# Coherent States

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We have shown that the number states $|n\rangle$ do not behave similarly as the classical fields. For example, the expectation value $\langle n|\hat{\mathbf{E}}|n\rangle$ is not only static but also zero. A classical field is a field whose amplitude is a harmonic function of $t$, i.e., $\exp(\pm i\omega t)$. Since the number states form a complete set of the basis vectors, all the photon states, including the classical field, can be written in the number state basis. Hence, we write a classical field $\mathbf{E}_{\mathrm{cl}}(\mathbf{r},t)$ as a superposition of the number states,

$$|\text{classical}\rangle = \sum_n C_n|n(t)\rangle = \sum_n C_n e^{-in\omega t}|n(0)\rangle. \tag{1}$$

The coefficients $C_n$ are to be determined to satisfy the following properties. The classical field $\mathbf{E}_{\mathrm{cl}}(\mathbf{r},t)$ is the expectation value of the electric field of the classical state,

$$\mathbf{E}_{\mathrm{cl}}(\mathbf{r},t) = \langle \text{classical}|\hat{\mathbf{E}}|\text{classical}\rangle, \tag{2}$$

where for a mode of frequency $\omega$, the classical field $\mathbf{E}_{\mathrm{cl}}(\mathbf{r},t)$ is sinusoidal,

$$\mathbf{E}_{\mathrm{cl}}(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_\omega(\mathbf{r})e^{-i\omega t+\phi}. \tag{3}$$

A classical field has the two features, the harmonic oscillation term $e^{-i\omega t}$ and the phase $\phi$. Although the expectation value by Eq. (3) define the exacts values of the amplitude and the phase, the amplitude and phase of the electric field of a state $|\psi\rangle$ in general have uncertainties. Hence, the amplitude and phase of a state should be described by probability distributions.

**Note 1: Coherent State**

A coherent state is a most classical state of which the amplitude is a finite constant, the phase grows as $\omega t$, and the uncertainties of the amplitude and phase are minimized.

Below, we first discuss how to obtain the phase distribution of a state $|\psi\rangle$, and find the coefficient $C_n$ of a coherent state.

## 1.1 Quantum Phase

In quantum optics, the electric field $\mathbf{E}$ of an arbitrary photon state $|\psi\rangle$ has the uncertainties in both its amplitude and phase, that is, $\langle E^2\rangle \neq 0$ and $\langle \phi^2\rangle \neq 0$. Indeed, we have not talked about how to obtain $\phi$ of a photon state $|\psi\rangle$. Note that the phase $\phi$ is not the phase of a wavefunction but the phase of the electric field. Since $\mathbf{E}$ is an operator but not a number, it turns out that there are many definitions of the phase $\phi$. Moreover, the phase $\phi$ of a state $|\psi\rangle$ is not a single value but a distribution with a finite variance. We will define a phase distribution $\mathcal{P}(\phi)$ where $\mathcal{P}(\phi)d\phi$ is the probability to find the state to have a phase $\phi$. Here, we follow the approach by Susskind and Glogower to obtain the phase distribution. The Susskind–Glogower operators are defined by

$$A \equiv (aa^\dagger)^{-\frac{1}{2}}a = (N+1)^{-\frac{1}{2}}a, \tag{4}$$

$$A^\dagger \equiv a^\dagger(aa^\dagger)^{-\frac{1}{2}} = a^\dagger(N+1)^{-\frac{1}{2}}. \tag{5}$$

If we temporarily treat $a$ as a complex number, $a = |a|\exp i\phi$, the operator $A$ will look as $A = \exp i\phi$. This is the motivation of the definitions, which is to make the operator $A$ taking out the phase factor $\exp i\phi$ of a state. The properties of the SG operators are

$$A|n\rangle = \begin{cases} |n-1\rangle, & n \neq 0, \\ 0, & n = 0, \end{cases} \tag{6}$$

$$A^\dagger|n\rangle = |n+1\rangle, \tag{7}$$

in the number state bases,

$$A = \sum_n |n\rangle\langle n+1|, \tag{8}$$

$$A^\dagger = \sum_n |n+1\rangle\langle n|, \tag{9}$$

$$AA^\dagger = 1, \tag{10}$$

$$A^\dagger A = 1 - |0\rangle\langle 0|. \tag{11}$$

The eigenstate of $A$ is $|\phi\rangle$,

$$A|\phi\rangle = e^{i\phi}|\phi\rangle. \tag{12}$$

The state $|\phi\rangle$ in the number states is

$$|\phi\rangle = \sum_n e^{in\phi}|n\rangle. \tag{13}$$

The state given by Eq. (13) is not normalized. The states $|\phi\rangle$ and $|\phi'\rangle$ are not orthogonal, that is, $\langle\phi'|\phi\rangle \neq 0$. Using the fact

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n')\phi}d\phi = \delta_{n,n'}, \tag{14}$$

we can show that

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi|\phi\rangle\langle\phi| = \mathbb{1}. \tag{15}$$

## Derivation 1: Identity with Phase States

Let $|\psi\rangle$ be an arbitrary state. In the number state bases, it is

$$|\psi\rangle = \sum_n C_n |n\rangle. \tag{16}$$

Applying the operator in Eq. (15) on the state, we obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} d\phi |\phi\rangle\langle\phi|\psi\rangle &= \sum_{C_n} \frac{1}{2\pi}\int_0^{2\pi} d\phi |\phi\rangle\langle\phi|n\rangle &(17)\\
&= \frac{1}{2\pi}\sum_n \int d\phi |\phi\rangle C_n e^{-in\phi} &(18)\\
&= \frac{1}{2\pi}\sum_{n,m} \int d\phi e^{im\phi}|m\rangle C_n e^{-in\phi} &(19)\\
&= \sum_{n,m} \delta_{mn} C_n |m\rangle &(20)\\
&= \sum_n C_n |n\rangle &(21)\\
&= |\psi\rangle, &(22)
\end{aligned}$$

which proves the operator in Eq. (15) is an identity.

The phase distribution $\mathcal{P}(\phi)$ of a state $|\psi\rangle$ is

$$\begin{aligned}
\mathcal{P}(\phi) &\equiv \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 &(23)\\
&= \frac{1}{2\pi}\left|\sum_n C_n e^{-in\phi}\right|^2. &(24)
\end{aligned}$$

The phase distribution $\mathcal{P}(\phi)$ is normalized,

$$\int_0^{2\pi} \mathcal{P}(\phi) d\phi = 1. \tag{25}$$

The phase distribution $\mathcal{P}(\phi)$ of an ensemble is

$$\mathcal{P}(\phi) = \frac{1}{2\pi}\langle\phi|\rho|\phi\rangle. \tag{26}$$

## Note 2: Phase of a Phase State

The phase distribution function $\mathcal{P}(\phi)$ reveals the phase distribution of a state $|\psi\rangle$. Since $N$ and $A$ does not commute ($[N, A] = -A$), a state can not have a single phase but a phase distribution. The phase state $|\phi'\rangle$ is supposed to have a specific phase $\phi'$. However, since the phase state is not normalized, it is not physical but a mathematical tool. We consider an approximate phase state which is normalized,

$$|\phi'\rangle_{\mathrm{app}} \equiv \sum_{n=0}^{N_{\mathrm{max}}} \frac{e^{in\phi'}|n\rangle}{\sqrt{N_{\mathrm{max}}+1}}. \tag{27}$$

The phase distribution function of $|\phi'\rangle_{\mathrm{app}}$ is

$$\mathcal{P}(\phi) = \frac{1}{2(N_{\mathrm{max}}+1)\pi} \left| \frac{\sin\left[\frac{(N_{\mathrm{max}}+1)(\phi-\phi')}{2}\right]}{\sin\left[\frac{\phi-\phi'}{2}\right]} \right|^2. \tag{28}$$

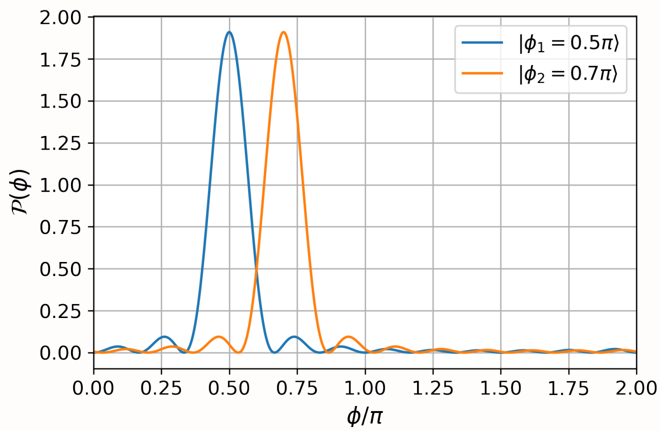


Figure 1: Phase distribution functions of $|\phi_1 = 0.5\pi\rangle$ and $|\phi_2 = 0.7\pi\rangle$. The maximum numeber is $N_{\mathrm{max}} = 12$.

```python
import matplotlib
import matplotlib.pyplot as plt
import numpy as np
# Data for plotting
phi1 = 0.5 * np.pi
phi2 = 0.7 * np.pi
Nmax = 12
phi = np.arange(0.0, 2.0 * np.pi, 0.01)
# define the phase distribution function
def phase_dist_func(x,y):
    return np.sin(Nmax*(x-y)/2)**2./np.sin((x-y)/2)**2/Nmax/(2*np.pi)
phase_dist_1 = phase_dist_func(phi,phi1)
phase_dist_2 = phase_dist_func(phi,phi2)
## plot
fig, ax = plt.subplots()
ax.plot(phi, phase_dist_1,label=r’$|\phi_1=0.5\pi\rangle$’)
## r: raw string
ax.plot(phi, phase_dist_2,label=r’$|\phi_2=0.7\pi\rangle$’)
## r: raw string
ax.set(xlabel=’$\phi$ ’, ylabel=’$\mathcal{P}(\phi)$’,
       title=’Phase Distribution Function of a Phase State’)
ax.grid()
plt.legend()
fig.savefig("phase_dist.png", dpi=300)
plt.show()
```

Figure 2: Python codes.

## Exercise 1: Phase Distribution Function

Show Eq. (28). Use Eq. (24). The summation is a geometric series.

## 1.2 Coherent States

We have shown that a phase state $|\phi\rangle$ has a well-defined phase. However, as a classical field, not only the phase but also the field amplitude should be well-defined, that is, we expect that $\langle\mathbf{E}\rangle$ does not vanish, and $\sigma(\mathbf{E})$ is small. Since the phase states are not normalized nor physical, we have to find other states.

The goal is the find the states $|\alpha\rangle$ such that the expectation of the electric field $\langle\alpha|\mathbf{E}|\alpha\rangle$ is proportional to the classical field $\mathcal{E}_\omega(\mathbf{r}) + \mathcal{E}^*_\omega(\mathbf{r})$. By observing that

$$\mathbf{E}_\omega(\mathbf{r}) = \frac{\left[\mathcal{E}_\omega(\mathbf{r})a + \mathcal{E}^*_\omega(\mathbf{r})a^\dagger\right]}{2}, \tag{29}$$

one finds that if the states $|\alpha\rangle$ are the eigenstates of the annihilation operator $a$,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{30}$$

with the eigenvalues $\alpha$, the expectation value $\langle\alpha|\mathbf{E}|\alpha\rangle$ is the same as the classical field. Since the operator $a$ is not hermitian, the eigenvalues $\alpha$ can be complex numbers in general. It turns out that the states $|\alpha\rangle$, called "coherent states", are the most classical states. Let's find out the coherent states in the number state bases. We expand the coherent states as

$$|\alpha\rangle = \sum_c C_n |n\rangle, \tag{31}$$

and plug it in Eq. (30),

$$a|\alpha\rangle = \sum_c C_n a|n\rangle = \alpha \sum_c C_n |n\rangle \tag{32}$$

$$\Rightarrow \sum_c C_n \sqrt{n}|n-1\rangle = \alpha \sum_c C_n |n\rangle. \tag{33}$$

We obtain

$$C_{n+1} = \alpha \frac{C_n}{\sqrt{n+1}}, \tag{34}$$

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0, \tag{35}$$

and thus

$$|\alpha\rangle = C_0 \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{36}$$

The coefficient $C_0$ is fixed by the normalization condition,

$$\langle\alpha|\alpha\rangle = |C_0|^2 \sum_{m,n} \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!m!}} \langle m|n\rangle, \tag{37}$$

where one finds

$$C_0 = e^{-\frac{|\alpha|^2}{2}}. \tag{38}$$

The coherent states are

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{39}$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle \tag{40}$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle. \tag{41}$$

### Exercise 2: Normalization Constant

Show Eq. (38). Begin with Eq. (37).

The expectations are

$$\langle\alpha|\mathbf{E}|\alpha\rangle = \left\langle \alpha \left| \frac{\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})a + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})a^\dagger\right]}{2} \right| \alpha \right\rangle \tag{42}$$

$$= \mathrm{Re}[\alpha\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})] \tag{43}$$

$$\langle\alpha|\mathrm{abs}[\mathbf{E}]^2|\alpha\rangle = \left\langle \alpha \left| \frac{\mathrm{abs}\left[\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})a + \boldsymbol{\mathcal{E}}^*_\omega(\mathbf{r})a^\dagger\right]^2}{4} \right| \alpha \right\rangle$$

$$= \mathrm{abs}[\mathrm{Re}[\alpha\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})]]^2 + \frac{|\mathcal{E}_\omega(\mathbf{r})|^2}{4}. \tag{44}$$

The standard deviation of the electric field is

$$\sigma(\mathbf{E}) = \frac{|\mathcal{E}_\omega(\mathbf{r})|}{2}. \tag{45}$$

The standard deviation is relatively small compared to the field amplitude when $|\alpha|$ is large. We can see this by dividing $\sigma(\mathbf{E})$ with $\langle\alpha|\mathbf{E}|\alpha\rangle$,

$$\frac{\sigma(\mathbf{E})}{\langle\alpha|\mathbf{E}|\alpha\rangle} = \frac{|\mathcal{E}_\omega(\mathbf{r})|}{2\mathrm{Re}[\alpha\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})]}. \tag{46}$$

The coherent states $|\alpha\rangle$ indeed have the minimum uncertainty. Using the quadrature operators $X$ and $Y$, one can show that the coherent states have

$$\sigma(X) = \sigma(Y) = \frac{1}{2}. \tag{47}$$

## Exercise 3: Uncertainty Relations

Show Eq. (47). Hints:

(a) $\langle\alpha|X|\alpha\rangle = \frac{\alpha+\alpha^*}{2}$

(b) $\langle\alpha|X^2|\alpha\rangle = \left(\frac{\alpha+\alpha^*}{2}\right)^2 + \frac{1}{4}$. Note that $(a+a^\dagger)^2 = a^2 + 2a^\dagger a + \left(a^\dagger\right)^2 + 1$

The physical meaning of $\alpha$ is the dimensionless amplitude, which is seen form that the average number $\bar{n}$ of a coherent state $|\alpha\rangle$ is

$$\bar{n} = \langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2. \tag{48}$$

The standard deviation $\sigma(N)$ is

$$\sigma(N) = |\alpha| = \bar{n}^{\frac{1}{2}}. \tag{49}$$

The standard deviation $\sigma(N)$ over the average number $\bar{n}$ is

$$\frac{\sigma(N)}{\bar{n}} = \bar{n}^{\frac{-1}{2}}. \tag{50}$$

The probability $p_n$ of measuring the number state $|n\rangle$ is a Poisson distribution

$$p_n = |C_n|^2 = e^{-|\alpha^2|}\frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}}\frac{\bar{n}^n}{n!}. \tag{51}$$

The phase distribution function $\mathcal{P}(\phi)$ of a coherent state is

$$\mathcal{P}(\phi) = \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_n \frac{\alpha^n}{\sqrt{n!}}\right|^2. \tag{52}$$

Let $\alpha = |\alpha|^{i\bar{\phi}}$. One can show that as $\bar{n} = |\alpha|^2$ is large, the distributions becom approximately the Gaussian distributions (See Ref. [1]),

$$p_n \simeq (2\pi\bar{n})^{-1/2} e^{-\frac{(n-\bar{n})^2}{2\bar{n}}}, \tag{53}$$

$$\mathcal{P}(\phi) \simeq \sqrt{\frac{2\bar{n}}{\pi}} e^{-2\bar{n}(\phi-\bar{\phi})^2}. \tag{54}$$

## 1.3 Displaced Vacuum States

The physical meaning of $\alpha$ is the dimensionless (complex) amplitude of a coherent state. The vacuum state is indeed a coherent state in the limit $\alpha \to 0$. Conversely, a coherent state is obtained by changing the complex amplitude $\alpha$ of the vacuum state. Mathematically, such a shift of $\alpha$ is done by the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{55}$$

The displacement operator $D(\alpha)$ has the explicit form

$$D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right). \tag{56}$$

To show this, first consider the special case of Baker–Campbell–Hausdorff formula, if

$$[A,[A,B]] = [B,[A,B]] = 0, \tag{57}$$

we have

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^A e^B \tag{58}$$

$$= e^{\frac{1}{2}[B,A]}e^B e^A. \tag{59}$$

With $A = \alpha a^\dagger$, $B = -\alpha^* a$, and $[A, B] = |\alpha|^2$, the displacement operator $D(\alpha)$ becomes

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}. \tag{60}$$

Using the relations

$$e^{-\alpha^* a}|0\rangle = \left(\mathbb{1} - \alpha^* a + \frac{(-\alpha^* a)^2}{2!} + \ldots\right)|0\rangle = |0\rangle, \tag{61}$$

we obtain

$$\begin{align} D(\alpha)|0\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle \tag{62} \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}|0\rangle \tag{63} \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}|0\rangle \tag{64} \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n (a^\dagger)^2}{n!}|0\rangle \tag{65} \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{66} \\ &= |\alpha\rangle. \tag{67} \end{align}$$

The displacement operator $D(\alpha)$ is unitary and satisfies the relation

$$\begin{align} D(\alpha)D^\dagger(\alpha) = D^\dagger(\alpha)D(\alpha) = \mathbb{1}, \tag{68} \\ D^\dagger(\alpha) = D(-\alpha). \tag{69} \end{align}$$

The displacement operators satisfy the law of addition; operations by two subsequent displacement operator $D(\alpha)$ and $D(\beta)$ give a total displacement operator

$$D(\alpha)D(\beta) = e^{i\mathrm{Im}[\alpha\beta^*]} D(\alpha + \beta). \tag{70}$$

We see that the total displacement is $\alpha + \beta$, that is, the sum of the displacements of the individual displacement operators. An extra phase $\mathrm{Im}[\alpha\beta^*]$ is the quantum feature, and note that although the total displacement does not depend on the order of the operators, the phase does depend.

## Note 3: Displacement Operator

For now, a displacement operator is just a mathematical tool. Later, as we learn light-matter interaction, we will know that a displacement operator is the evolution operator of a sinusoidal driving source, $\mathcal{H}_i(t) \sim \sin(\omega t + \phi)$. That is, if we turn on a sinusoidal driving source, the vacuum state will be shifted in the complex $\alpha$ space. This is one method to generate coherent states.

## 1.4 Dynamics of Coherent States

The dynamics of a coherent state $|\alpha\rangle$ is given by the Schrödinger's picture,

$$|\alpha(t)\rangle = e^{-i\frac{\mathcal{H}t}{\hbar}}|\alpha(0)\rangle \tag{71}$$

$$= e^{-i\frac{\omega}{2}t}e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n e^{-in\omega t}}{\sqrt{n!}}|n\rangle \tag{72}$$

$$= e^{-i\frac{\omega}{2}t}|\alpha(0)e^{-i\omega t}\rangle. \tag{73}$$

Thus, the amplitude $\alpha(t)$ is

$$\alpha(t) = \alpha(0)e^{-i\omega t}. \tag{74}$$

Although every photon mode $\boldsymbol{\mathcal{E}}_\omega(\mathbf{r})$ can be quite different from one system to another system, we can use the dimensionless quadrature operators $\hat{X}$ and $\hat{Y}$ to describe the dynamics. Recall that $\hat{X}$ is analogous to the position operator, and $\hat{Y}$ is analogous to the momentum operator. We can express a coherent state in the $X$ basis,

$$\psi_\alpha(X) = \langle X|\alpha\rangle, \tag{75}$$

where $|X\rangle$ is the eigenvector of $X$

$$\hat{X}|X\rangle = X|X\rangle. \tag{76}$$

To find $\psi_\alpha(X)$, we begin with

$$\langle X|a|\alpha\rangle = \alpha\langle X|\alpha\rangle \tag{77}$$

$$\Rightarrow \langle X|\hat{X} + i\hat{Y}|\alpha\rangle = \alpha\langle X|\alpha\rangle \tag{78}$$

$$\Rightarrow \left(X + \frac{\partial}{\partial X}\right)\langle X|\alpha\rangle = \alpha\langle X|\alpha\rangle \tag{79}$$

$$\Rightarrow \frac{\partial \psi_\alpha(X)}{\partial X} = (\alpha - X)\psi_\alpha(X) \tag{80}$$

$$\Rightarrow \psi_\alpha(X) = \sqrt{\frac{2}{\pi}}e^{-\frac{(X-\mathrm{Re}[\alpha])^2}{2}}e^{i\mathrm{Im}[\alpha]X}, \tag{81}$$

where we have used the normalization condition to derive the last step. The wavefunction $\psi_\alpha(X)$ is a Gaussian distribution, and it's peak position is

$$X_p(t) = \mathrm{Re}[\alpha(t)] \tag{82}$$

$$= |\alpha(0)|\cos(\phi_0 - \omega t). \tag{83}$$

with $\alpha(0) = |\alpha(0)|e^{i\phi_0}$. The peak position $X_p(t)$ is the same as that of a classical harmonic oscillator. Also the wavefunction $\psi_\alpha(X)$ has an minimum spreads of $X$ and $P$. Thus, a coherent state is the most classical state.

## Summary 1: Coherent States

Coherent states are

- eigenstates of the annihilation operator $a$.
- displaced vacuum states.
- most classical states whose phase and amplitude distributions are narrow.
- most classical states whose $X$ and $Y$ distributions are narrow.
- minimum uncertainty states.

## References


[1] S. M. Barnett and D. T. Pegg, *J. Mod. Opt.*, 36 (1989), 7.