

Coherent States

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We have shown that the number states $|n\rangle$ do not behave similarly as the classical fields. For example, the expectation value $\langle n|\hat{\mathbf{E}}|n\rangle$ is not only static but also zero. A classical field is a field whose amplitude is a harmonic function of t , i.e., $\exp(\pm i\omega t)$. Since the number states form a complete set of the basis vectors, all the photon states, including the classical field, can be written in the number state basis. Hence, we write a classical field $\mathbf{E}_{\text{cl}}(\mathbf{r}, t)$ as a superposition of the number states,

$$|\text{classical}\rangle = \sum_n C_n |n(t)\rangle = \sum_n C_n e^{-in\omega t} |n(0)\rangle. \quad (1)$$

The coefficients C_n are to be determined to satisfy the following properties. The classical field $\mathbf{E}_{\text{cl}}(\mathbf{r}, t)$ is the expectation value of the electric field of the classical state,

$$\mathbf{E}_{\text{cl}}(\mathbf{r}, t) = \langle \text{classical} | \hat{\mathbf{E}} | \text{classical} \rangle, \quad (2)$$

where for a mode of frequency ω , the classical field $\mathbf{E}_{\text{cl}}(\mathbf{r}, t)$ is sinusoidal,

$$\mathbf{E}_{\text{cl}}(\mathbf{r}, t) = \mathcal{E}_\omega(\mathbf{r}) e^{-i\omega t + \phi}. \quad (3)$$

A classical field has the two features, the harmonic oscillation term $e^{-i\omega t}$ and the phase ϕ . Although the expectation value by Eq. (3) define the exact values of the amplitude and the phase, the amplitude and phase of the electric field of a state $|\psi\rangle$ in general have uncertainties. Hence, the amplitude and phase of a state should be described by probability distributions.

Note 1: Coherent State

A coherent state is a most classical state of which the amplitude is a finite constant, the phase grows as ωt , and the uncertainties of the amplitude and phase are minimized.

Below, we first discuss how to obtain the phase distribution of a state $|\psi\rangle$, and find the coefficient C_n of a coherent state.

1.1 Quantum Phase

In quantum optics, the electric field \mathbf{E} of an arbitrary photon state $|\psi\rangle$ has the uncertainties in both its amplitude and phase, that is, $\langle E^2 \rangle \neq 0$ and $\langle \phi^2 \rangle \neq 0$. Indeed, we have not talked about how to obtain ϕ of a photon state $|\psi\rangle$. Note that the phase ϕ is not the phase of a wavefunction but the phase of the electric field. Since \mathbf{E} is an operator but not a number, it turns out that there are many definitions of the phase ϕ . Moreover, the phase ϕ of a state $|\psi\rangle$ is not a single value but a distribution with a finite variance. We will define a phase distribution $\mathcal{P}(\phi)$ where

$\mathcal{P}(\phi)d\phi$ is the probability to find the state to have a phase ϕ . Here, we follow the approach by Susskind and Glogower to obtain the phase distribution. The Susskind–Glogower operators are defined by

$$A \equiv (aa^\dagger)^{-\frac{1}{2}}a = (N+1)^{-\frac{1}{2}}a, \quad (4)$$

$$A^\dagger \equiv a^\dagger(aa^\dagger)^{-\frac{1}{2}} = a^\dagger(N+1)^{-\frac{1}{2}}. \quad (5)$$

If we temporarily treat a as a complex number, $a = |a|\exp i\phi$, the operator A will look as $A = \exp i\phi$. This is the motivation of the definitions, which is to make the operator A taking out the phase factor $\exp i\phi$ of a state. The properties of the SG operators are

$$A|n\rangle = \begin{cases} |n-1\rangle, & n \neq 0, \\ 0, & n = 0, \end{cases} \quad (6)$$

$$A^\dagger|n\rangle = |n+1\rangle, \quad (7)$$

in the number state bases,

$$A = \sum_n |n\rangle\langle n+1|, \quad (8)$$

$$A^\dagger = \sum_n |n+1\rangle\langle n|, \quad (9)$$

$$AA^\dagger = 1, \quad (10)$$

$$A^\dagger A = 1 - |0\rangle\langle 0|. \quad (11)$$

The eigenstate of A is $|\phi\rangle$,

$$A|\phi\rangle = e^{i\phi}|\phi\rangle. \quad (12)$$

The state $|\phi\rangle$ in the number states is

$$|\phi\rangle = \sum_n e^{in\phi}|n\rangle. \quad (13)$$

The state given by Eq. (13) is not normalized. The states $|\phi\rangle$ and $|\phi'\rangle$ are not orthogonal, that is, $\langle\phi'|\phi\rangle \neq 0$. Using the fact

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-n')\phi} d\phi = \delta_{n,n'}, \quad (14)$$

we can show that

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi |\phi\rangle\langle\phi| = \mathbb{1}. \quad (15)$$

Derivation 1: Identity with Phase States

Let $|\psi\rangle$ be an arbitrary state. In the number state bases, it is

$$|\psi\rangle = \sum_n C_n |n\rangle. \quad (16)$$

Applying the operator in Eq. (15) on the state, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi |\phi\rangle \langle \phi | \psi\rangle = \sum_n \frac{1}{2\pi} \int_0^{2\pi} d\phi |\phi\rangle \langle \phi | n\rangle \quad (17)$$

$$= \frac{1}{2\pi} \sum_n \int d\phi |\phi\rangle C_n e^{-in\phi} \quad (18)$$

$$= \frac{1}{2\pi} \sum_{n,m} \int d\phi e^{im\phi} |m\rangle C_n e^{-in\phi} \quad (19)$$

$$= \sum_{n,m} \delta_{mn} C_n |m\rangle \quad (20)$$

$$= \sum_n C_n |n\rangle \quad (21)$$

$$= |\psi\rangle, \quad (22)$$

which proves the operator in Eq. (15) is an identity.

The phase distribution $\mathcal{P}(\phi)$ of a state $|\psi\rangle$ is

$$\mathcal{P}(\phi) \equiv \frac{1}{2\pi} |\langle \phi | \psi\rangle|^2 \quad (23)$$

$$= \frac{1}{2\pi} \left| \sum_n C_n e^{-in\phi} \right|^2. \quad (24)$$

The phase distribution $\mathcal{P}(\phi)$ is normalized,

$$\int_0^{2\pi} \mathcal{P}(\phi) d\phi = 1. \quad (25)$$

The phase distribution $\mathcal{P}(\phi)$ of an ensemble is

$$\mathcal{P}(\phi) = \frac{1}{2\pi} \langle \phi | \rho | \phi \rangle. \quad (26)$$

Note 2: Phase of a Phase State

The phase distribution function $\mathcal{P}(\phi)$ reveals the phase distribution of a state $|\psi\rangle$. Since N and A does not commute ($[N, A] = -A$), a state can not have a single phase but a phase distribution. The phase state $|\phi'\rangle$ is supposed to have a specific phase ϕ' . However, since the phase state is not normalized, it is not physical but a mathematical tool. We consider an approximate phase state which is normalized,

$$|\phi'\rangle_{\text{app}} \equiv \sum_{n=0}^{N_{\text{max}}} \frac{e^{in\phi'}|n\rangle}{\sqrt{N_{\text{max}} + 1}}. \quad (27)$$

The phase distribution function of $|\phi'\rangle_{\text{app}}$ is

$$\mathcal{P}(\phi) = \frac{1}{2(N_{\text{max}} + 1)\pi} \left| \frac{\sin \left[\frac{(N_{\text{max}} + 1)(\phi - \phi')}{2} \right]}{\sin \left[\frac{\phi - \phi'}{2} \right]} \right|^2. \quad (28)$$

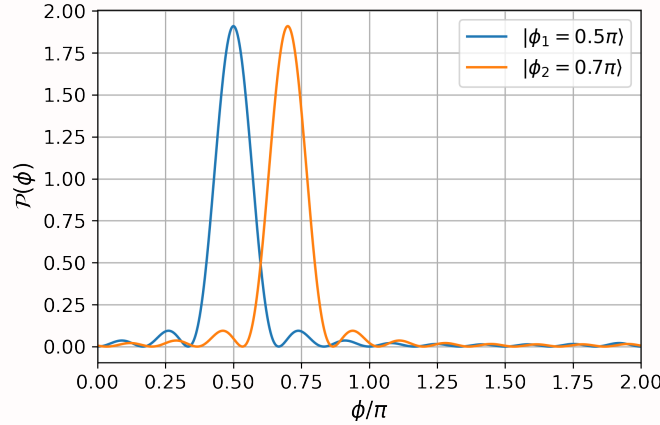


Figure 1: Phase distribution functions of $|\phi_1 = 0.5\pi\rangle$ and $|\phi_2 = 0.7\pi\rangle$. The maximum number is $N_{\text{max}} = 12$.

```

1 import matplotlib
2 import matplotlib.pyplot as plt
3 import numpy as np
4 # Data for plotting
5 phi1 = 0.5 * np.pi
6 phi2 = 0.7 * np.pi
7 Nmax = 12
8 phi = np.arange(0.0, 2.0 * np.pi, 0.01)
9 # define the phase distribution function
10 def phase_dist_func(x,y):
11     return np.sin(Nmax*(x-y)/2)**2./np.sin((x-y)/2)**2/Nmax/(2*np.pi)
12 phase_dist_1 = phase_dist_func(phi,phi1)
13 phase_dist_2 = phase_dist_func(phi,phi2)
14 ## plot
15 fig, ax = plt.subplots()
16 ax.plot(phi, phase_dist_1,label=r'$|\phi_1=0.5\pi\rangle$')
17 ## r: raw string
18 ax.plot(phi, phase_dist_2,label=r'$|\phi_2=0.7\pi\rangle$')
19 ## r: raw string
20 ax.set(xlabel='$\phi$', ylabel='$\mathcal{P}(\phi)$',
21       title='Phase Distribution Function of a Phase State')
22 ax.grid()
23 plt.legend()
24 fig.savefig("phase_dist.png", dpi=300)
25 plt.show()

```

Figure 2: Python codes.

Exercise 1: Phase Distribution Function

Show Eq. (28). Use Eq. (24). The summation is a geometric series.

1.2 Coherent States

We have shown that a phase state $|\phi\rangle$ has a well-defined phase. However, as a classical field, not only the phase but also the field amplitude should be well-defined, that is, we expect that $\langle \mathbf{E} \rangle$ does not vanish, and $\sigma(\mathbf{E})$ is small. Since the phase states are not normalized nor physical, we have to find other states.

The goal is to find the states $|\alpha\rangle$ such that the expectation of the electric field $\langle \alpha | \mathbf{E} | \alpha \rangle$ is proportional to the classical field $\mathcal{E}_\omega(\mathbf{r}) + \mathcal{E}_\omega^*(\mathbf{r})$. By observing that

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}_\omega^*(\mathbf{r})a^\dagger]}{2}, \quad (29)$$

one finds that if the states $|\alpha\rangle$ are the eigenstates of the annihilation operator a ,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad (30)$$

with the eigenvalues α , the expectation value $\langle \alpha | \mathbf{E} | \alpha \rangle$ is the same as the classical field. Since the operator a is not hermitian, the eigenvalues α can be complex numbers in general. It turns out that the states $|\alpha\rangle$, called “coherent states”, are the most classical states. Let’s find out the coherent states in the number state bases. We expand the coherent states as

$$|\alpha\rangle = \sum_c C_n |n\rangle, \quad (31)$$

and plug it in Eq. (30),

$$a|\alpha\rangle = \sum_c C_n a|n\rangle = \alpha \sum_c C_n |n\rangle \quad (32)$$

$$\Rightarrow \sum_c C_n \sqrt{n} |n-1\rangle = \alpha \sum_c C_n |n\rangle. \quad (33)$$

We obtain

$$C_{n+1} = \alpha \frac{C_n}{\sqrt{n+1}}, \quad (34)$$

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0, \quad (35)$$

and thus

$$|\alpha\rangle = C_0 \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (36)$$

The coefficient C_0 is fixed by the normalization condition,

$$\langle\alpha|\alpha\rangle = |C_0|^2 \sum_{m,n} \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!m!}} \langle m|n\rangle, \quad (37)$$

where one finds

$$C_0 = e^{-\frac{|\alpha|^2}{2}}. \quad (38)$$

The coherent states are

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (39)$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle \quad (40)$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle. \quad (41)$$

Exercise 2: Normalization Constant

Show Eq. (38). Begin with Eq. (37).

The expectations are

$$\langle \alpha | \mathbf{E} | \alpha \rangle = \left\langle \alpha \left| \frac{[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}_\omega^*(\mathbf{r})a^\dagger]}{2} \right| \alpha \right\rangle \quad (42)$$

$$= \text{Re}[\alpha \mathcal{E}_\omega(\mathbf{r})] \quad (43)$$

$$\begin{aligned} \langle \alpha | \text{abs}[\mathbf{E}]^2 | \alpha \rangle &= \left\langle \alpha \left| \frac{\text{abs}[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}_\omega^*(\mathbf{r})a^\dagger]^2}{4} \right| \alpha \right\rangle \\ &= \text{abs}[\text{Re}[\alpha \mathcal{E}_\omega(\mathbf{r})]]^2 + \frac{|\mathcal{E}_\omega(\mathbf{r})|^2}{4}. \end{aligned} \quad (44)$$

The standard deviation of the electric field is

$$\sigma(\mathbf{E}) = \frac{|\mathcal{E}_\omega(\mathbf{r})|}{2}. \quad (45)$$

The standard deviation is relatively small compared to the field amplitude when $|\alpha|$ is large. We can see this by dividing $\sigma(\mathbf{E})$ with $\langle \alpha | \mathbf{E} | \alpha \rangle$,

$$\frac{\sigma(\mathbf{E})}{\langle \alpha | \mathbf{E} | \alpha \rangle} = \frac{|\mathcal{E}_\omega(\mathbf{r})|}{2\text{Re}[\alpha \mathcal{E}_\omega(\mathbf{r})]}. \quad (46)$$

The coherent states $|\alpha\rangle$ indeed have the minimum uncertainty. Using the quadrature operators X and Y , one can show that the coherent states have

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \quad (47)$$

Exercise 3: Uncertainty Relations

Show Eq. (47). Hints:

(a) $\langle \alpha | X | \alpha \rangle = \frac{\alpha + \alpha^*}{2}$

(b) $\langle \alpha | X^2 | \alpha \rangle = \left(\frac{\alpha + \alpha^*}{2} \right)^2 + \frac{1}{4}$. Note that $(a + a^\dagger)^2 = a^2 + 2a^\dagger a + (a^\dagger)^2 + 1$

The physical meaning of α is the dimensionless amplitude, which is seen from that the average number \bar{n} of a coherent state $|\alpha\rangle$ is

$$\bar{n} = \langle \alpha | N | \alpha \rangle = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2. \quad (48)$$

The standard deviation $\sigma(N)$ is

$$\sigma(N) = |\alpha| = \bar{n}^{\frac{1}{2}}. \quad (49)$$

The standard deviation $\sigma(N)$ over the average number \bar{n} is

$$\frac{\sigma(N)}{\bar{n}} = \bar{n}^{-\frac{1}{2}}. \quad (50)$$

The probability p_n of measuring the number state $|n\rangle$ is a Poisson distribution

$$p_n = |C_n|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}} \frac{\bar{n}^n}{n!}. \quad (51)$$

The phase distribution function $\mathcal{P}(\phi)$ of a coherent state is

$$\mathcal{P}(\phi) = \frac{e^{-|\alpha|^2}}{2\pi} \left| \sum_n \frac{\alpha^n}{\sqrt{n!}} \right|^2. \quad (52)$$

Let $\alpha = |\alpha|e^{i\bar{\phi}}$. One can show that as $\bar{n} = |\alpha|^2$ is large, the distributions become approximately the Gaussian distributions (See Ref. [1]),

$$p_n \simeq (2\pi\bar{n})^{-1/2} e^{-\frac{(n-\bar{n})^2}{2\bar{n}}}, \quad (53)$$

$$\mathcal{P}(\phi) \simeq \sqrt{\frac{2\bar{n}}{\pi}} e^{-2\bar{n}(\phi-\bar{\phi})^2}. \quad (54)$$

1.3 Displaced Vacuum States

The physical meaning of α is the dimensionless (complex) amplitude of a coherent state. We can think the vacuum state is the limit of a coherent state as $\alpha \rightarrow 0$. Conversely, a coherent state is obtained by shifting α of the vacuum state. The operator doing this task is the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \quad (55)$$

The displacement operator $D(\alpha)$ has the explicit form

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \quad (56)$$

To show this, first consider the special case of Baker–Campbell–Hausdorff formula, if

$$[A, [A, B]] = [B, [A, B]] = 0, \quad (57)$$

we have

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \quad (58)$$

$$= e^{\frac{1}{2}[B,A]} e^B e^A. \quad (59)$$

With $A = \alpha a^\dagger$, $B = -\alpha^* a$, and $[A, B] = |\alpha|^2$, the displacement operator $D(\alpha)$ becomes

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}. \quad (60)$$

Using the relations

$$e^{-\alpha^* a}|0\rangle = \left(\mathbb{1} - \alpha^* a + \frac{(-\alpha^* a)^2}{2!} + \dots \right) |0\rangle = |0\rangle, \quad (61)$$

we obtain

$$D(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle \quad (62)$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle \quad (63)$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle \quad (64)$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle \quad (65)$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (66)$$

$$= |\alpha\rangle. \quad (67)$$

The displacement operator $D(\alpha)$ is unitary and satisfies the relation

$$D(\alpha)D^\dagger(\alpha) = D^\dagger(\alpha)D(\alpha) = \mathbb{1}, \quad (68)$$

$$D^\dagger(\alpha) = D(-\alpha). \quad (69)$$

Operations by two subsequent displacement operator $D(\alpha)$ and $D(\beta)$ gives a total displacement operator

$$D(\alpha)D(\beta) = e^{i\text{Im}[\alpha\beta^*]} D(\alpha + \beta). \quad (70)$$

The total displacement of the displacement operators satisfies the law of addition.

1.4 Dynamics of Coherent States

The dynamics of a coherent state $|\alpha\rangle$ is given by the Schrödinger's picture,

$$|\alpha(t)\rangle = e^{-i\frac{\mathcal{H}t}{\hbar}} |\alpha(0)\rangle \quad (71)$$

$$= e^{-i\frac{\omega}{2}t} e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}} |n\rangle \quad (72)$$

$$= e^{-i\frac{\omega}{2}t} |\alpha(0)\rangle e^{-i\omega t}. \quad (73)$$

Thus, the amplitude $\alpha(t)$ is

$$\alpha(t) = \alpha(0)e^{-i\omega t}. \quad (74)$$

Although every photon mode $\mathcal{E}_\omega(\mathbf{r})$ can be quite different from one system to another system, we can use the dimensionless quadrature operators \hat{X} and \hat{Y} to describe the dynamics. Recall that \hat{X} is analogous to the position operator, and \hat{Y} is analogous to the momentum operator. We can express a coherent state in the X basis,

$$\psi_\alpha(X) = \langle X|\alpha\rangle, \quad (75)$$

where $|X\rangle$ is the eigenvector of X

$$\hat{X}|X\rangle = X|X\rangle. \quad (76)$$

To find $\psi_\alpha(X)$, we begin with

$$\langle X|a|\alpha\rangle = \alpha\langle X|\alpha\rangle \quad (77)$$

$$\Rightarrow \langle X|\hat{X} + i\hat{Y}|\alpha\rangle = \alpha\langle X|\alpha\rangle \quad (78)$$

$$\Rightarrow \left(X + \frac{\partial}{\partial X}\right)\langle X|\alpha\rangle = \alpha\langle X|\alpha\rangle \quad (79)$$

$$\Rightarrow \frac{\partial\psi_\alpha(X)}{\partial X} = (\alpha - X)\psi_\alpha(X) \quad (80)$$

$$\Rightarrow \psi_\alpha(X) = \sqrt{\frac{2}{\pi}} e^{-\frac{(X - \text{Re}[\alpha])^2}{2}} e^{i\text{Im}[\alpha]X}, \quad (81)$$

where we have used the normalization condition to derive the last step. The wavefunction $\psi_\alpha(X)$ is a Gaussian distribution, and its peak position is

$$X_p(t) = \text{Re}[\alpha(t)] \quad (82)$$

$$= |\alpha(0)| \cos(\phi_0 - \omega t). \quad (83)$$

with $\alpha(0) = |\alpha(0)|e^{i\phi_0}$. The peak position $X_p(t)$ is the same as that of a classical harmonic oscillator. Also the wavefunction $\psi_\alpha(X)$ has a minimum spread of X and P . Thus, a coherent state is the most classical state.

Summary 1: Coherent States

Coherent states are

- eigenstates of the annihilation operator a .
- displaced vacuum states.

- most classical states whose phase and amplitude distributions are narrow.
- most classical states whose X and Y distributions are narrow.
- minimum uncertainty states.

References

- [1] S. M. Barnett and D. T. Pegg, *J. Mod. Opt.*, 36 (1989), 7.