Holonomic D-modules and Riemann-Hilbert correspondence

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Abstract

In this article, we introduce the Riemann-Hilbert correspondence between regular holonomic D-modules and perverse sheaves. The classical Riemann-Hilbert correspondence addresses flat connections of $\mathcal{O}$ vector bundles on a Riemann surface. D-modules represents a generalization of flat connections on vector bundles to an algebraic framework. However, if we regard only vector bundles in D-modules, they do not exhibit desirable behavior in a functorial way. For instance, the pushforward of a vector bundle is not a vector bundle in general. To address this, we introduce the category of holonomic D-modules with Kashiwara's constructibility theorem after dealing with some basic properties of D-modules. Subsequently, we will define the de Rham functor (solution functor), which is a functor from the derived category of holonomic D-modules to the category of constructible sheaves on X. We will then state the Riemann-Hilbert correspondence between regular holonomic D-modules(complex concentrated in degree 0) and perverse sheaves via this functor.

1 D-modules

1.1 Linear PDE systems

Example 1. Let $X = \mathbb{C}$ and $Pu = 0$ be the differential equation for some differential operator P. We can represent the solution set of the differential equation algebraically. Let D_X be the ring of differential operators and $\mathcal{M} = D_X/D_X P$ be the left D_X-module. Then the solution sheaf of $Pu = 0$ is given by

$$\begin{aligned} U \mapsto &\{f \in \mathcal{M}(U) \mid Pf = 0\} \\ &= \{\phi \in Hom_{D_X}(D_X, \mathcal{O}_X) \mid \phi(P) = 0\} \\ &= \mathcal{H}om_{D_X}(D_X/D_X P, \mathcal{O}_X) \end{aligned}$$

We called the derived functor $\mathrm{Sol}(-) := R\mathcal{H}om_{D_X}(-, \mathcal{O}_X)[\dim X]$ the "solution functor" with degree shifting. As an analogy to the Cauchy-Kowalevski theorem, we want the solution functor to map to the local systems that contain information on the monodromies of the differential systems.

Let's pick a specific operator $P = 1 + \partial$. For an analytic setting $\mathcal{O}_X^{an}$, the solution sheaf becomes a constant sheaf $\mathbb{C}_X[1]$ with a generator e^{-x}. However, in an algebraic setting, it has a trivial kernel which is not desirable. Thus we will do some analytification process for Sol functor later. Throughout this article, the coherent D-modules and their solutions are our main interests.

Simply, a notion of D-modules generalizes linear PDE systems in a purely algebraic sense.

Definition 1.1. Let X be a smooth variety over k. If we regard a sheaf of derivation $\mathcal{D}er_X(\mathcal{O}_X)$, and $\mathcal{O}_X$ as a subsheaf of $\mathcal{E}nd_k(O_X)$, D_X is defined to be a subsheaf of k-algebra generated by $\mathcal{O}_X$

and $\mathcal{D}er_X(\mathcal{O}_X)$. Quasi-coherent D_X-modules and coherent D_X-modules are similarly defined as $\mathcal{O}_X$ case.

For example, if we set $X = \mathbb{A}^n$, D_X is a Weyl algebra over k. We denote denote the category of left D_X-modules by $\mathrm{DMod}(X)$, and right D_X-modules by $\mathrm{DMod}^r(X)$.

Example 2. A locally free sheaf $\mathcal{F}$ with a flat left(right) covariant derivative is a coherent left(right) D-module.

Example 3. $\mathcal{O}_X$-coherent D_X-modules are coherent $\mathcal{D}_X$-modules but not conversely. For example, $\mathcal{D}_X$ is a coherent D_X-module but not a coherent $\mathcal{O}_X$-module.

Example 4. Let $X = \mathbb{A}^1$, and $k = k[x]/(x)$ be a $\mathcal{O}_X$-coherent module. If we try to give a D_X-module structure on M like $\partial \cdot c = 0$ for $c \in k$, the Leibniz rule does not hold since $(\partial x - x\partial) \cdot 1 = 0$. In fact, we cannot give any D_X-module structure on k.

This example shows that a D_X-module structure is somewhat more restrictive than we think. We will demonstrate later that $\mathcal{O}_X$-coherent D_X-module should be a locally free sheaf with a flat connection. Although it is quite interesting, if we regard only locally free sheaves with flat connection, it does not nicely behave in a functorial way. For example, a pushforward of locally free sheaf is not necessarily locally free. Indeed, we should consider a bigger category, which has a nice funtoriality, the derived category of holonomic D_X-modules. To do so, we need to introduce some properties of D-modules.

1.2 Filtrations on D-modules

One of our goals is an understanding of coherent D_X-modules. But as we have seen in Example 3, coherent D_X-module is not $\mathcal{O}_X$ coherent in general. There is a nice way to paraphrase D_X coherency in terms of $\mathcal{O}_X$ coherency via good filtration.

It is natural to think of a filtration over D_X so that $F^l D_X$ has differential operators of order at most l. This filtration can be given canonically if we set $F^0 D_X = \mathcal{O}_X$ and define F^l inductively by

$$F^l D_X = \{D \in D_X \mid [D, f] \in F^{l-1} D_X \text{ for all } f \in \mathcal{O}_X\}. \tag{1}$$

It is called a *geometric filtration* of D_X. Since $[F^l D_X, F^m D_X] \in F^{l+m-1} D_X$, multiplication in a graded module $\mathrm{gr}^F D_X = \bigoplus_{l \geq 0} F^l D_X / F^{l-1} D_X$ is commutative. Thus we have $\mathrm{gr}^F D_X \cong \mathrm{Sym}_{\mathcal{O}_X} \mathcal{T}_X \cong \mathcal{O}_{\mathbb{T}^*_X}$. Moreover, every quotient module $F^l D_X / F^{l-1} D_X$ is locally isomorphic to $\mathcal{O}_X < \partial^\alpha \mid |\alpha| = l >$ which is a free $\mathcal{O}_X$-module of finite rank. For some filtration on a D_X-modules $\mathcal{M}$ compatible with $F^l D_X$, a graded module $\mathrm{gr}^F \mathcal{M} = \bigoplus_{l \geq 0} F^l \mathcal{M} / F^{l-1} \mathcal{M}$ became a module over $\mathrm{gr}^F D_X = \mathcal{O}_{\mathbb{T}^*_X}$.

We call a filtration is *good* if $\mathrm{gr}^F \mathcal{M}$ is $\mathcal{O}_X$ coherent in each degree. For a coherent D_X module, there is an easy way to construct a good filtration. Let $\mathcal{M}$ be a coherent D_X-module and let $m_1, \ldots, m_r$ be finite local generators of $\mathcal{M}$. Then we can construct a good filtration as $F^l \mathcal{M} = \sum\limits_i (F^{l+k_i} D_X) m_i$ for arbitrary integers k_i. Also, it is not hard to show that all good filtrations can be constructed in the above way. As we can expect from the construction, although good filtration is not unique, for two good filtrations F, G, there exists $k_0, k_1 \in \mathbb{Z}$ such that $F^{l+k_0} \mathcal{M} \subset G^l \mathcal{M} \subset F^{l+k_1} \mathcal{M}$. This implies that two good filtrations are not so different. We give the following proposition without proof.

Proposition 1.2. Let $\mathrm{SS}^F(\mathcal{M}) := \mathrm{supp}(\mathrm{gr}^F\mathcal{M}) \subset \mathbb{T}^*_X$ be a *singular support* of a coherent D_X-module. A singular support does not depend on the choice of good filtration. So, we can denote it by $\mathrm{SS}(\mathcal{M})$ without confusion.

Recall that the support of a graded module of a polynomial ring is a cone. As an analogy of this, the singular support is a conical closed subscheme of $\mathbb{T}^*_X$. Thus image of the singular support in X is equal to the usual support of X.

Example 5. Let $X = \mathbb{A}^1$, and $U = \mathbb{A}^1 \setminus \{0\}$. Also, we let ξ denote the coordinate of dx.

(a) The simplest example is D_X itself. Since $\mathrm{gr}^F D = \mathcal{O}_{\mathbb{T}_X}$, $\mathrm{SS}(D_X) = \mathbb{T}_X$.

(b) Consider $\mathcal{O}_X$ with a natural left D_X-module structure: $\mathcal{O}_X \cong D_X/D_X\partial$. There is a trivial good filtration $F^l\mathcal{O}_X = \mathcal{O}_X$. $\mathrm{gr}^F\mathcal{O}_X = \mathcal{O}_X = \mathbb{T}_X/(\xi)$ concentrated in degree 0, and $\mathrm{SS}(\mathcal{O}_X) = X$. We could think O_X is generated by 1 which is a solution of a differential equation $\partial f = 0$. On the other hand, there is another filtration:

$$F^l\mathcal{O}_X = \begin{cases} (x) & l = 0, \\ \mathcal{O}_X & \text{otherwise} \end{cases}$$

Then $\mathrm{gr}^F\mathcal{O}_X = (x) \oplus O_X/(x) \cong (x) \oplus k$ is not isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_{\mathbb{T}^*_X}$-module, while $\mathrm{SS}(\mathcal{O}_X) = X$ since $\xi^2\mathrm{gr}^F\mathcal{M} = 0$.

(c) Now, suppose $\mathcal{M} = D_X/D_Xx$. For a good filtration $F^l\mathcal{M} = k < \partial^\alpha \mid |\alpha| \leq l >$, we get graded module $\mathrm{gr}^F\mathcal{M} = k[\xi] = \mathcal{O}_{\mathbb{T}_X}/(x)$. This singular support is given by $x = 0$. One can think of a generator $1 \in \mathcal{M}$ as a Dirac delta function.

(d) Let $\mathcal{M} = j_*\mathcal{O}_U = k[x, x^{-1}] = D_X/D_Xx\partial$ be a left D_X-module. A good filtration $F^l = \frac{1}{x^l}\mathcal{O}_X$ gives a graded module

$$\mathrm{gr}^F\mathcal{M} = k[x] \oplus k\frac{1}{x} \oplus k\frac{1}{x^2} \oplus \cdots \cong k[x] \oplus k[\xi].$$

The singular support is given by $x = 0$ and $\xi = 0$. Also one can think of the generator $1/x$ as a solution of a differential equation $(\partial x)f = 0$.

Remark 1.3. Suppose that the singular support is a subset of $X = \mathbb{A}^1$. Then for arbitrary $m \in \mathcal{M}$, $m\xi^n$ vanishes for sufficiently large n. For some good filtration on $\mathcal{M}$, $F^l\mathcal{M} = F^{l+1}\mathcal{M} = \cdots$ for sufficiently large l. It means that ξ^n annihilates $\mathrm{gr}\mathcal{M}$ for some n. Thus $\mathrm{gr}\mathcal{M}$ is a coherent O_X-module.

1.3 Kashiwara's theorem

To motivate defining functors for D-modules, we first introduce the statement of Kashiwara's theorem that distinguishes D_X-modules from $\mathcal{O}_X$-modules. We define all functors for the derived category of D_X-modules $D\mathrm{DMod}(X)$.

Theorem 1.4. (Kashiwara) Let $Z \subset X$ be a smooth closed subvariety of X. The pushforward functor:

$$H^0 i_\star : \mathrm{DMod}(Z) \to \mathrm{DMod}_Z(X)$$

is an equivalence of categories.

Remark 1.5. In the category of $\mathcal{O}_X$, pushforward functor i_* is not an equivalence of categories. Let $i : \{0\} \to \mathbb{A}^1$ be a smooth closed immersion then we have $i_* k = k[x]/(x)$. Since $k[x]/(x^2)$ is supported at 0 and is not a direct sum of $k[x]/(x)$, i_* is not essentially surjective.

Now, we define functors for the D_X-modules(Those functors can be naturally extended to the derived category of D_X-modules).

Definition 1.6. Let $f : X \to Y$. We call D_X-$f^{-1}D_Y$-bimodule $\underset{X\to Y}{D} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y$ on X a *transfer bimodule.*

Since $\mathcal{O}_X$ is not a D_X-O_X bimodule, we need to give a left D_X-module structure as follows:

$$\theta \cdot (x \otimes D) = \theta(x) \otimes D + x \otimes \tilde{\theta} D$$

for $\theta \in D_X$, $x \in \mathcal{O}_X$, $D \in f^{-1}D_Y$, and $\tilde{\theta}$ comes from a natural map $T_X \to f^{-1}T_Y$.

Definition 1.7. Let $f : X \to Y$. We define a functor $f^\dagger$ by

$$\begin{aligned} f^\dagger : D^-\mathrm{DMod}(Y) &\to D^-\mathrm{DMod}(X) \\ \mathcal{M} &\mapsto \underset{X\to Y}{D} \otimes^L_{f^{-1}D_Y} f^{-1}\mathcal{M}. \end{aligned}$$

Since $\underset{X\to Y}{D} \otimes^L_{f^{-1}D_Y} f^{-1}\mathcal{M} = \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$, $f^\dagger$ is the usual O_X-pullback functor with a left D_X-module structure.

Similarly, a pushforward functor is defined for a right D_X-module.

Definition 1.8. Let $f : X \to Y$. We define a functor $f_\star$ by

$$\begin{aligned} f_\star : \mathrm{DMod}^r(X) &\to \mathrm{DMod}^r(Y) \\ \mathcal{M} &\mapsto Rf_*(\mathcal{M} \otimes^L_{D_X} \underset{X\to Y}{D}). \end{aligned}$$

Remark 1.9. A tensor product is right exact, and f_* is left exact. Therefore, we cannot employ homological techniques directly in the classical setting. This is one reason to consider the derived category of D_X-modules.

Example 6. Let $X = \mathbb{A}^1$, and $i : \{0\} \to \mathbb{A}^1$ be a smooth closed immersion. Let's do the same calculation of Remark 1.5 in a classical setting.

(a) We have $H^0 i_\star k = i_*(k \otimes_{O_{X,0}} D_{X,0}) = D_X/xD_X = k[\partial]$ similar to previous remark. However, D_X/x^2D_X splits into $xk[\partial] \oplus (x\partial - 1)k[\partial] \cong (D_X/xD_X)^2$. From this example, we can guess that $i_\star$ could be an equivalence of categories.

A natural side change functor is desirable to deal with the left D_X-module, To do so, we introduce a natural right D_X-module structure of the dualizing sheaf ω_X. Lie derivative $\mathcal{L}_X$ gives a hint for a right D_X-module structure on ω_X.

$$\mathcal{L}_\theta(\omega) = \iota_\theta d\omega + d\iota_\theta\omega$$

For $f \in \mathcal{O}_X$, we have $\mathcal{L}_\theta(f) = \iota_\theta df = \theta(f)$. However, for a top form $\omega \in \omega_X$, $\mathcal{L}_\theta(\omega) = d\iota_\theta\omega$ implies that $\mathcal{L}_{f\theta}(\omega) = d(\iota_{f\theta}\omega) = \mathcal{L}_\theta(f\omega)$. This signifies that L_θ is a left covariant derivative on a sheaf of a 0-form, and $-\mathcal{L}_\theta$ is a right covariant derivative on a sheaf of a top form. Thus, we have a induced right D_X-module structure on ω_X form Example 2.

Definition 1.10. We define a side change functor $\Omega : \mathrm{DMod}(X) \to \mathrm{DMod}^r(X)$ by

$$\Omega(\mathcal{M}) := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

with a right D_X-module structure as

$$(\omega \otimes m) \cdot f = \omega \otimes (fm) = \omega f \otimes m$$
$$(\omega \otimes m) \cdot \theta = \frac{1}{2}(\omega \otimes (\theta m) - \omega\theta \otimes m.)$$

Similarly, we define a inverse side change functor $\Omega^{-1} : \mathrm{DMod}^r(X) \to \mathrm{DMod}(X)$ by

$$\Omega^{-1}(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

with a left D_X-module structure as

$$f \cdot (m \otimes \eta) = (mf) \otimes \eta = m \otimes f\eta$$
$$\theta \cdot (m \otimes \eta) = \frac{1}{2}((m\theta) \otimes \eta - m \otimes \theta\eta.)$$

Now, we can define a $f^{-1}D_Y$-D_X transfer bimodule using the dualizing sheaf.

$$\underset{Y \leftarrow X}{D} := \omega_X \otimes_{O_X} \underset{X \to Y}{D} \otimes_{f^{-1}O_Y} f^{-1}\omega_Y^{-1}$$

Finally, we get a functor $f_\star$ for a left D-modules.

$$\begin{aligned} f_\star : \mathrm{DMod}(X) &\to \mathrm{DMod}(Y) \\ \mathcal{M} &\mapsto f_*(\underset{Y \leftarrow X}{D} \otimes_{D_X} \mathcal{M}). \end{aligned}$$

We give a proof sketch of Kashiwara's theorem.

Proof. Suppose the smooth closed subvariety Z has codimension 1. Choose a local chart $\mathcal{O}_Z = \mathcal{O}_X/(x)$ and assume that X is affine space. Consider the differential operator $D = \partial x$. Let $\mathcal{M}$ be a D_X-module. For any $m \in \mathcal{M}$, there is a minimal k such that $x^k m = 0$. Then, $x^{k-1}m$ has D-eigenvalue 0 and $\partial^{k-1}x^{k-1}$ has eigenvalue $-k+1$. Also, $m' = m - \partial^{k-1}x^{k-1}m$ satisfies $x^{k-1}m' = 0$. Thus m can be decomposed into the sum of eigenvectors of D. Then we can use the same techniques for solving a simple harmonic oscillator in physics. x acts as a raising operator, and ∂ acts as a lowering operator. Thus there are only non-positive D-eigenvalues, and x, ∂ is an isomorphism of adjacent eigenspaces. Therefore, $\mathcal{M} = \mathcal{M}_0 \otimes_k k[\partial]$ where $\mathcal{M}_0$ is a D-module with eigenvalue 0.

$$\begin{aligned} \underset{Z\to X}{D} &= i^{-1}(\mathcal{O}_X/(x)) \otimes_{i^{-1}\mathcal{O}_X} i^{-1}D_X \\ &= i^{-1}(\mathcal{O}_X/(x) \otimes_{\mathcal{O}_X} D_X) \\ &= i^{-1}(D_X/xD_X) \end{aligned}$$

side changing is as follows:

$$\underset{Z\leftarrow X}{D} = i^{-1}(D_X/D_X x)$$

Thus,

$$\begin{aligned} i_\star\mathcal{M}_0 &= i_*(i^{-1}(D_X/xD_X) \otimes_{D_X} \mathcal{M}_0) \\ &= D_X/D_X x \otimes_{D_X} \mathcal{M}_0 \\ &= \mathcal{M}_0 \otimes_k k[\partial] \cong \mathcal{M}. \end{aligned}$$

Therefore, we have equivalence $\mathcal{M}_0 = \ker(x) = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, \mathcal{M})$, and $\mathcal{M} = i_\star\mathcal{M}_0$. We can use induction on codimension for arbitrary subvariety Z. □

By Kashiwara, if support of a D_X-module is a proper subvariety of X, the tensor term $k[\partial]$ makes it non-coherent. Furthermore, we have the following proposition.

Proposition 1.11. A D_X-module is $\mathcal{O}_X$-coherent if and only if it is a $\mathcal{O}_X$-locally free sheaf with a flat connection.

2 Holonomic D-modules

Our functors behave nicely for singular supports of D_X-modules. We give the following proposition without proof.

Proposition 2.1. Let $i : Z \to X$ be a closed embedding and $\mathcal{M}$ be a coherent D_Z-module. $\mathrm{SS}(i_\star\mathcal{M})$ is given by the inverse image and then the direct image of $\mathrm{SS}(\mathcal{M})$:

$$\mathbb{T}^*_Z \leftarrow \mathbb{T}^*_X \times_X Z \to \mathbb{T}^*_X.$$

Proposition 2.2. Let $f : X \to Y$ be a smooth morphism and $\mathcal{M}$ be a coherent D_Y-module. $\mathrm{SS}(f^\dagger\mathcal{M})$ is given by the inverse image and then the direct image of $\mathrm{SS}(\mathcal{M})$:

$$\mathbb{T}^*_Y \leftarrow \mathbb{T}^*_Y \times_Y X \to \mathbb{T}^*_X.$$

Then we have the inequality for the dimension of singular support from Kashiwara.

Theorem 2.3. Let $\mathcal{M} \in \mathrm{DMod}(X)$. Then

$$\dim \mathrm{SS}(\mathcal{M}) \geq \dim X.$$

Recall each dimension of singular support in Example 5 is greater than or equal to $\dim X = 1$. Also, note that the pushforward of a locally free sheaf has singular support of dimension $\dim X$. In (d), the irreducible component $x = 0$, $\xi = 0$, of singular support gives information about where the locally free sheaves come from. Now we define a holonomic D-module and characteristic cycle which has more finer information than singular support.

Definition 2.4. A D-module is *holonomic* if $\dim \mathrm{SS}(\mathcal{M}) = \dim X$. Let $I = \{C_i\}$ be the set of irreducible components of $\mathrm{SS}(\mathcal{M})$ and ξ_i be a generic point of C_i. Then, m_i is defined to be the multiplicity of $\mathcal{M}$ at ξ_i. The characteristic cycle of a holonomic D-module $\mathcal{M}$ is defined to be the formal sum

$$\mathrm{CC}(\mathcal{M}) := \sum m_i [C_i]$$

For the short exact sequence $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ of a coherent D-modules, if we give an induced filtration to $\mathcal{M}'$, $\mathcal{M}''$, we have a short exact sequence of graded modules $\mathrm{gr}^F \mathcal{M}' \to \mathrm{gr}^F \mathcal{M} \to \mathrm{gr}^F \mathcal{M}''$. The characteristic cycle is additive if we regard only the top dimension of characteristic cycles. Since irreducible components of a holonomic D-module have dimension $\dim X$, the following proposition holds.

Proposition 2.5. The category of holonomic D-modules $\mathrm{DMod}_h(X)$ is abelian, and closed under extensions. Furthermore, every holonomic D-module is generically O_X-locally free with a flat connection.

Proof. Clear from the above discussion and Remark 1.3. □

2.1 Six functors for holonomic D-modules

To state and sketch a proof of constructibility theorem, we introduce six functors, a duality functor.

Definition 2.6. Let $f : X \to Y$ be a smooth morphism. We define

$$f^! := Lf^\dagger[\dim(X) - \dim(Y)]$$
$$f_\star := Rf_*(\underset{Y \leftarrow X}{D} \otimes^L_{D_X} -)$$

Example 7. Let $i : \{0\} \to \mathbb{A}^1$ and $\mathcal{M} \in \mathrm{DMod}(X)$.

$$i^\dagger \mathcal{M} = \underset{\{0\}\to X}{D} \otimes^L_{\mathcal{D}_{X,0}} \mathcal{M}_0$$

Since $\underset{\{0\}\to X}{D} = k \otimes_{k[x]_0} D_{X,0} = D_{X,0}/D_{X,0}x$, We have a flasque representation of $\underset{\{0\}\to X}{D}$:

$$0 \to D_{X,0} \xrightarrow{x} D_{X,0} \to 0$$

Thus we have $i^\dagger \mathcal{M} = \mathcal{M} \xrightarrow{x} \mathcal{M}$ in degrees $[-1, 0]$. On the other hand, $R\mathrm{Hom}(i_\star k, \mathcal{M}) = R\mathrm{Hom}(D/Dx, \mathcal{M}) = \mathcal{M} \xrightarrow{x} \mathcal{M}$ in degrees $[0, 1]$. For adjunction, we need to shift $i^\dagger$ by $-1 = \dim(\{0\}) - \dim(X)$.

Example 8. Let $p : X \to \{\text{pt}\}$. We can compute $\underset{\text{pt}\leftarrow X}{D} = \omega_X \otimes_{\mathcal{O}_X} \underset{X\to\text{pt}}{D} \otimes_k (p^{-1}\omega_Y^{-1}) = \omega_X$ and

$$f_\star \mathcal{M} = R\Gamma(\omega_X \otimes^L_{D_X} \mathcal{M})$$

Also we have a locally free Spencer resolution $\Omega^\bullet_X \otimes_{O_X} D_X \to \omega_X$ where differential in local coordinates:

$$\omega \otimes P \mapsto d\omega \otimes P + \sum_i dx_i \wedge \omega \otimes \partial_i P$$

So the pushforward of $\mathcal{O}_X$ is given by

$$\begin{aligned} p_\star \mathcal{O}_X &= R\Gamma(\Omega^\bullet)[\dim X] \\ &= H^\bullet(\mathbb{C}_X)[\dim X] \end{aligned}$$

For this reason, some author denotes $f_\star$ by $Rd^{\text{dR}}_\star$.

Definition 2.7. The duality functor $\mathbb{D}_X : D^-(\text{DMod}(X)) \to D^+(\text{DMod}(X))^{op}$ is given by

$$\mathbb{D}_X(\mathcal{M}) := \Omega^{-1}(R\mathcal{H}om_{D_X}(\mathcal{M}, D_X))[\dim(X)].$$

The following proposition tells the degree shift in the duality functor is necessary.

Proposition 2.8. There is a natural isomorphism $\mathbb{D}^2 \cong \text{id}_{D_c(\text{DMod}(X))}$. Furthermore, $\mathbb{D}$ is an autoequivalence of $\text{DMod}_h(X)$.

Definition 2.9. We define the functors $f^\star$, $f_!$ for the derived category of quasi-coherent D_X-modules as follows

$$\begin{aligned} f^\star &:= \mathbb{D}_X \circ f^! \circ \mathbb{D}_Y \\ f_! &:= \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X. \end{aligned}$$

Remark 2.10. Note that $\mathbb{D}$ is equivalence in the derived category of *coherent* D_X-modules. However $f^!$, $f_\star$ does **not preserve** the coherency. For example, in the setting of Example 5, $j_\star D_U$ is not a coherent D_X module. Thus we cannot expect some nice properties in the derived category of coherent D_X-modules.

However, unexpectedely, $f^!$ and $f_\star$ **preserve** the holonomicity. We present the following fundamental adjunctions without proof.

Proposition 2.11. For the derived category of holonomic D-modules, the functors $(f^\star, f_\star)$, $(f_!, f^!)$ are adjoint pairs.

2.2 Kashiwara's constructibility theorem

Now we can prove the following Kashiwara's constructibility theorem.

Theorem 2.12. Let $\mathcal{M} \in D_c\text{DMod}(X)$ be a coherent complex. The following are equivalent.

(a) $\mathcal{M}$ is holonomic.

(b) There is a stratification $i_\alpha : S_\alpha \hookrightarrow X$ such that $i_\alpha^! M$ is a locally free sheaf with a flat connection.

(c) For every closed point $i_x : \{x\} \hookrightarrow X$, we have $i_x^!(\mathcal{M})$ is finite dimensional.

Proof. For (a) $\Rightarrow$ (b), choose an open subset U of X such that $\mathcal{M}|_U$ is locally free by Proposition 2.5. Add U to the stratification. Let $j : Z = X \setminus U \hookrightarrow X$ be a closed embedding. Then, $j^!\mathcal{M}$ is a holonomic D_Z-module and induction on the dimension of Z gives proof. (b) $\Rightarrow$ (c) uses the commutativity of the pullback functor. We omit the proof of (c) $\Rightarrow$ (a). $\square$

Remark 2.13. In fact, there is a problem with this proof because a proof of the preservation of holonomicity using constructibility.

3 Riemann-Hilbert correspondence

3.1 de Rham functor

Definition 3.1. Let $\mathcal{M} \in \mathrm{DMod}(X)$. We define the analytic de Rham functor:

$$\begin{aligned} D_h^b\mathrm{DMod}(X) &\to D_c^b\mathrm{Sh}_{\mathbb{C}}(X^{an}) \\ \mathrm{dR}_X^{an}(\mathcal{M}) &:= (\omega_X \otimes_{D_X}^L \mathcal{M})^{an} = \Omega_X^{\bullet,an} \otimes_{\mathcal{O}_{X^{an}}} a^{-1}\mathcal{M}[\dim X]. \end{aligned}$$

where $a : X^{an} \to X$ is the de Rham functor.

If we think a D_X-flat resolution of $\mathcal{O}_X$ similar to the Spencer resolution, $\mathrm{dR}_X(\mathcal{M}) = R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathcal{M})[n]$ holds. Also by the Verdier duality,

$$R\mathcal{H}om_{D_X}(\mathcal{O}_X, \mathcal{M})[n] = R\mathcal{H}om(\mathbb{D}_X\mathcal{M}, \mathbb{D}_X\mathcal{O}_X)[n] = \mathrm{Sol}_X(\mathbb{D}_X, \mathcal{M}).$$

Therefore there is an identification

$$\mathrm{dR}(\mathcal{M}) = \mathrm{Sol}_X(\mathbb{D}\mathcal{M})$$

Remark 3.2. Note that $\mathrm{dR}_X(\mathcal{O}_X) \cong \mathbb{C}_X[\dim X]$. This is why the above functor is called the de Rham functor.

Kashiwara proved the following theorem in his PhD thesis. By Kashiwara's constructibility theorem and the above remark, we get the following theorem.

Theorem 3.3. Let $\mathcal{M}$ be a holonomic D_X-module. Then,

(a) The cohomologies of $\mathrm{dR}^{an}(\mathcal{M})$ are constructible.

(b) The de Rham functor is an essentially surjective t-exact functor between $D_h^b\mathrm{DMod}(X)$ and $D_c^b(X^{an})$.

We may take a definition of perverse t-structure on $D_c^b(X^{an})$ through the above functor.

Example 9. Recall the definition of perverse sheaves:

$$\mathcal{F} \in \mathrm{Perv}(X) \iff \begin{cases} \dim \mathrm{supp} H^i \mathcal{F} \leq -i \text{ for all } i \\ \dim \mathrm{supp} H^i \mathbb{D}\mathcal{F} \leq -i \text{ for all } i \end{cases}$$

Consider the subvariety $i : Z \hookrightarrow X$ of dimension m. Take a local coordinate of X such that $Z = \{x_j = 0 \,|\, j \geq m+1\}$. Then locally, $i_\star \mathcal{O}_Z = D_X/D_X(\partial_1, \cdots, \partial_m, x_{m+1}, \cdots, x_n) = D_X/D_X I$ (remind Remark 1.5) Apply the de Rham functor, we get

$$\mathrm{dR}^{an}(i_\star \mathcal{O}_Z) = \mathbb{C}_Y[\dim Y - \dim X][\dim X] = \mathbb{C}_Y[\dim Y].$$

Also if we believe the first part and use the fact that the Verdier dual commutes with the pushforward functor, the second part is also true.

Example 10. There is the only trivial local system on $\mathbb{A}^1$. However, there is nontrivial connection $\mathcal{O}_X \cong D_X/D_X(\partial - \lambda)$ on $\mathbb{A}^1$. Thus the functor is not fully faithful.

As we see in the above example, the de Rham functor is not fully faithful without regularity(In this case, it is not regular at ∞) condition.

Definition 3.4. (a) Let C be a curve. Then a flat connection $\mathcal{M}$ on C is *regular* if its analytification is regular in a compact manifold C^{an} with poles.

(b) Let X be a smooth variety along D. Then a flat connection $\mathcal{M}$ on X is *regular* if $i^*\mathcal{M}$ is regular for every smooth curve $i : C \hookrightarrow X$ if $i^{-1}Z$ is a divisor of C.

(c) $\mathcal{M} \in \mathrm{DMod}_h(X)$ is called regular if $i^!_{S_\alpha}\mathcal{M}$ is regular for some stratification of X.

Indeed, we get the main theorem of this article.

Theorem 3.5. (Riemann-Hilbert correspondence) The de Rham functor is an equivalence of categories between $D\mathrm{DMod}^{rs}_h(X)$ and $D^b_c(X)$. Also, the de Rham functor is an equivalence of categories between $\mathrm{DMod}^{rs}_h(X)$ and $\mathrm{Perv}(X)$.

$$\begin{tikzcd} D\mathrm{DMod}^{rs}_h(X) \arrow[rr, leftrightarrow, "\text{Riemann-Hilbert}"] & & D^b_c(X) \\ \mathrm{DMod}^{rs}_h(X) \arrow[u, hook, "\text{degree } 0"] \arrow[rr, leftrightarrow, "\cong"] & & \mathrm{Perv}(X) \arrow[u, hook, "\text{perverse heart}"'] \end{tikzcd}$$

Before calculations, we introduce the following theorem.

Theorem 3.6. *(Goresky-MacPherson extension.)* (a) Let $Y \subseteq X$ be a locally closed smooth connected subvariety of X such that the inclusion $i : Y \hookrightarrow X$ is affine. Let $\mathcal{M}$ be a simple holonomic D_X-module. Then the unique simple submodule of $i_\star\mathcal{M}$ is the image $i_{!\star}$ of the canonical morphism $i_!\mathcal{M} \to i_\star\mathcal{M}$.
(b) Any simple holonomic D_X-module is isomorphic to the minimal extension $\mathcal{M}_{!\star}$, where $\mathcal{M}$ is the vector bundle over Y.

Example 11. Let $X = \mathbb{A}^1$ with $i : \{0\} \hookrightarrow X$ and $j : U = X \setminus \{0\} \hookrightarrow X$. Although we introduce the de Rham functor for covariance, we will use the Sol functor for calculation. We will give some examples of regular holonomic D-modules on X and their corresponding perverse sheaves. For a D-module $\mathcal{M} = D_X/D_X P$ for $P \in D_X$, we get the corresponding perverse sheaf using a free resolution of $\mathcal{M}$.

$$\mathrm{Sol}(D_X/D_X P) = \underset{\underline{-1}}{\mathcal{O}_X^{\mathrm{an}}} \xrightarrow{P^t} \underset{\underline{0}}{\mathcal{O}_X^{\mathrm{an}}}.$$

(a) Let $\mathcal{M}$ be D_X itself. Then $H^{-1}(\mathrm{Sol}(D_X)) = \mathcal{O}_X$ is not a constructible sheaf. So we cannot expect $\mathrm{Sol}(\mathcal{M})$ to be a perverse sheaf for non-holonomic D-modules.

(b) Recall the holonomic D-modules of Example 5 (b), (c).

D-module $\mathcal{M}$	P	$H^{-1}(\mathrm{Sol}(\mathcal{M}))$	$H^0(\mathrm{Sol}(\mathcal{M}))$
$\mathcal{O}_X$	∂	$\underline{\mathbb{C}}_X$	0
δ_0	x	0	$i_\star\underline{\mathbb{C}}_{\{0\}}$

Their corresponding cohomologies are constructible and satisfy the dimension conditions of perverse sheaves. Also, we can check $\mathrm{id}_{\star!}\,\mathcal{O}_X = \mathcal{O}_X$, $i_{\star!}\underline{\mathbb{C}}_{\{0\}} = i_\star\underline{\mathbb{C}}_{\{0\}} = \delta_0$ and those are simple.(See Theorem 3.6) In this case, one possible stratification is $X = \{0\} \bigcup U$.

(c) Consider the vector bundle $D_U/D_U(x\partial - \lambda) \cong \mathcal{O}_U$ for the case of nontrivial monodromy around 0. Let $\mathcal{M}_\lambda = D_X/D_X(x\partial - \lambda)$ be the $\star$ pushforward of the bundle by j. Note that the kernel of an operator $P = x\partial - \lambda$ is cx^λ in the analytic setting.

   (i) For $\lambda \notin \mathbb{Z}$, the -1th cohomology of the corresponding perverse sheaf is a pushforward of locally constant sheaf with monodromy $e^{2\pi i\lambda}$ on U. On the other hand, for any nonzero element $m \in \mathcal{M}_\lambda$, we can generate 1 by applying ∂ and x finitely many times. For example, $x^2 \xrightarrow{\partial} (2-\lambda)x \xrightarrow{\partial} (2-\lambda)(1-\lambda)$. This implies that $\mathcal{M}_\lambda$ is a simple D-module, and $\mathrm{Sol}(\mathcal{M}_\lambda)$ is also simple. Also, the Riemann-Hilbert correspondence says that $\mathcal{M}_\lambda \cong \mathcal{M}_\mu$ are isomorphic if $\lambda - \mu \in \mathbb{Z}$ since the corresponding perverse sheaves are isomorphic. The minimal extension $j_{\star!}\mathcal{M}_\lambda$ is M_λ itself. The preciese map is as follows:

$$\mathcal{M}_\lambda \to \mathcal{M}_{\lambda+1} \qquad\qquad \mathcal{M}_{\lambda+1} \to \mathcal{M}_\lambda$$

$$1 \to \partial \qquad\qquad 1 \to \frac{x}{\lambda+1}$$

   Note that this isomorphism holds only for $\lambda \neq -1$.

   (ii) For $\lambda \in \mathbb{Z}_{\geq 0}$, there is a global solution x^λ. From the above isomorphism, we only need to check the case $\lambda = 0$. In this case, $H^{-1}(\mathrm{Sol}(\mathcal{M}_0)) = \underline{\mathbb{C}}_X$ and $H^0(\mathrm{Sol}(\mathcal{M}_0)) = i_\star\underline{\mathbb{C}}_{\{0\}}$ hold. The following short exact sequence says that $\mathcal{M}_0$ is not simple.

$$0 \to \mathcal{O}_X \to \mathcal{M}_0 \to \delta_0 \to 0$$

   On the perverse sheaf side, we have the following short exact sequence and the corresponding long exact sequence.

$$
\begin{tikzcd}
\cdots \arrow[r] & \underline{\mathbb{C}}_X \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots & \cdots & \underline{\mathbb{C}}_X \arrow[d, "\cong"] & 0 \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{O}_X^{\mathrm{an}} \arrow[r, "x\partial"] \arrow[d, "\partial"] & \mathcal{O}_X^{\mathrm{an}} \arrow[r] \arrow[d, "1"] & \cdots & \cdots & \underline{\mathbb{C}}_X \arrow[d] & i_\star\underline{\mathbb{C}}_{\{0\}} \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \mathcal{O}_X^{\mathrm{an}} \arrow[r, "x"] & \mathcal{O}_X^{\mathrm{an}} \arrow[r] & \cdots & \cdots \arrow[uur] & 0 \arrow[uur] & i_\star\underline{\mathbb{C}}_{\{0\}} \arrow[uur] & \cdots
\end{tikzcd}
$$

Note that the direction of arrows in SES is reversed in the perverse sheaf side. This is because we used the contravariant equivalence functor Sol.

(iii) For $\lambda \in \mathbb{Z}_{<0}$, we only need to check the case $\lambda = -1$. In this case, we have only one nontrivial cohomology $H^{-1}(\mathrm{Sol}(\mathcal{M}_{-1})) = j_\star\underline{\mathbb{C}}_U$. Similarly, we have the following short exact sequence for D_X-modules.

$$0 \to \delta_0 \to \mathcal{M}_{-1} \to \mathcal{O}_X \to 0$$

On the perverse sheaf side, we have

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & i_\star\underline{\mathbb{C}}_{\{0\}} \arrow[r] \arrow[d, "\cong"] & \cdots & \cdots & 0 \arrow[d] & i_\star\underline{\mathbb{C}}_{\{0\}} \arrow[d] & \cdots \\
\cdots \arrow[r] & \underline{\mathbb{C}}_X \arrow[r] \arrow[d, "\cong"] & i_\star\underline{\mathbb{C}}_{\{0\}} \arrow[r] \arrow[d] & \cdots & \cdots & j_\star\underline{\mathbb{C}}_U \arrow[d] & 0 \arrow[d] & \cdots \\
\cdots \arrow[r] & \underline{\mathbb{C}}_X \arrow[r] & 0 \arrow[r] & \cdots & \cdots \arrow[uur] & \underline{\mathbb{C}}_X \arrow[uur] & 0 \arrow[uur] & \cdots
\end{tikzcd}
$$

$\mathcal{M}_0 \not\cong \mathcal{M}_{-1}$ comes from the nonisomorphic perverse sheaves. We verified that both $\mathcal{M}_0$ and $\mathcal{M}_{-1}$ have their unique simple submodule $\mathcal{O}_X$ and δ_0 respectively.

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