $\epsilon = eP$ is written as $\Omega' = P^{-1}\Omega P + P^{-1}dP = \dfrac{B_0'}{z}dz$. Thus, we can assume that the connection matrix has a Jordan normal form.

Definition 1.6. (Elementary regular model) We define an elementary regular model as a (k, ∇)-vector space equipped with a connection matrix:

$$\Omega(z) = (\alpha \mathrm{Id} + N)\frac{dz}{z}$$

where $\alpha \in \mathbb{C}$ and N is a single Jordan block. We denote it by $\mathcal{N}_{\alpha,d}$.

We get Step 3 from the following lemma.

Lemma. Let a matrix N be a single Jordan block of size $d \geq 1$.
(a) On any simply connected open set U of D^*, the horizontal section s is obtained by the linear combination of the columns of the matrix

$$z^{-\alpha \mathrm{Id}+N} = z^{-\alpha}\left(\mathrm{Id} - N\log z + N^2\frac{(\log z)^2}{2!} + \cdots + (-1)^{d-1}N^d\frac{(\log z)^d}{(d)!}\right)$$

(b) The horizontal sections have moderate growth near the origin. Therefore, if s is the section of D^*, s has the unique extension to D.
(c) It has the nonzero global section if and only if $\alpha \in \mathbb{Z}$.

The next theorem follows Step 1.

Theorem 1.7. Let $(\mathcal{M}, \nabla)$ be a (k, ∇)-vector space with a connection matrix $\Omega(z) = \dfrac{A(z)}{z}dz$, $A(z) \in GL_d(k)$. Then there exists some base change $P(z) \in \mathrm{GL}_d(k)$ such that, after the base change of matrix P, the matrix Ω' takes the form

$$\Omega'(z) = \frac{B_0}{z}dz$$

where $B_0 \in \mathrm{Mat}_d(\mathbb{C})$ is constant.

We will first show that there exists a formal series $P(z)$ satisfies the condition, then show that $P(z)$ converges. Let us denote $\mathbb{C}[[z]]$ as a formal power series ring and $\widehat{k}$ as a formal meromorphic series.

Proposition 1.8. Let $\Omega(z) = \dfrac{\widehat{A}(z)}{z}dz$ be a connection matrix with $\widehat{A}(z) \in \mathrm{Mat}_d(\mathbb{C}[[z]])$. We moreover assume that any two eigenvalues of the matrix $\widehat{A}(0)$ do not differ by a nonzero integer. Then there exists a matrix $\widehat{P} \in \mathrm{GL}_d(C[[z]])$ such that

$$\widehat{P}^{-1}\frac{\widehat{A}(z)}{z}\widehat{P} + \widehat{P}^{-1}\frac{d\widehat{P}}{dz} = \frac{\widehat{A}_0}{z}$$

Proof.

$$z\frac{d\widehat{P}}{dz} = \widehat{P}A_0 - \widehat{A}\widehat{P}$$

where $\widehat{A} = A_0 + zA_1 + z^2A_2 + \cdots$. If we put the series $\widehat{P} = \mathrm{Id} + zP_1 + z^2P_2 + \cdots$, we get the following recursive equation:

$$lP_l = P_lA_0 - A_0P_l + \Phi(A_0, \cdots, A_{l-1}, P_1, \cdots, P_{l-1})$$

where Φ is a polynomial of $A_0, \cdots, A_{l-1}, P_1, \cdots, P_{l-1}$. using the following lemma, P_l is uniquely determined. □

Lemma. Let $\phi(P) = XP - PY$ be a linear map for some square matrices X, Y. Then, X, Y has no common eigenvalues $\iff$ ϕ is invertible. □

The following proposition is believable.

Proposition 1.9. The matrix of the base change $P(z)$ obtained in Proposition 1.8 converges if $\widehat{A}(z) \in \mathrm{Mat}_d(\mathbb{C}\{z\})$.

After the following proposition, we can transform the eigenvalues of $\widehat{A}(0)$ such that they do not differ by a nonzero integer.

Proposition 1.10. Let $\Omega(z) = \dfrac{\widehat{A}(z)}{z}dz$ be a connection matrix with $\widehat{A}(z) \in \mathrm{Mat}_d(\mathbb{C}[[z]])$. Then, there exists a transition matrix $Q \in \mathrm{GL}_d(\mathbb{C}[z, z^{-1}])$ such that the transformed connection matrix has eigenvalues that do not differ by a nonzero integer.

Proof. Using the base change with constant matrix, reduce $\widehat{A}(0)$ to the Jordan normal form. Then the first group of eigenvalues are added by 1 after changing the basis through $R = \begin{pmatrix} z\,\mathrm{Id} & \\ & \mathrm{Id} \end{pmatrix}$. We get a matrix satisfying the condition by recursively applying this process. $\square$

The above propositions obtain the proof of the Theorem 1.7.

Proof. **(Theorem 1.7)** Let $\Omega(z) = \dfrac{A(z)}{z}dz$ be a connection matrix with $A(z) \in \mathrm{Mat}_d(\mathbb{C}\{z\})$. Then, there exists a transition matrix $Q \in \mathrm{GL}_d(\mathbb{C}[z, z^{-1}])$ such that the transformed connection matrix has eigenvalues that do not differ by a nonzero integer. Let $P(z) \in \mathrm{GL}_d(\mathbb{C}\{z\})$ be a base change matrix obtained in Proposition 1.9. Then, the transformed connection matrix $\Omega'(z) = \dfrac{B_0}{z}dz$ has a logarithmic pole. $\square$

It completes the proof of the Riemann-Hilbert correspondence.

Note. In Proposition 1.8, we assume that the eigenvalues of A(0) do not differ by a nonzero integer. Otherwise, we cannot expect the monodromy of the solution sheaf to be conjugate to $\exp(-2i\pi A(0))$.

2 Algebraizations

We will introduce some of the results of the Riemann-Hilbert correspondence.

2.1 Deligne lattices

Corollary 2.1. (Deligne lattices). Let $(\mathcal{M}, \nabla)$ be a (k, ∇)-vector space with regular singularitiy.

(a) Let $\sigma\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto z - [\mathrm{Re}(z)]$ be a map. There is a unique (up to isomorphism) logarithmic lattice $\mathcal{V}$ of $(\mathcal{M}, \nabla)$ such that the eigenvalues of $\mathrm{Res}_{\mathcal{V}}\nabla$ are contained in $\mathrm{Im}\sigma$. Moreover, for any homomorphism $\phi : (\mathcal{M}, \nabla) \to (\mathcal{M}', \nabla')$,

$$\phi(\mathcal{V}) = \mathcal{V}(\mathrm{Im}(\phi), \nabla') = \mathcal{V}(\mathcal{M}', \nabla') \cap \mathrm{Im}(\phi).$$

(b) Conversely, let T be an automorphism of $\mathbb{C}^d$. There exists then a unique (up to isomorphism) bundle with meromorphic connection on D with logarithmic pole at 0 for which the local system it defines on D^* is that associated to T, and such that the real part of each eigenvalue of a residue of its connection is contained in $[0, 1)$.

(c) The functor from (k, ∇)-vector space $(\mathcal{M}, \nabla)$ with regular singularity to the $\mathbb{C}$-vector space $H = \mathcal{V}/z\mathcal{V}$ with the automorphism $T = \exp(-2i\pi \mathrm{Res}_{\mathcal{V}}\nabla)$ is an equivalence of categories. $\square$

Remark 2.2. (Algebraization of Deligne lattices). For any $\lambda \in \mathbb{C}$, let us set

$$\mathbb{M}^\lambda = \{e \in \mathcal{M}\colon \exists n, (z\nabla_{\partial/\partial z} - \lambda\,\mathrm{Id})^n e = 0\}.$$

If we consider the basis of the elementary model $\mathcal{N}_{\lambda',d}$, the solution is a constant vector $\mathbb{M}^\lambda$ is a finite $\mathbb{C}$-vector space since,

$$0 = (z\nabla_{\partial/\partial z} - \lambda\,\mathrm{Id})^n e = (z\frac{\partial}{dz} + N + (\lambda' - \lambda)\,\mathrm{Id})^n e$$

has a solutions iff $\lambda' - \lambda \in \mathbb{Z}$, i.e. $\exp(-2i\pi\lambda)$ is an eigenvalue of T. Also, multiplication by z induces an isomorphism from $\mathbb{M}^\lambda \to \mathbb{M}^{\lambda+1}$. Therefore, $\mathbb{M} := \bigoplus_\lambda \mathbb{M}^\lambda$ is a free $\mathbb{C}[z, z^{-1}]$-module contained in $\mathcal{M}$, stable under the action of ∇ with the natural isomorphism

$$k \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{M} \to \mathcal{M}$$

Then, $\mathbb{V} := \bigoplus_{\lambda \in \mathrm{Im}\sigma} \bigoplus_{k\in\mathbb{N}} \mathbb{M}^{\lambda+k}$ is the Deligne lattice of $\mathbb{M}$ and we have the isomorphism

$$\mathbb{C}\{z\} \otimes_{\mathbb{C}[z]} \mathbb{V} \to \mathcal{V}$$

Lastly, the natural mapping

$$\bigoplus_{\lambda \in \mathrm{Im}\sigma} \mathbb{M}^\lambda \to \mathcal{V}/z\mathcal{V}$$

is an isomorphism compatible with the action of T, if we define it on the left-hand side by the $\exp(-2i\pi z \nabla_{\partial/\partial z})$. Thus $\mathbb{M}$ can be identified to the graded $(\mathbb{C}[z, z^{-1}], \nabla)$-module

$$\mathrm{gr}_{\mathcal{V}}\mathcal{M} := \bigoplus_{k\in\mathbb{Z}} (z^k\mathcal{V}/z^{k+1}\mathcal{V}).$$

2.2 Partial Riemann-Hilbert correspondence

In this subsection, we will algebraize $(\mathcal{M}, \nabla)$ on $\mathbb{P}^1$. The following useful equivalence immediately follows from the Riemann-Hilbert correspondence.

Theorem 2.3. (Partial Riemann-Hilbert correspondence). There is an equivalence of category of meromorphic bundles with connection on M with poles along $\Sigma \cup \Sigma'$ having regular singularities on Σ to the category of meromorphic bundles on U having poles at Σ' where $\Sigma' \subseteq U \subseteq M \setminus \Sigma$ is connected, and the inclusion map induces an isomorphism of fundamental groups.

Let U be some small disc centered at the origin Σ'.

Corollary 2.4. Let $(\mathcal{M}, \nabla)$ be a k-vector space with connection. There exists a unique (up to isomorphism) meromorphic bundle $(\mathscr{M}, \nabla)$ with connection on $\mathbb{P}^1$, having singularity at 0, and regular singularity at ∞, and with germ at 0 isomorphic to $(\mathcal{E}, \nabla)$.

We can express this corollary more concretely.

Corollary 2.5. Given $A(z) \in \mathrm{GL}_d(k)$, there exists a holomorphic base change $P \in \mathrm{GL}_d(\mathbb{C}\{z\})$ such that $B(z) = P^{-1}AP + P^{-1}\dfrac{dP}{dz}$ has entries in the ring $\mathbb{C}[z, z^{-1}]$, and such that the connection with matrix $B(z)dz$ has a regular singularity at ∞.

Proof. Let $\mathcal{E}$ be the lattice $\mathbb{C}\{z\}^d$ with meromorphic connection $A(z)$. Let $\mathscr{M}$ be the meromorphic bundle given by Corollary. We can then construct a meromorphic subbundle $\mathscr{E}(*\infty)$ of $\mathscr{M}$ by gluing $\mathscr{M}|_{\mathbb{P}^1\setminus\{0\}}$ and a lattice $\mathscr{E}|_D$ on a small disc D defined by $\mathcal{E}$. This meromorphic bundle has a pole at ∞ only. It is known that this bundle is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(*\infty)^d$ (if we allow the meromorphicity of connections at 0). If we take a global basis of $\mathscr{E}(*\infty)$ using this isomorphism, the connection matrix has entries in the ring $\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(*\{0, \infty\})) = \mathbb{C}[z, z^{-1}]$. ☐

References

[1] Dragan Milicic. "Lectures on differential equations in complex domains". In: (1989).

[2] Claude Sabbah. *Isomonodromic Deformations and Frobenius Manifolds*. Springer London, 2008. ISBN: 978-1-84800-054-4.