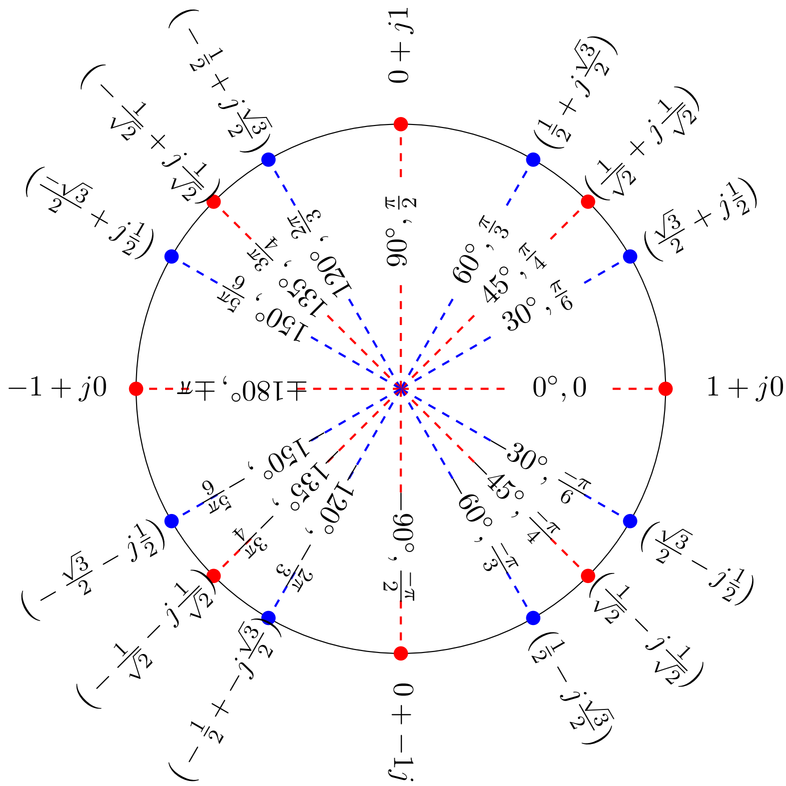
Contents

Chapter 1

Mathematical Preliminaries



1.1 Trigonometric Identities

It will be useful to memorize $\sin\theta, \cos\theta, \tan\theta$ values for $\theta = 0, \pi/3, \pi/4, \pi/2$ and $\pi \pm \theta$, $2\pi - \theta$ for the above values of θ. The values of $\sin\theta$ and $\cos\theta$ for these values are given below

Table 1.1: Some sine and cosine values to memorize

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\pi/6 = 30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\pi/4 = 45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\pi/3 = 60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\pi/2 = 90°$	1	0	∞

The following identities involving sine and cosine functions will be useful

$$\begin{aligned}
\sin(\theta \pm \phi) &= \sin\theta\ \cos\phi \pm \cos\theta\ \sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta\ \cos\phi \mp \sin\theta\ \sin\phi \\
\sin\theta\sin\phi &= \frac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right] \\
\cos\theta\cos\phi &= \frac{1}{2}\left[\cos(\theta-\phi) + \cos(\theta+\phi)\right] \\
\sin\theta\cos\phi &= \frac{1}{2}\left[\sin(\theta-\phi) + \sin(\theta+\phi)\right] \qquad (1.1)
\end{aligned}$$

The following special case of the above formulas are also useful

$$\begin{aligned}
\sin(\pi/2 \pm \phi) &= \cos\phi \\
\cos(\pi/2 \pm \phi) &= \mp\sin\phi \\
\sin(\pi \pm \phi) &= \mp\sin\phi \\
\cos(\pi \pm \phi) &= -\cos\phi \\
\cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta) \\
\sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta) \\
\cos(2\theta) &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta
\end{aligned}$$

1.2 Magnitude and Angle Representation

Any two real numbers a and b can be written as $a = r\cos\theta$ and $b = r\sin\theta$, where $r \geq 0$ and $0 \leq \theta < 2\pi$, where r and θ are given by

$$r = \sqrt{a^2 + b^2}, \qquad (1.2)$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right). \qquad (1.3)$$

1.3 Complex Numbers

1.3.1 History - (Veritasium's video, KRN's video)

Most students encounter complex numbers for the first time when trying to solve quadratic equations. The roots of the quadratic eqeation $ax^2 + bx + c = 0$ are given by the famous formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$. When $b^2 - 4ac < 0$, we need to take the square root of negative numbers in order to compute the roots. There is a popular misconception that complex numbers were invented in order to obtain roots to such quadratic equations. This is inaccurate and it was the roots to cubic equations that prompted the introduction of complex numbers.

The early history of complex numbers is fascinating and it contains all the elements of a good story including competition, intrigue, and deceit. I would highly suggest that you begin by understanding their history before learning about how to do arithmetic with complex numbers. In addition to the fact that it is an interesting story, there is a lesson to learn from this history about how we use complex numbers today. Either of the two videos linked above is a good starting point.

We will use the letter j to refer to the imaginary number $\sqrt{-1}$. Even though j is not a real number, we can perform all arithmetic operations such as addition, subtraction, multiplication, division with j using the algebra of real numbers keeping in mind that $j^2 = -1$.

1.3.2 Cartesian Form - (Video, Python notebook)

A complex number z is any number of the form $z = x + jy$, where x is called the real part of z and y is called the imaginary part of z. The real and imaginary parts are denoted by $\Re\{z\}$ and $\Im\{z\}$, respectively. Note: The imaginary part is not jy, rather it is only y. It is important to stick to this terminology, otherwise computations can go wrong. Just like how real numbers can be represented on the number line, a complex number can be represented as a point in a 2-D plane called the complex plane. The complex plane, shown in Fig. 1.1, is called the Argand diagram (named after a mathematician). The complex number z can be thought of as a point with the X-coordinate given by the real part of z and the Y-coordinate given by the imaginary part of z. Often, it is useful to think of a complex number $z = x + jy$ as a two-dimensional vector where x is the X-coordinate and y is Y-coordinate of the vector.

1.3.3 Magnitude and Phase (Video)

The X and Y coordinates of z can be expressed in terms of the length of the vector r and the angle made by this vector with the positive X-axis, namely θ. Since $x = r\cos\theta$ and $y = r\sin\theta$, z can be expressed as

$$z = r\cos\theta + jr\sin\theta, \tag{1.4}$$

where θ can be in degrees or radians (usually radians) and recall that 2π rad $= 360^\circ$. r is called the magnitude of z, denoted by $|z|$ and θ is called the phase of the complex number

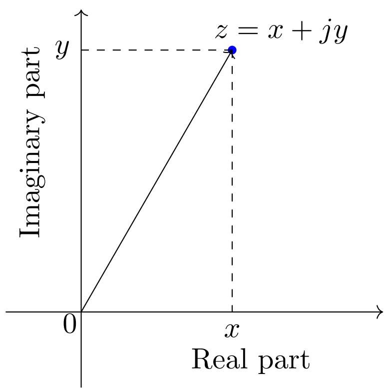


Figure 1.1: Complex plane showing a complex number as a point/vector in the 2-D complex plane

z, denoted by $\arg\{z\}$ or $\angle z$.

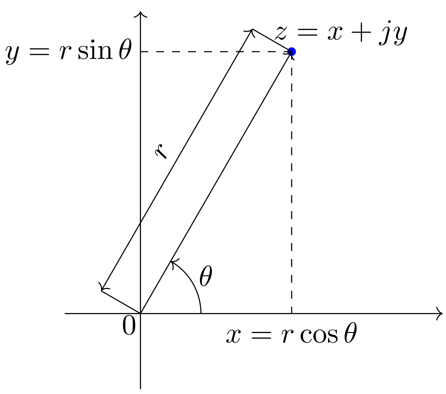


Figure 1.2: Magnitude and phase representation of a complex number $z = r\cos\theta + jr\sin\theta$

Magnitude and phase representation

x, y, r and θ are related according to

$$\begin{aligned} x &= \Re\{z\} = r\ \cos\theta, & y &= \Im\{z\} = r\ \sin\theta \\ r &= |z| = \sqrt{x^2 + y^2}, & \theta &= \angle z = \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \tag{1.5}$$

Magnitude is always positive

The magnitude r represents the *length* of the vector and hence, has to be non-negative. It does not make sense for the magnitude of a complex number to be negative.

Correctly computing the angle

Typical software packages and calculators will give a result for $\tan^{-1}\left(\frac{y}{x}\right)$ that lies in $(-\pi/2, \pi/2)$. Thus, we cannot directly distinguish between the angle of $x + jy$ and $-x - jy$ if we simply compute $\tan^{-1}\left(\frac{y}{x}\right)$. To get the correct angle θ, we must add π or subtract π to the result if x is negative. Both adding and subtracting π will give the correct answer; however, by convention, we will add or subtract π such that the result is in the range $(-\pi, \pi]$.

1.3.4 Euler's Formula - (Video)

The cosine, sine and exponential functions have infinite series (Maclaurin's series) expansions given by

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \ldots \tag{1.6}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots \tag{1.7}$$

$$e^\theta = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \ldots \tag{1.8}$$

where θ is in radians.

By replacing θ by $j\theta$ and $-j\theta$, respectively, in (1.8), we get the following two equations.

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \ldots \tag{1.9}$$

$$e^{-j\theta} = 1 - \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} - \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} - \frac{(j\theta)^5}{5!} + \ldots \tag{1.10}$$

From (1.9), (1.10), (1.6) and (1.7) the following relationship can be seen to be true. Equation 1.11 is called Euler's formula. We will repeatedly use this identity in the notes and, hence, you should memorize and develop a familiarity with these four formulae.

Euler's formula

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \qquad (1.11) \\ e^{-j\theta} &= \cos\theta - j\sin\theta \\ \cos\theta &= \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta &= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{aligned}$$

When $\theta = \pi$, (1.11) reduces to $e^{j\pi} = -1$ or, equivalently

$$\boxed{e^{j\pi} + 1 = 0},$$

which is known as Euler's identity. The famous physicist Richard Feynman called this "our jewel" and "the most remarkable formula in mathematics" https://en.wikipedia.org/wiki/Mathematical_beauty.

1.3.5 Polar form or Exponential form (Video)

We already saw that a complex $z = x + jy$ can be written as $z = r\cos\theta + jr\sin\theta$. Using Euler's identities z can be written as

$$z = x + jy = r\cos\theta + jr\sin\theta = re^{j\theta}. \tag{1.12}$$

$re^{j\theta}$ is known as the polar form or exponential form.

In summary, the same complex number z can be written in Cartesian form as $x + jy$ or in polar form as $re^{j\theta}$ and it is very important to be able to convert a complex number from Cartesian form to exponential form and vice versa. When expressing a complex number in polar form, make sure r is positive. It will be useful to get familiarized with the Cartesian and polar forms of several commonly encountered complex numbers on the unit circle as shown in Figure 1.3.

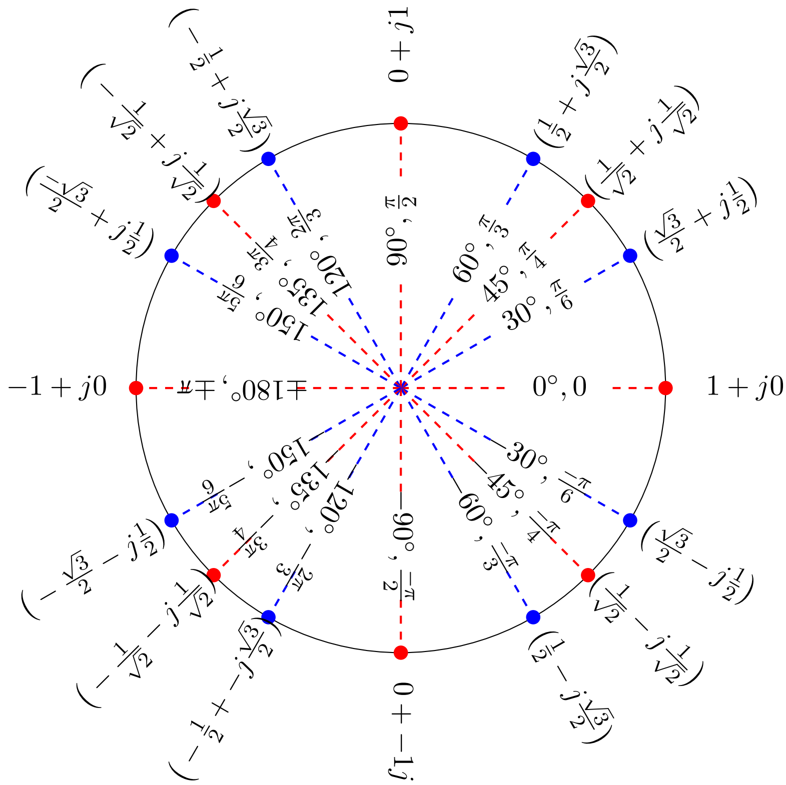


Figure 1.3: Unit circle showing the Cartesian and polar representation of commonly encountered complex numbers

Example 1.3.1. *It is very useful to know the polar form for often used complex numbers such as* $1, j, -j, -1$. *They are given by*

$$\boxed{1 = e^{j0}, -1 = e^{j\pi}, j = e^{j\frac{\pi}{2}}, -j = e^{-j\frac{\pi}{2}}}.$$

Example 1.3.2. *Let* $z = 2e^{j\frac{\pi}{4}}$. *Find the magnitude and angle of* z. *The answer is* $|z| = 2$ *and* $\theta = \pi/4$. *Many students will first convert* z *to Cartesian form, write it as* $z = \sqrt{2} + j\sqrt{2}$

and then compute $|z|$ *as* $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$*. While this is not wrong, this is time consuming and misses the point. You should train yourself to see that* z *is already given in polar form, i.e., as* $re^{j\theta}$ *with* $r = 2$ *and* $\theta = \pi/4$*. Since* r *represents the magnitude of* z*, you can directly read off the magnitude as 2 and angle as* $\pi/4$*. In my experience, students have difficulty with this sometimes well into the course.*

Example 1.3.3. *Let* $z = -2e^{j\frac{\pi}{4}}$*. Find* $|z|$ *and* $\angle z$*.*

The point of this example is to emphasize that the magnitude of a complex number r *must be positive. It would be wrong to say that* $|z| = -2$*. We must rewrite* z *as* $z = 2e^{j\pi}e^{j\frac{\pi}{4}} = 2e^{j\frac{5\pi}{4}}$ *and interpret* r *as* 2 *instead of* -2 *and the angle as* $\pi + \pi/4 = 5\pi/4$*.*

Caution: The expression for θ in (1.5) does not identify θ uniquely, since $\tan(\theta) = \frac{y}{x}$ also implies that $\tan(\theta \pm \pi) = \frac{y}{x}$. It is best think of the vector (x, y) and determine which quadrant this vector lies in based on the signs of x, y and then make sure θ corresponds to an angle in the correct quadrant.

Example 1.3.4. *Suppose* $z_1 = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ *and* $z_2 = -\frac{\sqrt{3}}{2} - j\frac{1}{2}$*. It is easy to see that* $\tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$*. However,* z_1 *is complex number in the first quadrant, whereas* z_2 *is a complex number in the 3rd quadrant. Therefore,* θ_1 *should be* $\pi/6$ *and* θ_2 *should be* $7\pi/6$*.*

Periodicity of complex exponentials

One important aspect of the polar form for a complex number is that adding 2π to the angle does not change the complex number. Particularly,

$$re^{j\theta} = re^{j(\theta + 2k\pi)} \quad \text{for any integer } k$$

This fact will be repeatedly used in the course. An immediate example of where this is useful is given in Section 1.3.12.

Example 1.3.5. *Express* $e^{j2\pi}, e^{-j\pi}, e^{j\frac{3\pi}{2}}, e^{j\frac{9\pi}{2}}, e^{j\pi}$ *in Cartesian form.*

1.3.6 Conjugate - (Video)

The conjugate of a complex number $z = x + jy$ is given by $z^* = x - jy$. When z is written in polar form as $z = re^{j\theta}$, the complex conjugate is given by $z^* = re^{-j\theta}$. In general, to compute the conjugate of a complex number, replace j by $-j$ everywhere.

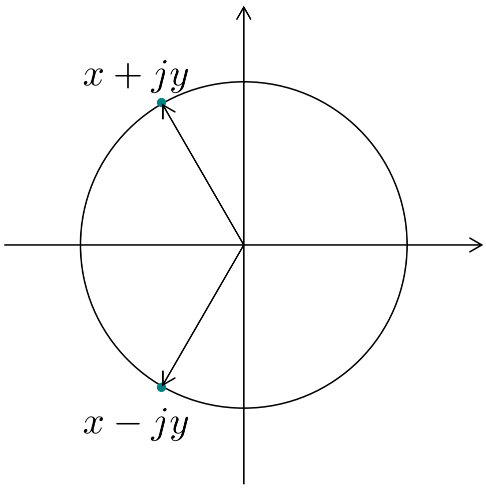


Figure 1.4: Conjugate of a complex number

1.3.7 Arithmetic with two complex numbers - (Video)

Let $z = x + jy = re^{j\theta}$, $z_1 = x_1 + jy_1 = r_1e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2e^{j\theta_2}$. We can perform arithmetic operations on complex numbers such as addition, subtraction, multiplication, division, exponentiation (raising a complex number to powers) etc by treating j as a real constant and then simplifying or reducing the result using

$$j^2 = -1,\ j^3 = -j,\ j^4 = 1,\ j^5 = j,\ \ldots.$$

The real and imaginary parts of the result of the computation are grouped and simplified together to yield a complex number in Cartesian form as the end result. While it is relatively easy to express the results of addition, subtraction, multiplication, and raising a complex number to integer powers in Cartesian form, a slightly more involved computation is required in order to express the result of division in Cartesian form. The following examples will make this more concrete

Example 1.3.6. *Let $z_1 = 1 + j$ and $z_2 = 2 + 3j$. Compute $z_1 + z_2$, $z_1 - z_2$, and z_1z_2 and express the result in Cartesian form.*

$$\begin{aligned}
z_1 + z_2 &= (1 + j) + (2 + 3j) = (1 + 2) + (j + 3j) = 3 + j(1 + 3) = 3 + 4j \\
z_1 - z_2 &= (1 + j) - (2 + 3j) = (1 - 2) + (j - 3j) = -1 + j(1 - 3) = -1 - 2j \\
z_1z_2 &= (1 + j)(2 + 3j) = 1 \cdot 2 + 1 \cdot 3j + j \cdot 2 + j \cdot (3j) = 2 + 3j + 2j + 3\underbrace{j^2}_{=-1} \\
&= (2 - 3) + j(3 + 2) = -1 + 5j
\end{aligned}$$

A special case of multiplying two complex numbers is multiplying a complex number z with its conjugate z^*. We can see that zz^* is given by

$$zz^* = (x + jy)(x - jy) = x^2 + jyx - jxy - j^2y^2 = (x^2 + y^2) + j(yx - xy) = x^2 + y^2 = |z|^2.$$

If the complex number is specified in polar form $z = re^{j\theta}$, then zz^* can be computed as

$$zz^* = re^{j\theta}re^{-j\theta} = r^2e^{j\theta}e^{-j\theta} = r^2 = |z|^2.$$

Therefore, multiplying z by its conjugate, z^* always results in $|z|^2$, which is a *real* number. This is a useful fact to keep in mind.

Example 1.3.7. *Let $z_1 = 1 + j$ and $z_2 = 2 + 3j$. Compute z_1/z_2 and express the result in Cartesian form.*

We begin by noticing that $\frac{z_1}{z_2} = \frac{1+j}{2+3j}$ and that it is not directly in Cartesian form. To convert this to Cartesian form, we multiply the numerator and the denominator by the conjugate of z_2 yielding

$$\frac{z_1}{z_2} = \frac{1+j}{2+3j} = \frac{(1+j)(2-3j)}{(2+3j)(2-3j)} = \frac{5-j}{2^2+3^2} = \frac{5}{13} - j\frac{1}{13}.$$

Arithmetic with complex numbers

Building on these examples, the student should verify the more general results given below.

$$\begin{aligned}
z_1 \pm z_2 &= (x_1 \pm x_2) + j(y_1 \pm y_2) && (1.13)\\
z_1 z_2 &= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) = r_1r_2e^{j(\theta_1+\theta_2)} && (1.14)\\
|z| &= \sqrt{zz^*} = r\\
\frac{z_1}{z_2} &= \frac{(x_1 + jy_1)}{(x_2 + jy_2)} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}
\end{aligned}$$

Typically, adding and subtracting complex numbers will be easier when the Cartesian form is used whereas, multiplication and division will be easier when the polar form is used. It will be important to be able to use both these representations in order to make computations easier.

1.3.8 Geometric interpretation of arithmetic operations - (Video, Python notebook)

It is often useful to understand geometrically what we are doing when we are adding, subtracting, multiplying or dividing two complex numbers. If we think of complex numbers are two-dimensional vectors, then adding and subtracting two complex numbers is the same as adding and subtracting vectors. Standard geometrical interpretations such as the parallelogram method or tip-to-toe method are useful to gain visual insight. However, multiplying and dividing vectors are typically not valid or well-defined operations. Multiplying a complex number $z_1 = r_1e^{j\theta_1}$ by $z_2 = r_2ie^{j\theta_2}$ corresponds to doing two operations to z_1 - we

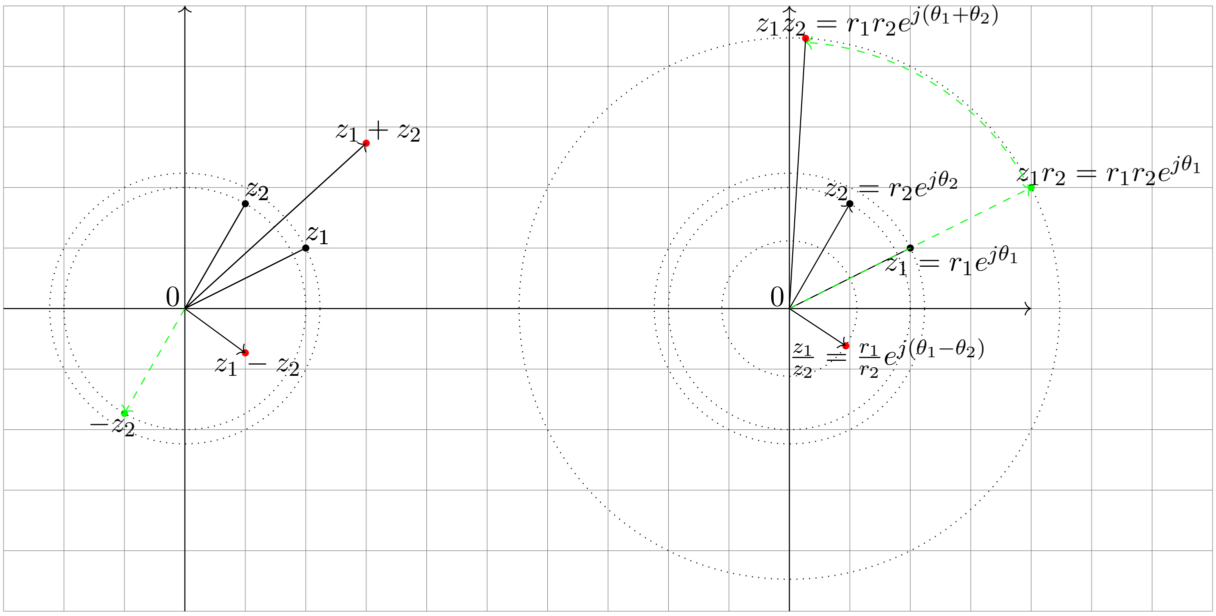


Figure 1.5: What are we doing geometrically when we are adding, subtracting, multiplying and dividing complex numbers? Left panel show addition and subtraction of complex numbers. Right panel shows multiplication and division of complex numbers. $z_1 z_2$ is obtained by first scaling z_1 by r_2 and then rotating the result by θ_2 counter-clockwise.

scale the magnitude of z_1 by r_2 and then rotate the result by θ_2 in the counter-clockwise direction. Similarly, dividing z_1 by z_2 corresponds to scaling the magnitude of z_1 by $1/r_2$ and rotating the result by θ_2 in the clockwise direction. These are illustrated in Fig. 1.5.

1.3.9 More properties of complex numbers

The student should verify that these properties hold for complex numbers.

$$
\begin{aligned}
(z_1 + z_2)^* &= z_1^* + z_2^* \\
(z_1 \ z_2)^* &= z_1^* \ z_2^* \\
\left(\frac{z_1}{z_2}\right)^* &= \frac{z_1^*}{z_2^*} \\
|z_1 - z_2| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
|z_1 z_2| &= |z_1||z_2| = r_1 r_2 \\
\left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \\
z z^* &= x^2 + y^2 = r^2 = |z|^2 \\
z + z^* &= 2\Re\{z\} \\
z - z^* &= 2j\Im\{z\}
\end{aligned}
$$

1.3.10 Fields, Fundamental theorem of algebra, and algebraic closure

We can add, subtract, multiply, and divide two complex numbers and the result will be another complex number. Further, there are identities for addition and multiplication of complex numbers, i.e.. $0 + z = z$ and $(1 + j0)z = z$ for any complex number z. For every complex number z, there is another complex number $1/z$ which is the inverse of z. Formally, in mathematics, we call such a set of numbers with the associated addition and multiplication operations as a field. Thus complex numbers form a field, typically denoted by $\mathbb{C}$.

There are many fields. For example, the set of real numbers with conventional addition and multiplication form a field, the set of rational numbers is also a field. However, there is something special about the field of complex numbers. The fundamental theorem of algebra states that every non-zero, single-variable, degree-n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots. Such a statement is not true for real numbers or rational numbers. For example, $x^2+1 = 0$ is polynomial equation with real coefficients; but it's roots are not real. However, with complex numbers, we are guaranteed that every degree-n equation has roots within the complex field. Because of this, we say that complex numbers are algebraically closed. Numberphile's video on the fundamental theorem of algebra has more details.

The fact that we can add, subtract, multiply, divide, invert complex numbers and solve equations with coefficients and stay within the complex field makes the complex number system quite rich in what we can do with complex numbers. Hence, it is used a lot in mathematics and engineering.

1.3.11 n^{th} power of a complex number - (Video 1)

Let $z_0 = x_0 + jy_0 = r_0 e^{j\theta_0}$. For any integer n, the nth power of z, z^n is simply obtained by using (1.14) n times. In the polar form, $z_0^n = r_0^n e^{jn\theta_0}$. Just like how the two real numbers 1 and -1 have the same square, different complex numbers can have the same nth power.

Consider the set of distinct complex numbers $z_k = e^{j\theta_0 + \frac{2\pi(k-1)}{n}}$. All the z_ks are different have the same nth power for $k = 1, 2, \ldots, n$. We can see this by raising z_k to the nth power to get

$$z_k^n = \left(e^{j\theta_0 + \frac{2\pi(k-1)}{n}}\right)^n = e^{jn\theta_0 + 2\pi(k-1)} = e^{jn\theta_0}. \tag{1.15}$$

1.3.12 n^{th} roots of a complex number - (Video 1, Video 2, Python notebook)

The nth root of z_0 is a bit more interesting and tricky. Any complex number z which is the solution to the nth degree equation

$$z^n - z_0 = 0$$

is an nth root of z_0. The fundamental theorem of algebra states that an nth degree equation has exactly n (possibly complex and possibly repeated) roots. Hence, every complex number

z_0 has exactly n, nth roots. Let $z_1, z_2, \ldots, z_n$ denote the n roots. These roots can be found using the fact $e^{j\theta} = e^{j(\theta+2\pi k)}$. The kth root $z_k = r_k e^{j\theta_k}$ can be found by setting

$$\begin{aligned} z_k^n = z_0 \quad &\Rightarrow \quad r_k^n e_k^{jn\theta} = r_0 e^{j\theta_0} = r_0 e^{j(\theta_0+2k\pi)} \\ &\Rightarrow \quad r_k = \sqrt[n]{r_0}, \quad \theta_k = \frac{\theta_0 + 2\pi(k-1)}{n} \text{ for } k = 1, 2, \ldots n \end{aligned} \tag{1.16}$$

Here $\sqrt[n]{r_0}$ refers to the positive nth root. Note that for all the k roots, the magnitude is the same whereas the angles are different. Clearly, computing nth roots is much easier in the polar form than in the Cartesian form.

Example 1.3.8. *Find the third roots of unity* $\sqrt[3]{1}$

Since $1 = 1e^{j0}$, *this corresponds to* $r_0 = 1, \theta_0 = 0$. *Hence, the three roots of unity are given by*

$$r = 1, \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

In cartesian coordinates, they are $(1 + j0)$, $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$. *These are referred to as* $1, \omega, \omega^2$ *sometimes. The three roots are shown in Figure 1.6.*

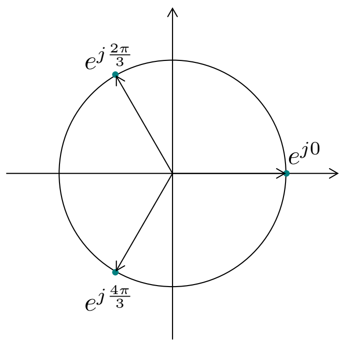


Figure 1.6: Cube roots of unity

1.3.13 Functions of a complex variable - (Video)

Let $f(z)$ be a complex function of a complex variable z, i.e., for every z, $f(z)$ is a complex number. Note that a real number is also considered as a complex number and, hence, $f(z)$ could have a zero imaginary part. Examples of functions include $f(z) = |z|, f(z) = \arg(z), f(z) = z^n, f(z) = \exp(z)$, etc. The exponential functions can be interpreted using Euler's identity as follows.

$$f(z) = \exp(z) = e^x e^{jy} = e^x \cos y + je^x \sin y$$

The logarithm of a complex number $\ln z$ can be also interpreted using Euler's identity as

$$\ln(z) = \ln\left(re^{j(\theta+2k\pi)}\right) = \ln r + j(\theta + 2k\pi).$$

It can be seen from the above expression that $\ln z$ is not a function (i.e., there are many possible values of $\ln z$ for a given value of iz). However, if we set $k = 0$ in the above expression, then we get what is called the principal value of $\ln z$, denoted by Ln z, which is a function.

Similarly, it is important to realize that for any integer n, the nth power of a complex number is a function of the complex number, i.e., for every complex number z, there is only one complex number z^n. However, for an integer n, the nth root of a complex number is not uniquely defined and hence, is not a function. Often, one may take the root corresponding to $k = 0$ in (1.16) as the default root and hence the principal value. Then, the principal value becomes a function. This is similar to square roots of positive real numbers being defined as the positive numbers. There are interesting examples where careless use of just the principal value as the nth root can lead to fallacious arguments.

1.3.14 Complex functions of a real variable - (Video, Python notebook)

You may be used to dealing with functions of a variable such as $y = f(x)$, where x is called the independent variable and y is called the dependent variable and typically, y takes real values when x takes real values. In this course, we will be interested in complex functions of a real variable. Often the real variable will represent time or frequency. Such a function, normally denoted as $x(t)$ or $X(\omega)$ is a function which takes a complex value for every real value of the independent variable t or ω. Pay attention to the notation carefully - t or ω now becomes the independent variable and $x(t)$ or $X(\omega)$ now becomes the dependent variable. We can also think of the complex function as the combination of two real functions of the independent variable, one for the real part of $x(t)$ and one for the imaginary part of $x(t)$.

When dealing with real functions of a real variable, you may be used to plotting the function $x(t)$ as a function of t. However, when $x(t)$ is a complex function, there is a problem in plotting this function since for every value of t, we need to plot a complex number. In this case, we do one of two things - either we plot the real part of $x(t)$ versus t and plot the imaginary part of $x(t)$ versus t, or we plot $|x(t)|$ versus t and $\arg(x(t))$ versus t. Either of these is fine, but we do need two plots to effectively understand how $x(t)$ changes with t.

Example 1.3.9. *Consider the function $x(t) = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$ for all real values of t. This is clearly a complex function of a real variable t. $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, arg(x(t))$ are all real functions of the real variable t. Hence, we can plot $\Re\{x(t)\}$ versus t and $\Im\{x(t)\}$ versus t or we can plot $|x(t)|$ versus t and $\angle x(t)$ versus t as shown in Fig. 1.7*

Example 1.3.10. *Consider the function $x(t) = e^{st}$ where $s = \sigma + j\omega$ is some complex number. This is a complex function of a real variable t. The real part and imaginary part of $x(t)$ are each real functions of a real variable t and can be obtained as follows. Notice that $x(t)$ can be written as*

$$
\begin{aligned}
x(t) &= e^{\sigma+j\omega}t = e^{\sigma t}e^{j\omega t} \\
&= e^{\sigma t}\left(\cos(\omega t) + j\sin(\omega t)\right) \\
&= \underbrace{e^{\sigma t}\cos(\omega t)}_{\text{Real part}} + j\underbrace{e^{\sigma t}\sin(\omega t)}_{\text{imaginary part}}
\end{aligned}
$$

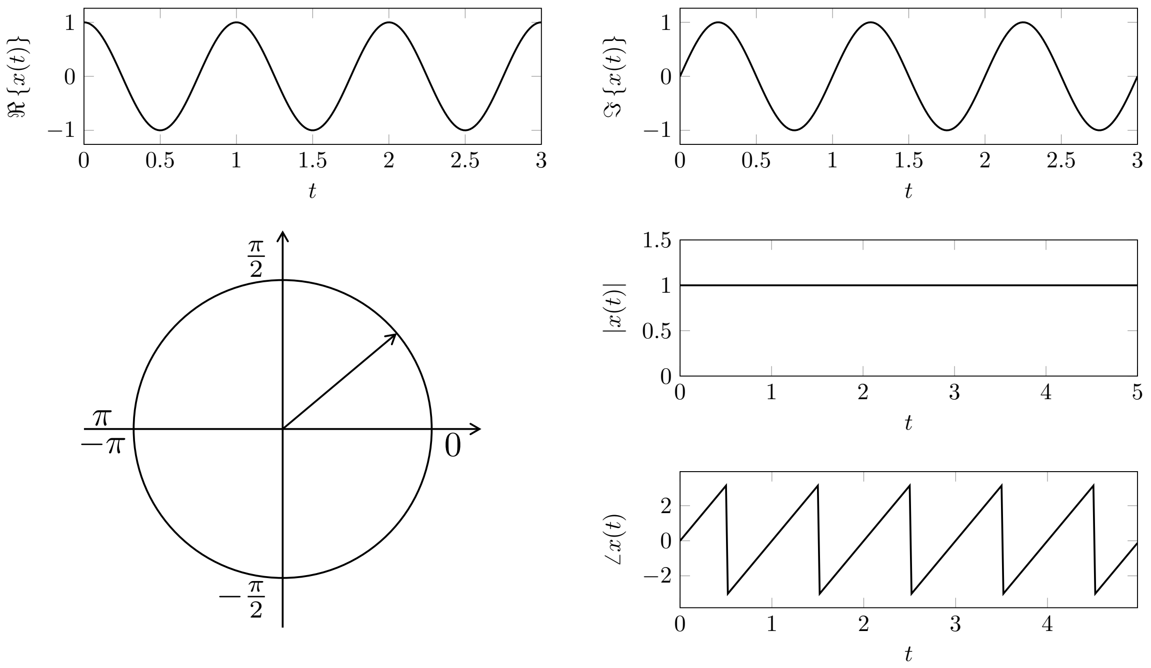


Figure 1.7: Plot of $\Re\{x(t)\}, \Im\{x(t)\}, |x(t)|, \angle x(t)$ versus t for $x(t) = e^{j2\pi t}$.

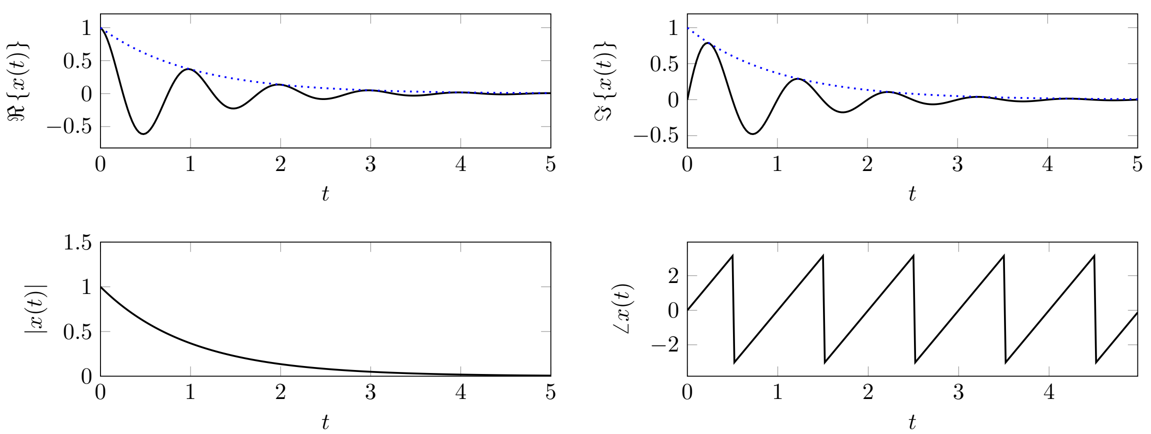


Figure 1.8: Plots of $\Re x(t)$ and $\Im x(t)$ versus t and $|x(t)|$ and $\angle x(t)$ versus t.

The magnitude of $x(t)$ and phase of $x(t)$ are both functions of t given by

$$\begin{aligned} |x(t)| &= e^{\sigma t} \\ \angle x(t) &= \omega t \end{aligned}$$

These are plotted in Fig. 1.8 for $\sigma = -1$ and $\omega = 2\pi$. The dotted blue lines in the figures show $|x(t)|$ and it can be seen that it defines the envelope of the real and imaginary parts of $x(t)$.

It is useful to get insight into what happens to $x(t) = e^{st}$ as $t \to \infty$. It can be seen that if $\sigma < 0$, $x(t) \to 0$ as $t \to \infty$ and if $\sigma > 0$, $x(t)$ becomes undefined at $t \to \infty$ with the $|x(t)| \to \infty$ as $t \to \infty$. Thus, the real part of s determines whether the function $x(t)$ is bounded or unbounded as $t \to \infty$.

Example 1.3.11. *Consider the function* $H(\omega) = \frac{1}{1+j\omega}$*, where* ω *is a real variable. Roughly sketch the magnitude and phase of* $H(\omega)$ *as a function of* ω*.*

$$
\begin{aligned}
H(\omega) &= \frac{1}{1+j\omega} \\
|H(\omega)| &= \frac{1}{\sqrt{1+\omega^2}} \\
\angle H(\omega) &= 0 - \tan^{-1}\omega
\end{aligned}
$$

A plot of $|H(\omega)|$ *versus* ω *and* $\angle(H(\omega))$ *versus* ω *is shown in Fig. 1.9.*

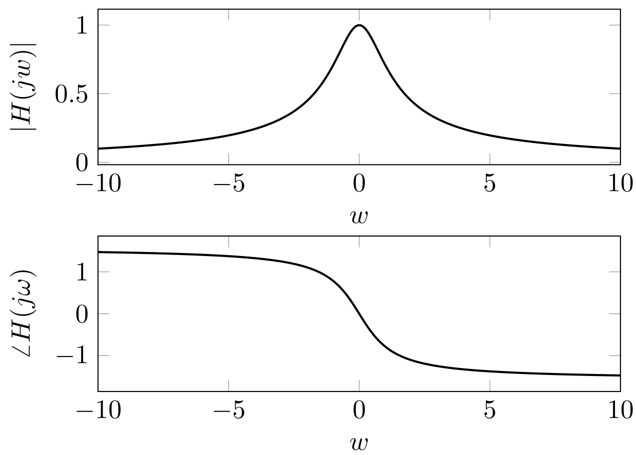


Figure 1.9: Plot of $H(\omega)$ vs ω and $\angle H(\omega)$ versus ω for $H(\omega) = \frac{1}{1+j\omega}$.

Example 1.3.12. *Consider the function* $X(\omega) = \frac{j\omega}{1+j\omega}$*, where* ω *is a real variable. Roughly sketch the magnitude and phase of* $X(\omega)$ *as a function of* ω*.*

$$
\begin{aligned}
X(\omega) &= \frac{j\omega}{1+j\omega} \\
|X(\omega)| &= \frac{|\omega|}{\sqrt{1+\omega^2}} \\
\angle(X(\omega)) &= \begin{cases} -\frac{\pi}{2} - \tan^{-1}\omega & ,\omega < 0 \\ \frac{\pi}{2} - \tan^{-1}\omega & ,\omega > 0 \end{cases}
\end{aligned}
$$

A plot of $|X(\omega)|$ *versus* ω *and* $\angle(X(\omega))$ *versus* ω *is shown in Fig. 1.10*

1.3.15 Plotting the magnitude and phase of $H(\omega) = e^{ja_1\omega} + e^{ja_2\omega}$ vs ω - (Video)

One of the tricks that is useful to get some insight into plotting the magnitude and phase of functions of the form $H(j\omega) = e^{ja_1\omega} + e^{ja_2\omega}$ vs ω is to express $e^{ja_1\omega} + e^{ja_2\omega}$ as follows

$$
\begin{aligned}
e^{ja_1\omega} + e^{ja_2\omega} &= e^{j\left(\frac{a_1+a_2}{2}\omega\right)}\left[e^{j\left(\frac{a_1-a_2}{2}\right)\omega} + e^{-j\left(\frac{a_1-a_2}{2}\right)\omega}\right] \qquad (1.17) \\
&= e^{j\left(\frac{a_1+a_2}{2}\omega\right)} 2\cos\left[\left(\frac{a_1-a_2}{2}\right)\omega\right]
\end{aligned}
$$

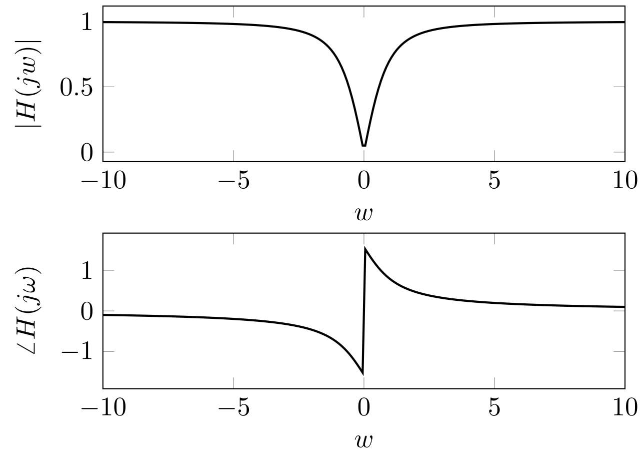


Figure 1.10: Plot of $X(\omega)$ vs ω and $\angle X(\omega)$ versus ω for $X(\omega) = \frac{j\omega}{1+j\omega}$.

Now, it is easy to see that $|H(\omega)| = 2\left|e^{j\left(\frac{a_1+a_2}{2}\omega\right)}\right| \cdot \left|\cos\left[\left(\frac{a_1-a_2}{2}\right)\omega\right]\right|$ which is simply $2\left|\cos\left[\left(\frac{a_1-a_2}{2}\right)\omega\right]\right|$.

Example 1.3.13. *Compute the magnitude and phase of $H(\omega) = e^{j\omega} + e^{j3\omega}$ and determine the values of ω for which $H(\omega) = 0$.*

$$H(\omega) = e^{j\omega} + e^{j3\omega} = e^{j2\omega}\left(e^{-j\omega} + e^{j\omega}\right) \tag{1.18}$$

$$|H(\omega)| = 2\cos(\omega) \tag{1.19}$$

$$\angle H(\omega) = 2\omega\ sign(\cos\omega) \tag{1.20}$$

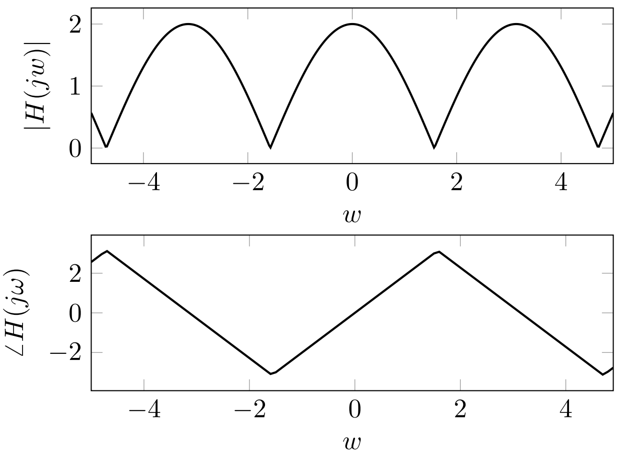


Figure 1.11: Plot of $|H(j\omega)|$ vs ω and $\angle H(j\omega)$ versus ω for $H(j\omega) = H(j\omega) = e^{j\omega} + e^{j3\omega}$.

The advantage computing the magnitude in this way lies in making it easy to compute values of ω for which $H(\omega) = 0$. The values of ω for which $|H(\omega)| = 0$ which are simply given by the values of ω for which $2\cos(\omega) = 0$. These are given by odd multiples of $\pi/2$, i.e., $\omega = (2i+1)\pi/2$ for any integer i.

1.4 Integrals of complex functions and integration by Parts - (Video 1, Video 2, Python notebook)

In this class, we will encounter integrals of complex functions of a real variable integrated with respect to the real variable. Let $x(t) = x_R(t) + jx_I(t)$ be a complex function of a real variable t. The integral can be evaluated as the sum of $x_R(t)$ and $jx_I(t)$, where $x_R(t)$ and $x_I(t)$ are real functions of a real variable. The following result is true

$$\int_a^b x(t)\ dt = \int_a^b x_R(t)\ dt + j\int_a^b x_I(t)\ dt \tag{1.21}$$

However, this approach will be very cumbersome in many cases as it will not be easy to split the integral into real and imaginary parts separately. It will be easier to directly integrate the complex function. All the rules of integration of real functions of real variables apply directly and j can simply be treated as a constant. In this class, we are typically interested in time t or frequency ω being the independent variable. Hence, the integrals will be with respect to t or ω.

For example,

Example 1.4.1. *Evaluate* $\int_0^{\pi/4} e^{j2t}\ dt$

$$\int_0^{\pi/4} e^{j2t}\ dt = \left[\frac{e^{j2t}}{2j}\right]_0^{\pi/4} = \frac{j-1}{2j}$$

Example 1.4.2. *Evaluate* $\int_0^{\infty} e^{(-2+j3\pi)t}\ dt$

$$\begin{aligned}\int_0^{\infty} e^{(-2+j3\pi)t}\ dt &= \left[\frac{e^{(-2+j3\pi)t}}{-2+j3\pi}\right]_0^{\infty}\\ &= 0 - \frac{1}{-2+j3\pi} = \frac{1}{2-j3\pi}\end{aligned}$$

Sometimes we will have to use Integration by parts to evaluate integrals. The main result to recall is

$$\int_a^b u(t)\ dv(t)\ dt = [u(t)\ v(t)]_a^b - \int_a^b v(t)\ du(t)\ dt \tag{1.22}$$

Example 1.4.3. *Evaluate* $\int_0^1 te^{-j\omega t}dt$*. Your answer should clearly be a function of* ω

We choose

$$\begin{aligned} u(t) = t \quad &, \quad dv(t) = e^{-j\omega t}\ dt\\ \implies du(t) = dt \quad &, \quad v(t) = \frac{e^{-j\omega t}}{-j\omega}\end{aligned}$$

$$\begin{aligned}\int_0^1 te^{-j\omega t}dt &= \left[\frac{te^{-j\omega t}}{-j\omega}\right]_0^1 - \int_0^1 \frac{e^{-j\omega t}}{-j\omega}\ dt\\ &= -e^{-j\omega} - \left[\frac{e^{-j\omega t}}{-\omega^2}\right]_0^1 = -e^{-j\omega} + \frac{e^{-j\omega}-1}{\omega^2}\end{aligned}$$

In some cases, when we need to evaluate $\int x_R(t)\ dt$ or $\int x_I(t)\ dt$, it may be easier to think of an $x(t)$ whose real part is $x_R(t)$ or imaginary part is $x_I(t)$ and to integrate $x(t)$ first and then take the real part or the imaginary part. We can then make use of the fact that in (1.21),

Integral of the real (imaginary) part is the real (imaginary) part of the integral

$$\begin{aligned}\int x_R(t)\ dt &= \Re\left\{\int x(t)\ dt\right\}\\ \int x_I(t)\ dt &= \Im\left\{\int x(t)\ dt\right\}\end{aligned}$$

The following example will highlight this

Example 1.4.4. *Suppose we want to evaluate* $\int e^{-at}\cos(bt)\ dt$ *where* a *and* b *are real numbers.*

One way to evaluate this integral is to apply integration by parts. There is an easier way using complex functions. From Euler's identity, we know that $\cos(bt) = \Re\{e^{jbt}\}$ *and hence,* $e^{-at}\cos(bt) = \Re\{e^{(-a+jb)t}\}$. *Since*

$$\int e^{-at}\cos(bt)\ dt = \int \Re\left\{e^{(-a+jb)t}\ dt\right\} = \Re\left\{\int e^{(-a+jb)t}\ dt\right\}$$

the easier approach is to evaluate $\int e^{(-a+jb)t}\ dt$ *and then just take the real part of the result. Since* $\int e^{(-a+jb)t}\ dt = \frac{e^{(-a+jb)t}}{-a+jb}$, *the final result is* $\Re\left\{\frac{e^{(-a+jb)t}}{-a+jb}\right\}$. *Therefore,*

$$\int e^{-at}\cos(bt)\ dt = \Re\left\{\frac{e^{(-a+jb)t}}{-a+jb}\right\} = \frac{-ae^{-at}\cos(bt) + be^{-at}\sin(bt)}{a^2+b^2}$$

1.5 Practice Problems - (Video solutions)

1. Express these numbers in Cartesian form

   a) $2e^{j\pi/4}$

   b) $(1+j)(2-3j)$

   c) $\frac{2}{e^{j\pi/2}}$

   d) $e^{j\pi/3} + e^{-j\pi/3}$

   e) $e^{j\pi/4} + e^{j3\pi/4}$

2. Express these numbers in polar form

   a) $1+j$,

   b) $\frac{1}{2} + j\frac{\sqrt{3}}{2}$, $\frac{1}{2} - j\frac{\sqrt{3}}{2}$, $-\frac{1}{2} + j\frac{\sqrt{3}}{2}$, $-\frac{1}{2} - j\frac{\sqrt{3}}{2}$

   c) $(1-j)\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$

d) $\frac{1+j}{2j}$

e) $e^{j\pi/4} + e^{j3\pi/4}$

f) $1 + e^{j\pi/4}$

3. Compute the magnitude and phase of the following complex numbers. Do not explicitly compute the numbers in Cartesian form unless it is required. Try to compute the magnitude and phase using what you know about the magnitude and phase of products of complex numbers.

a) $(1-j)\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$

b) $e^{j\pi/2}(1+j)(1+\sqrt{3}j)$

c) $je^{j\pi/3}$

d) $e^{j\pi/4} + e^{j3\pi/4}$

e) $(\sqrt{3}+j)^2$

f) $(1-\sqrt{3}j)/(1+\sqrt{3}j)^2$

g) $e^{j\pi/5} \times e^{j2\pi/5} \times e^{j3\pi/5} \ldots e^{j9\pi/5}$

h) $e^{j\pi/5} \times e^{j2\pi/5} \times e^{j3\pi/5} \ldots e^{j9\pi/5} \times e^{j10\pi/5}$

4. Let $z_1 = 1, z_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}, z_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

a) What are z_1^3, z_2^3 and z_3^3?

b) Show that $z_3 = z_2^2$

c) Show that $z_1 + z_2 + z_3 = 0$

Can you now see why z_1, z_2, z_3 can be called the cube roots of unity. They are usually expressed as $1, \omega, \omega^2$. Part c shows that the sum of the cube roots of unity is zero. In one of the homework problems, we will show that this true for nth roots of unity for any n.

5. Let $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$. Find and express your answer in Cartesian and polar form

a) $2z_1 - z_2$

b) $\frac{1}{z_1}$

c) $\frac{z_1}{z_2^2}$

d) $\sqrt[3]{z_2}$

6. Let z be any complex number. Is it true that $(e^z)^\star = e^{z^\star}$?

7. Prove that
$$\int e^{ax}\ \cos(bx)\ dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right)$$
You can use integration tricks you learned in your calculus class to solve this problem. That is not the point of the exercise. Try using Euler's identity and then using integration of exponentials to see if you can solve the problem.

8. Plot the magnitude and phase of the function $X(f) = e^{j\pi f} + e^{j5\pi f}$, for $-1 \leq f \leq 1$.

1.5.1 References

A good online reference for complex numbers is the wiki page http://en.wikipedia.org/wiki/Complex_number.

1.6 Geometric Series - (Video, Python notebook)

A sequence of the form $a, ar, \ldots, ar^n, \ldots$, where a and r can be any *complex* number is called a geometric sequence. The first term in the sequence is a and the ratio of any two adjacent terms is r, which is called the common ratio. The partial sum S_N defined as

$$S_N := a + ar + ar^2 + \ldots + ar^{N-1} = \sum_{n=0}^{N-1} ar^n$$

is the sum of the first N terms of the sequence and is called a geometric series. Notice that the sum starts at ar^0 and goes up to ar^{N-1}. Notice that lowercase n is used as an index for the summation and uppercase N is the number of terms. It might seem annoying as to why I used lower case n and upper case N to mean two different things. Throughout the course, we will have to sum signals which are indexed by time and it is common to use n to represent a time index and N is commonly used to refer to the number of terms. Therefore, it is better to get used to this notation.

A nice formula for S_N can be obtained as follows

$$S_N = a + ar + ar^2 + \ldots + ar^{N-1} \tag{1.23}$$

$$rS_N = ar + ar^2 + \ldots + ar^{N-1} + ar^N \tag{1.24}$$

$$(1-r)S_N = a - ar^N = a(1-r^N) \tag{1.25}$$

From this, we can see that

$$\boxed{S_N = \sum_{n=0}^{N-1} ar^n = \begin{cases} \frac{a(1-r^N)}{1-r}, & r \neq 1; \\ aN, & r = 1. \end{cases}} \tag{1.26}$$

Using the same idea as before, the following general formula can be derived

$$\boxed{\text{For } N_2 > N_1, \sum_{n=N_1}^{N_2} r^n = \begin{cases} \frac{r^{N_1} - r^{N_2+1}}{1-r}, & r \neq 1; \\ N_2 - N_1 + 1, & r = 1. \end{cases}} \tag{1.27}$$

The above formula is valid for any $N_2 > N_1$ regardless of whether N_1, N_2 are positive, zero or negative. While the formula for the sum a geometric sequence is straight forward, students often have difficulty recollecting this from memory. You *must memorize* this formula !

The following special cases of (1.27) are typically encountered

$$\boxed{\sum_{n=0}^{N} r^n = \begin{cases} \frac{1-r^{N+1}}{1-r}, & r \neq 1; \\ N+1, & r = 1. \end{cases}} \tag{1.28}$$

Another special case of the above result is when $N_2 \to \infty$. In this case, the sum of infinite terms converges or diverges depending on whether $|r| < 1$ or $|r| > 1$.

$$\boxed{\sum_{n=N_1}^{\infty} r^n = \begin{cases} \frac{r^{N_1}}{1-r}, & |r| < 1; \\ \text{does not converge}, & |r| \geq 1. \end{cases}} \tag{1.29}$$

Example 1.6.1. *For example,* $1, \frac{1}{2}, \frac{1}{4}, \ldots$ *is an infinite geometric series with* $a = 1$, $r = \frac{1}{2}$.

$$S_{11} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \ldots + \frac{1}{1024} = \frac{1 - \frac{1}{2^{11}}}{1 - \frac{1}{2}}$$

You may have seen these before, but in this class often we will be interested in the case when a and/or r are complex numbers. Luckily, nothing changes from when a and b are just real numbers.

An additional difficulty students often face is in recognizing that a given sum is actually a sum of a geometric sequence and in properly identifying a and r. The practice problems in this section as well as the homework problems should give you some practice.

The following identity is also true, although we will not use this often in this class.

$$\sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}, \quad |r| < 1 \tag{1.30}$$

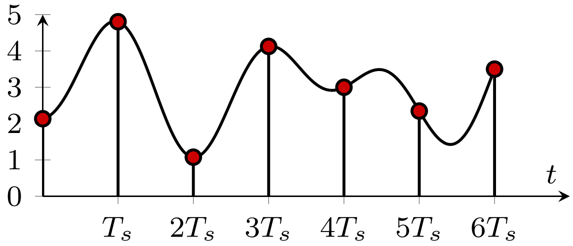
1.7 Practice Problems - (Video solutions)

1. Compute $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \ldots$
2. Compute $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$
3. Simplify $\sum_{n=2}^{9} 2^{3n} 3^{-2n}$
4. Compute $\sum_{n=2}^{\infty} 2^{3n} 3^{-2n}$
5. Compute $\sum_{n=-\infty}^{-2} 2^{-3n} 3^{2n}$. See if you can substitute $m = -n$ and obtain the expression in the previous problem
6. Simply $\sum_{n=2}^{\infty} x^n 3^{-n}$ an expression as a rational function of x. Evaluate this function for $x = 2$
7. Compute $\sum_{n=1}^{\infty} \cos^n(\pi t)$ and express as a function of t
8. Compute $\sum_{n=1}^{\infty} \frac{1}{2^n} \cos(n\pi t)$ and express the result as a function of t. Hint: Use Euler's formula to convert this sum of two geometric series.
9. Simplify $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n e^{j\omega n}$
10. Compute $e^{j\frac{\pi}{2}} + \frac{1}{2}e^{j\pi} + \frac{1}{4}e^{j\frac{3\pi}{2}} + \ldots + \frac{1}{2^9}e^{j\frac{10\pi}{2}}$. Simplify your answer into a complex number in Cartesian form
11. Prove the result in (1.28) and (1.29)
12. For any two given integers k and N, what is $\sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}}$?
13. Just for intellectual curiosity - Can you prove the results in (1.26) and (1.30)?

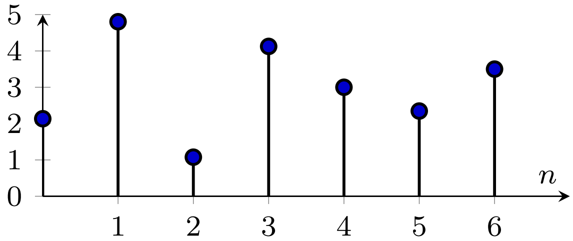
Chapter 2

Introduction to Signals

(a) CT signal $x(t)$



(b) DT signal $x[n] = x(nT_s)$



2.1 Introduction - what is a signal? - Python notebook

The word signal has several meanings in the English language and is used to denote various things in everyday conversation. In this course, we will use the term 'signal' to refer to a *function* of an independent variable that carries some information or describes some physical phenomenon. Often, the independent variable will be time and hence, the signal will describe some phenomenon or quantity that changes with time. In the second half of the course, the independent variable may denote frequency and hence, we will be studying quantities that change as a function of frequency. Occasionally, the independent variable may denote space or other things. The theory that we will develop in this course is valid, regardless of what the independent variable denotes. We will denote the independent variable time by t and the signal by $x(t)$ or $y(t)$ etc. Thus x is a function of t. In your calculus course, you are likely to have used x as the independent variable and $y = f(x)$ as the dependent variable. Please note that here x will often be the *dependent* variable in this course and get used to this new notation. Apart from this new notation, signals are fundamentally the same as functions that you have seen before.

Example 2.1.1. *Consider the scenario of a person singing into a microphone. The voltage at the output of the mic changes as a function of time and can be represented by the signal $x(t)$ as shown in Figure 2.1.*

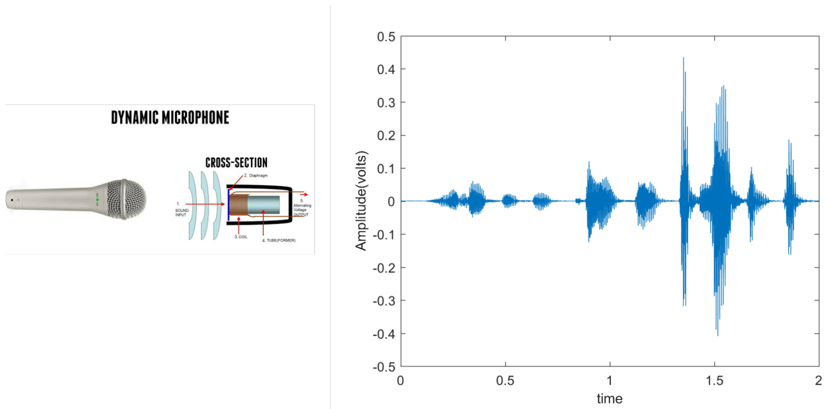


Figure 2.1: An audio signal which shows how the voltage across the microphone varies as a function time when a person speaks or sings.

2.2 Examples of signals

The following are some examples of signals that are commonly encountered

1. Audio signal (music files, speech waveforms) - voltage at the output of a microphone varies as a function of time.

2. Biomedical signals - electrocardiogram (ECG), EKG, etc - voltage at the output of a sensor varies as a function of time.

3. A signal transmitted from the antenna of your cellular phone/Wi-Fi - electric and magnetic fields vary as a function of time and/or space.

4. Closing price of a company's stock on each day - closing price varies as a function of time where the units is in days. This is also called as a time series.

5. Text prompts - a prompt that you use to ask a question to a chatbot and the response you get from the chatbot are both signals. The text prompt is often broken down in to tokens (for example, each word could be a token or each token could correspond to a part of a word). Each token takes a value between 1 and D where D is the total number of tokens. Thus, the text prompt we provide is converted to a time series of tokens, which is essentially a signal. ChatGPT uses a dictionary with roughly 50,000 tokens.

6. A photograph or image - intensity of reflected light varies as a function of spatial coordinates. This is a 2D signal.

7. Think about some signal that you are interested in and understand what is varying as a function of what.

2.3 Continuous-Time(CT) and Discrete-Time(DT) Signals - (Video)

2.3.1 Continuous-Time (CT) signals

We will encounter two kinds of signals in this course, namely continuous-time and discrete-time signals. As in Example 2.1.1, in some cases, the time variable t can change in a continuous manner and $x(t)$ can be defined for every t in some interval $[t_i, t_f]$. Such signals are called continuous-time signals. The signals in examples 1, 2, and 3 are CT signals. The photograph from a film camera can be considered as a continuous-space signal. If you don't understand why a film camera produces a continuous-space signal, ask your instructor. It is quite likely that you have never seen a film camera whereas your instructor grew up with one.

2.3.2 Discrete-Time (CT) signals

In contrast, in some cases, signals are defined only for *discrete* values of time. Often, we will normalize the time axis such that the independent variable is defined only for *integer* values of time. We will typically use n or k to denote the time variable and $x[n]$, $y[n]$, etc to denote discrete-time signals. The signal in example 4 is a discrete-time signal. A photograph from a digital camera would be a discrete-space signal.

Example 2.3.1. *If we want to construct a signal that shows how the maximum temperature in a day changes as a function of days in a year in College Station, TX, the independent variable would increase in increments of one day. Naturally, in this example, the time* *variable is defined only for integer values. Such a signal is an example of a discrete-time signal. An example of such as signal is shown in Figure 2.2.*

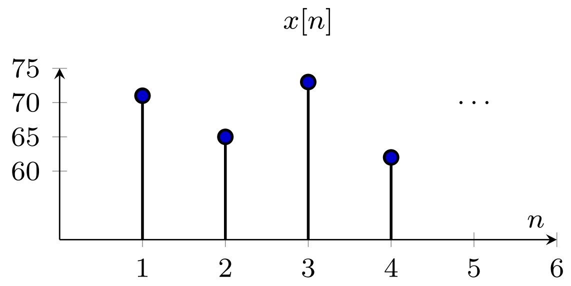


Figure 2.2: Example of a discrete time (DT) signal. Here the signal represents the maximum temperature in a day as function of days. As you can guess, this is likely to represent the month of January and not July.

2.3.3 Sampling

We will often obtain DT Signals by sampling CT signals as shown in Fig 2.3. Sampling means that we choose to keep or record the values of a continuous time signal only for some values of time called sampling instants. Often, these sampling instants are uniformly spaced, i.e., we take samples at time $t = 0, T_s, 2T_s, 3T_s, \ldots$ and T_s will be called the sampling duration. The resulting signal is a discrete-time signal. One of the remarkable results that we will study in this course is that even though we discarded the values of the CT signal except at the sampling instants, in many cases (of practical interest), we can perfectly recover the lost values. We will have to wait a bit longer to develop all the machinery that we need to understand this result. Sampling is an extremely important operation that acts as a bridge between the continuous-time world where many physical signals lie and the discrete-time world which is ideally suited for processing by modern digital computers. Thus, sampling enables processing of CT signals on digital computers vastly opening the possibilities for how we can analyze and process signals.

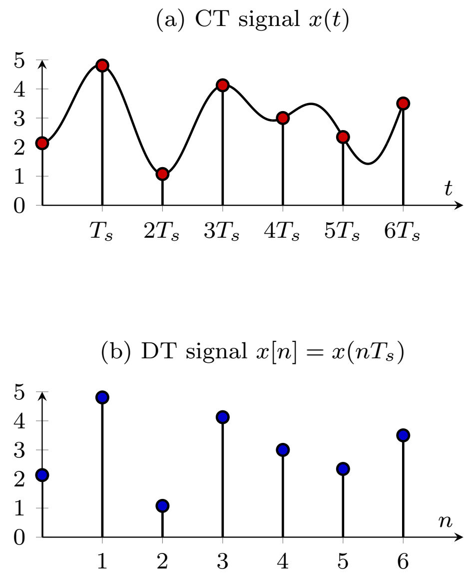


Figure 2.3: Discrete-time signal obtained by sampling a CT Signal every T_s seconds.

Example 2.3.2. *How does music get converted and stored as a file that you can stream on Spotify or iTunes?*

When music is produced, it is produced as a continuous-time signal. This signal is first sampled at the rate of 44.1 KHz, i.e., by taking 44100 samples/second. The sampled signal is a discrete-time signal i.e., the values are still real numbers. Then, we go through the process of quantization where each real number is represented by 16 bits. Sampling and quantization together are sometimes referred to as analog-to-digital or A/D conversion. I prefer to refer separately to sampling and quantization to be more precise.

The bit sequence is often compressed further to reduce the size at the loss of some quality. Different compression formats use different algorithms for compression and often provide parameters that can trade off file size for quality.

Example 2.3.3. *Approximately how many bits can a CD hold? A CD can hold 74 minutes of music recorded in stereo and sampled at 44.1 KHz. Further, they use 16-bits to represent each sample. So, how many bytes can a CD hold?*

2.4 How to specify or describe signals? - (Video)

Signals are typically described in a few different ways. Most commonly, an explicit mathematical formula will be given to describe the signal. Sometimes, a pictorial description of the signal may be given, i.e., the signal will be plotted without giving an explicit mathematical description. It will be useful to go back and forth between these representations, i.e., you should be able to roughly sketch a given signal and you should be able to write a mathematical description of a signal from a plot. Here are some examples.

2.4.1 Mathematical description

An example of providing a mathematical description is to say that $x(t) = \sin(2\pi t)$. This specifies the value taken by x for every value of t. In general, it is good practice to clearly specify the mathematical description of the signal and the range of values of time for which the description holds. For e.g. consider these three signals

$$\begin{aligned} x_1(t) &= \sin(2\pi t), \forall t \\ x_2(t) &= \begin{cases} \sin(2\pi t), & t \geq 0; \\ 0, & \text{otherwise.} \end{cases} \\ x_3(t) &= \sin(2\pi t), t \geq 0 \end{aligned}$$

These three signals are different signals and often, students are not careful in distinguishing them. The first signal $x_1(t)$ takes the form $\sin(2\pi t)$ for all values of t, $x_2(t)$ takes the form $\sin(2\pi t)$ only for $t \geq 0$ and is zero for $t < 0$, whereas $x_3(t)$ takes the form $\sin(2\pi t)$ for $t \geq 0$ and we do not know what it is for $t < 0$, i.e., it is undefined for $t < 0$. If the range of values of t for which a description holds is not specified, we will assume that the description is true for all values of t, i.e., if someone says consider the signal $x(t) = 4$, that would mean the signal which is constant for *all* values of time and takes the value 4.

Another issue that is potentially confusing is that typically $x(t)$ is used to denote two things - the signal $x(t)$, by which we mean the entire signal, and it also refers to the value of the signal $x(t)$ at time t. Sometimes, this language is confusing and you must pay attention to how this is being used. If this is confusing, talk to me and get your thoughts clarified. After a few times, you will get the hang of it.

Notation for CT and DT signals

$x(t)$ will denote continuous-time signals and $x[n]$ will denote discrete-time signals.

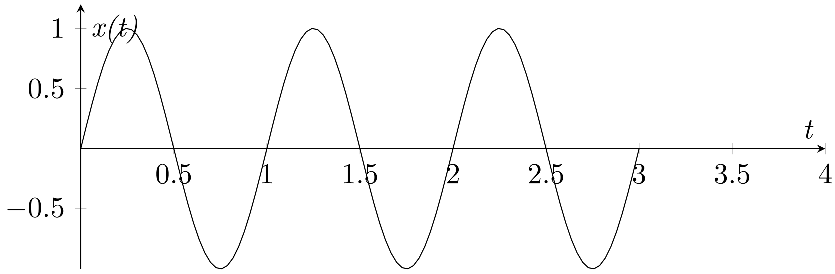


Figure 2.4: Sketch of the CT signal $x(t) = \sin(2\pi t), 0 \leq t \leq 3$.

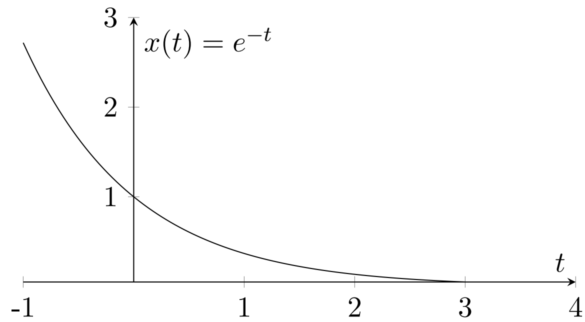


Figure 2.5: Sketch of the CT signal $x(t) = e^{-t}, t \geq -1$.

The DT signal $x[n] = n^2$ is sketched below.

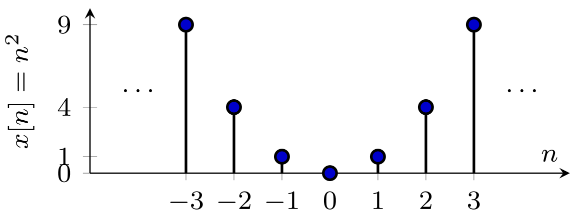


Figure 2.6: Sketch of the DT signal $x[n] = n^2$.

Piecewise description Sometimes when describing signals mathematically, we may have to describe the signal piecewise, i.e., the signal may take a different mathematical description for different range of values of t. Consider the RC circuit shown in Fig. 2.7. The capacitor is charged to 10V at time $t = -2$ and at time $t = 2$, the switch is closed. Let $x(t)$ represent the voltage across the capacitor at time t.

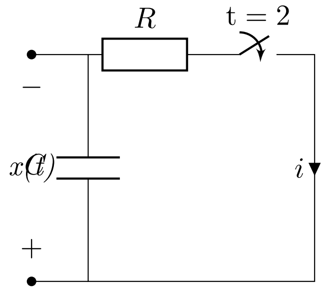


Figure 2.7: RC circuit.

The piecewise description of $x(t)$ is given by

$$x(t) = \begin{cases} 10 & \text{if } -2 \leq t < 2 \\ 10e^{-t/RC} & t \geq 2. \end{cases} \tag{2.1}$$

The corresponding plot is given below,

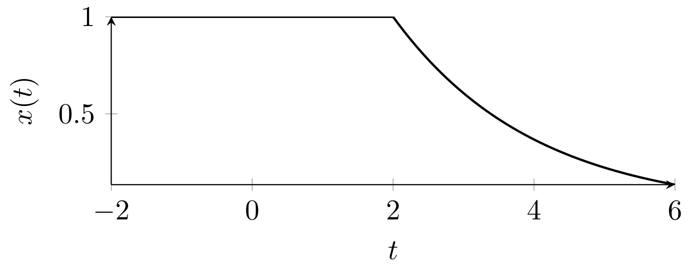


Figure 2.8: Signal showing discharge of a capacitor starting at $t = 2$.

2.4.2 Pictorial description

Sometimes, signals will be defined by sketching the signal as a function of time such as in Fig. 2.9

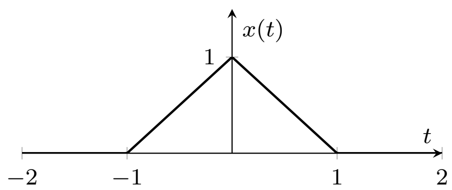


Figure 2.9: Pictorial description of a signal.

The above signal can be mathematically described as

$$x(t) = \begin{cases} 1 - |t| & \text{if } -1 < t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

It is important to be able to go back and forth between pictorial and mathematical descriptions of signals.

2.4.3 Complex signals

In general, we will be interested in complex signals in this class, i.e. a signal of the form

$$x(t) = x_R(t) + jx_I(t)$$

where $x_R(t)$ and $x_I(t)$ are two real functions of t. An example of a complex signal $x(t)$ is given below

$$x(t) = e^{j2\pi t} = \cos(2\pi t) + j\sin(2\pi t).$$

While we can always separately think of $x_R(t)$ and $x_I(t)$ as two real signals or functions, it will be more convenient to directly manipulate the signal $x(t)$ by thinking of $x(t)$ as a complex function of a real variable as the algebra will become less tedious. A typical student does not get a lot of exposure to complex signals prior to this course. It is difficult to gain intuition about them and often students are left wondering where complex signals occur in nature.

It is worth reminding ourselves that many objects we use in engineering, mathematics or sciences do not directly occur in nature. Most quantities that we can directly measure typically are scalar real numbers (and real functions) such as voltages, currents, acceleration along any direction. Yet, we are often interested in studying or manipulating multiple quantities together. For example, we may be interested in measuring the velocity or acceleration of an object and in your physics courses, you may have used 2-D vectors to represent them and manipulated vectors using dot products, scaling etc. 2-D vectors do not occur in nature; it was our choice to represent velocity or acceleration along the X and Y directions using a 2-D vector. For the purpose of doing what you wanted to do in those courses, that representation was convenient. In a similar fashion, complex numbers are convenient for representing some things. For example, 2-D vectors have one serious limitation - there is no concept of multiplying two vectors together and hence, they are not convenient for studying some phenomena.

There are two main reasons why we want to consider complex signals in this class - (i) We are interested in studying two quantities that occur together. The complex number system and their arithmetic properties provides a natural and convenient system for tracking these two quantities together, i.e., it will be easier to manipulate the quantities together by treating them as a complex signal instead of as two separate real signals, (ii) complex signals occur as intermediate steps in the processing of signals just like how they appeared in the solution to cubic equations; even when our eventual goal is to understand the behavior of some real quantities. The following examples will illustrate these two situations. Although the algebraic closure of complex numbers is not directly used much in this course, that is another reason complex numbers are studied extensively in mathematics.

Example 2.4.1. *Suppose we are interested in tracking the position of a particle that is moving along a circle of radius r at the rate of 10 revolutions/minute. One way to track the position would be the track the X-coordinate and the Y-coordinate separately. Then the X-coordinate will be given by the signal $x_R(t) = \cos\left(2\pi\frac{1}{6}t\right)$ and the Y-coordinate will be given by the signal $x_I(t) = \sin\left(2\pi\frac{1}{6}t\right)$, respectively. However, it is often easier to describe the position of the particle directly in the complex plane as $x(t) = e^{j2\pi\frac{1}{6}t}$.*

Example 2.4.2. *As another example, consider a RC circuit. Suppose the input voltage to the circuit is a sinusoid of frequency f Hz or $\omega = 2\pi f$ rads/s with unit amplitude i.e.,*

$v_{in}(t) = \sin \omega t$. *The voltage across the capacitor will also be a sinusoid of the same frequency* ω*; however, the amplitude may be scaled by* a *and the phase of the sinusoid may be shifted by* θ *radians, i.e.,* $v_{out}(t) = a \sin(\omega t + \theta)$. *We need two parameters* a *and* θ *to represent the gain or transfer function of the circuit. Further,* a *and* θ *will typically vary with* ω *and hence, we want to track* $a(\omega)$ *and* $\theta(\omega)$ *together to understand how an RC circuit transforms input signals. We can always track these two functions separately. However, mathematically, it will be more convenient to combine them in to one* complex *function of* ω.

Example 2.4.3. *When using Wi-Fi on your phone or computer, we transmit bits by converting them to an electromagnetic waveform (signal) as follows. We first split the bit stream* $\{b\}$ *into two separate streams* $\{b_I\}$ *and* $\{b_Q\}$ *and then we form two real signals* $x_I(t)$ *and* $x_Q(t)$ *that are dependent on* $\{b_I\}$ *and* $\{b_Q\}$*, respectively. The signal that is transmitted out of the antenna is given by*

$$s(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t),$$

where f_c *is 2.4 GHz, 3.6 GHz, 5 GHz etc. depending on the Wi-Fi version. The receiver typically observes a distorted version of* $s(t)$ *and its function is to recover* $x_I(t)$ *and* $x_Q(t)$ *and from that, recover* $\{b_I\}, \{b_Q\}$. *While* $x_I(t)$ *and* $x_Q(t)$ *are real signals, and the Wi-Fi receiver processes these real signals, often it is mathematically convenient to think of* $s(t)$ *as*

$$s(t) = \Re\{(x_I(t) + j x_Q(t)) e^{j2\pi f_c t}\},$$

and to think that the job of the receiver is to estimate the complex signal $x_I(t) + j x_Q(t)$.

Example 2.4.4. *Suppose we want to compute the voltage as a function of time at some point in a circuit which contains resistors, inductors and capacitors. The voltage waveform is typically a real signal and will be the solution to a differential equation. The mathematically convenient way to solve the differential equation will be through Laplace transforms, which are complex functions of a real variable (or, complex signals). The Laplace transforms themselves are only intermediate signals; our real interest may only be in determining the voltage or current waveform as a function of time. Accepting Laplace transforms as a complex signal and manipulating them makes it more convenient to arrive at the final answer which is a real signal. Sometimes, trying to obtain intuition about these intermediate signals may not be necessary, at least at first.*

2.5 Energy and Power - (Videos, Python notebook)

2.5.1 Energy and Power of CT signals

Imagine that a voltage source $x(t)$ (complex signal) is connected to a 1 Ω resistor as shown in Fig. 2.10 such that the voltage signal may change with time. At time instant t, the instantaneous power dissipated in the resistor is given by $|v|^2/R = |x(t)|^2$. Thus, we can define the instantaneous power of the signal $x(t)$ at time t as $|x(t)|^2$.

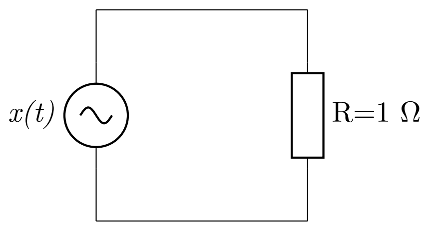


Figure 2.10: Think of signal $x(t)$ as a time-varying voltage source connected to a 1Ω resistor.

For a continuous-time complex signal $x(t)$, we will define the energy and (average) power of the signal as follows. Notice that a real signal is just a special case of a complex signal and hence, these definitions apply to both complex and real signals.

Energy and power of a CT signal

$$\text{Instantaneous Power} - P_x(t) := |x(t)|^2 \tag{2.3}$$

$$\text{Energy} - E_x := \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{2.4}$$

$$\text{Power} - P_x := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{2.5}$$

What we call as power in (2.5)should be thought of as the average power, which is the average of the energy over an infinitely long window. This can be thought of as the limit of the average energy over a window of length $2T$ from time $t = -T$ to $t = T$ as shown in Fig. 2.11.

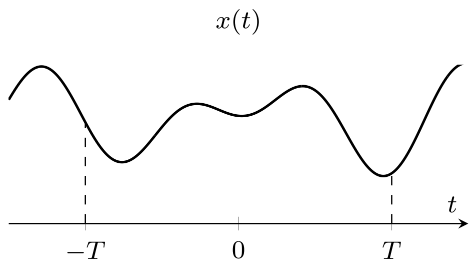


Figure 2.11: The power of a CT signal is computed as the limit of the average energy over a time window of length $2T$ from $t = -T$ to $t = T$.

2.5.2 Energy and Power of DT Signals

If $x[n]$ is a DT complex signal such as what is shown here

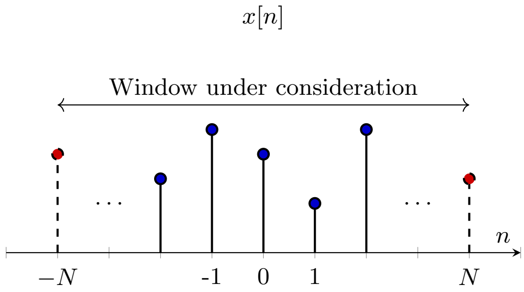


Figure 2.12: The power of a CT signal is computed as the limit of the average energy over a time window of length $2N + 1$ for $-N \leq n \leq N$.

Energy and power of a DT signal

$$\text{Instantaneous power} - P_x[n] := |x[n]|^2 \tag{2.6}$$

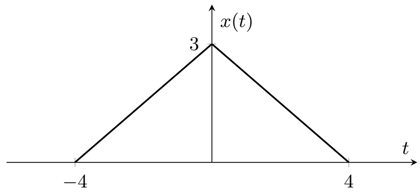
$$\text{Energy} - E_x := \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{2.7}$$

$$\text{Power} - P_x := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{2.8}$$

2.5.3 Energy and Power type signals

- A signal is said to be of energy type or an energy signal if $0 < E_x < \infty$ (i.e., E_x is bounded)
- A signal is said to be of power type if $0 < P_x < \infty$, i.e., P_x is bounded
- Sometimes a signal may be of neither type - for example $x(t) = e^t$ or $x[n] = 3^n$ are neither energy type nor power type signals.

Example 2.5.1. *Find the energy of the signal $x(t)$ given below and determine if it is an energy signal, power signal or neither*



Solution:

$$x(t) = \begin{cases} 3\left(1 - \frac{t}{4}\right) & \textit{if } 0 \leq t \leq 4 \\ 3\left(1 + \frac{t}{4}\right) & \textit{if } -4 \leq t \leq 0 \\ 0 & \textit{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt \overset{(a)}{=} \int_{-\infty}^{\infty} x^2(t) dt \\ &= \int_{-4}^{4} \left(3\left(1 - \frac{|t|}{4}\right)\right)^2 dt \\ &= 9\int_{-4}^{0} \left(1 + \frac{t}{4}\right)^2 dt + 9\int_{0}^{4} \left(1 - \frac{t}{4}\right)^2 dt \\ &= 9\frac{(1+\frac{t}{4})^3}{\frac{3}{4}}\Bigg|_{-4}^{0} + 9\frac{(1-\frac{t}{4})^3}{\frac{-3}{4}}\Bigg|_{0}^{4} \\ &= 9 \cdot \frac{4}{3} + 9 \cdot \frac{4}{3} \\ &= 24 \end{aligned}$$

(a) *is true since $x(t)$ in this example is strictly real.*

Since the energy is finite, $x(t)$ is an energy signal.

Example 2.5.2. *Find the energy of the signal $x(t)$ given by*

$$x(t) = \begin{cases} e^{-(2+j3)t}u(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Solution: The energy of the signal is defined as $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$. Since $x(t) = 0$ for $t < 0$, we only need to compute the integral for $t \geq 0$. The energy becomes:

$$E = \int_0^{\infty} |x(t)|^2 \, dt.$$

Substitute $x(t) = e^{-(2+j3)t}$:

$$|x(t)|^2 = \left| e^{-(2+j3)t} \right|^2.$$

The magnitude of $e^{-(2+j3)t}$ is:

$$\left| e^{-(2+j3)t} \right| = e^{-2t}.$$

Thus:

$$|x(t)|^2 = \left(e^{-2t}\right)^2 = e^{-4t}.$$

The energy is now:

$$E = \int_0^{\infty} e^{-4t} \, dt. = \left[\frac{e^{-4t}}{-4} \right]_0^{\infty}.$$

Evaluate the limits:

$$\lim_{t \to \infty} \frac{e^{-4t}}{-4} = 0, \quad \textit{and} \quad \frac{e^{-4 \cdot 0}}{-4} = -\frac{1}{4}.$$

Therefore,

$$E = 0 - \left(-\frac{1}{4} \right) = \frac{1}{4}.$$

Example 2.5.3. *Find the energy of the signal $x(t) = e^{-|t|}$*

Solution: The energy of the signal is defined as:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

Substitute $x(t) = e^{-|t|}$:

$$|x(t)|^2 = \left(e^{-|t|}\right)^2 = e^{-2|t|}.$$

Thus, the energy becomes:

$$E = \int_{-\infty}^{\infty} e^{-2|t|} \, dt.$$

Since $|t|$ is symmetric about $t = 0$, we can split the integral:

$$E = 2 \int_{0}^{\infty} e^{-2t} \, dt.$$

Compute the integral:

$$\int_{0}^{\infty} e^{-2t} \, dt = \left[\frac{e^{-2t}}{-2} \right]_0^{\infty}.$$

Evaluate the limits:

$$\lim_{t \to \infty} \frac{e^{-2t}}{-2} = 0, \quad \textit{and} \quad \frac{e^{-2 \cdot 0}}{-2} = -\frac{1}{2}.$$

So:

$$\int_{0}^{\infty} e^{-2t} \, dt = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.$$

Multiply by 2 for the symmetric region:

$$E = 2 \cdot \frac{1}{2} = 1.$$

Thus, the energy of the signal is:

$$E = 1.$$

Example 2.5.4. *What is the energy of the signal $x[n]$ given by*

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

Solution:

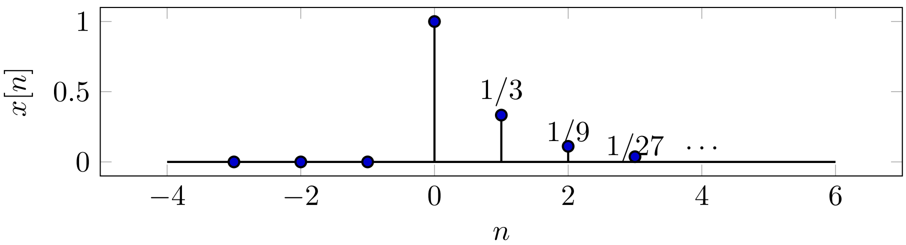


Figure 2.13: Plot of the signal $x[n]$ in (2.9)

Notice that the energy of the signal is

$$E_x = 1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \dots$$

Clearly this is a sum of infinitely many terms of a geometric sequence

$$\begin{aligned} E_x &= \sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=0}^{\infty} x^2[n] = \sum_{n=0}^{\infty} \left(\left(\frac{1}{3}\right)^n\right)^2 \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = \frac{1}{1-\frac{1}{9}} \\ &= \frac{9}{8} \end{aligned}$$

Remarks on the definition of energy

It is natural to wonder why the energy and power of a signal are defined as in equations (2.4)-(2.8). The energy of a signal is a measure of the strength of the signal and if we think of the DT signals as vectors, one natural notion of the strength is the square of the length of the vector and it can be seen that the energy of a DT signal is exactly the same as the square of the length of the vector. i.e. think of $x[n]$ as the vector $\underline{x} = [\dots, x[-2], x[-1], x[0], x[1], x[2], \dots]$ then the energy of $x[n]$ is the same as the square of the length of the vector or square of the magnitude of the vector $\underline{x}^T \underline{x}$. In linear algebra, this is called the square of the 2-norm of the vector $\underline{x}$. By extension, we can think of the energy of the CT signal as the integral defined in (2.4). In mathematics, $\int_{-\infty}^{\infty} |x(t)|^2 dt$ is called the square of the 2-norm of $x(t)$.

In some cases, the energy of the signal $x(t)$ directly corresponds to the energy needed to either generate or transmit such waveforms and hence, it has the physical units of energy. For example, when $x(t)$ is a voltage waveform that is passed through a 1Ω resistor, the *instantaneous* power dissipated in the resistor at time t is $P(t) = |x(t)|^2$. The total energy dissipated is the integral of the power given by $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$.

Similarly, the energy required to transmit a signal $x(t)$ out of your mobile phone antenna would be directly related to the energy of $x(t)$ as we have defined it. If you are a communications engineer and you have the choice of transmitting one of two waveforms $x_1(t)$ or $x_2(t)$ as shown in Fig. 2.5.3 and if the performance of your system were identical for both these waveforms, you should choose the one that has lower energy to maximize battery life and to reduce interference to other users.

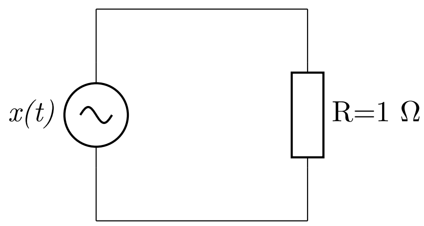
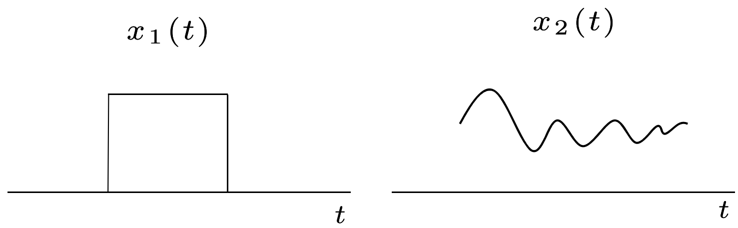


Figure 2.14: The total energy dissipated across the resistor is $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$.



Even though we used physical examples from voltage waveforms, circuits and antennas to motivate the use of our definition of energy, this is not required and it turns out that the way we defined energy and power (particularly, that of energy) is meaningful as a measure of the strength of a signal or a measure of how big a signal is even when the signals do not correspond to physical waveforms. Therefore, we will use these definitions for all signals regardless of whether they are physically meaningful or not (for example, if the signal refers to stock prices, we will still use the term energy of the signal).

This concept becomes particularly useful in deciding which approximation error is smaller. Often in engineering, we try to approximate one signal $x(t)$ by another signal and typically an algorithm tries to produce the approximation. Say we would like to a robot to follow a particular path. The path can be defined by a signal that specifies the position of the robot as a function of time. We may have two different control algorithms that try to make the robot follow the desired path $x(t)$. Since control algorithms are never exact, they might give you an approximation to $x(t)$ given by $\hat{x}_1(t)$ and $\hat{x}_2(t)$. How would you decide which of these two algorithms gives you a better approximation? One way to decide this is to compute the error signals $e_1(t) = x(t) - \hat{x}_1(t)$ and $e_2(t) = x(t) - \hat{x}_2(t)$ and then compute the energies of $e_1(t)$ and $e_2(t)$ and to decide based on which of the two energies is lower. This iPython notebook on energy of error signals has a hands-on example where you can try this.

Which signals have zero energy?

The only DT signal that has zero energy is the zero signal, i.e., $x[n] = 0, \forall n$. However, this is not true for CT signals. Consider the signal defined as

$$x(t) = \begin{cases} a, & \text{if } t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

where a is finite. While this signal is clearly not identically equal to the zero signal, the energy of this signal is indeed zero. This can be seen by looking $|x(t)|^2$ and realizing that the area under the signal $|x(t)|^2$ is the energy of $x(t)$. In this example, this is indeed zero. The reason this happens is that the signal $x(t)$ is non-zero only for one value of t out of uncountably many values; in mathematics, we say that this is a set of measure zero. If you were listening to some music and you had a disturbance for just one time instant (not a small window, but exactly one instant), would you be able to listen to it? If you see disturbance in video for exactly one time instant would that register in your eye? The answer is no. This is why it makes sense for us to use a defintion of energy that would assign zero energy to these signals even though the signals themselves are not identically zero for all t. In general, if the signal is non-zero for any countable number of values of t, the energy would still be zero.

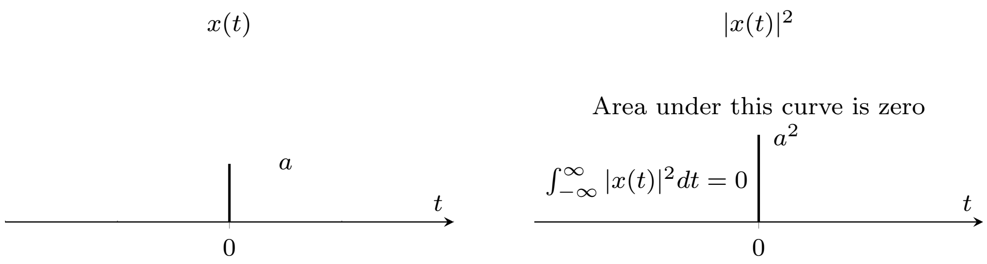


Figure 2.15: Non-zero CT signal with zero energy

For all practical purposes, this signal is no different from the zero signal and hence, we will treat this signal as the zero signal. There are infinitely many such signals and we say that they all belong in an equivalence class of being the zero signal.

When are two signals the same?

We have argued that the example shown above and the zero signal should be treated as the same even though they are not identical for all t. Now, this leads us to an interesting question of when can we consider two signals to be the same? There are many notions in mathematics for defining when we can think of two signals as being the same. One easy notion to understand is when $x_1(t) = x_2(t), \forall t$. But this is too strict.

When are two signals considered to be equal to each other?

In this class,

1. Two DT signals are the same if and only if $x_1[n] = x_2[n], \forall n$.
2. We will say that two CT signals are the same if the energy of their difference is zero, i.e. $\int_{-\infty}^{\infty} |x_1(t) - x_2(t)|^2 dt = 0$. Notice that this does not mean that the signals are the same for every value of t.

2.6 Periodic signals

Roughly speaking, a periodic signal is a signal where a pattern repeats indefinitely. Many real life signals can be approximated well by periodic signals - for example, under ideal conditions, a person's ECG signal would be a pattern that repeats itself. Continuous-time sinusoids are also examples of periodic signals. More precisely, a CT signal $x(t)$ is said to be periodic if there exists a positive (non-zero) real number T_0 such that

$$x(t) = x(t+T_0)\ \forall t \quad (2.10)$$

$$x(t) = x(t+lT_0)\ \forall t \text{ and } \forall \text{integer } l \quad (2.11)$$

The smallest positive number T_0 for which the above are true is called the fundamental time period. Another way to think about a periodic signal is that for a periodic signal with fundamental time period T_0, a shift to the left by T_0 will leave the signal unchanged. An example of a periodic signal is shown below.

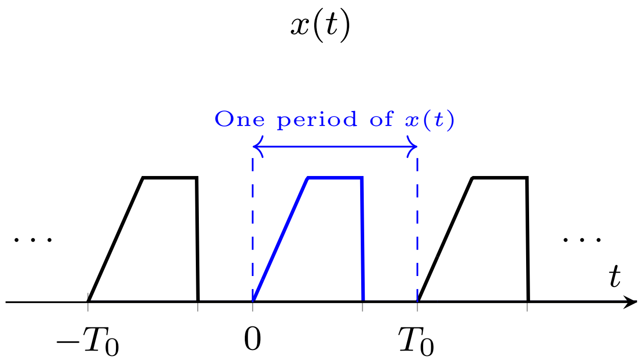


Figure 2.16: Example of a CT periodic signal with fundamental time period T_0.

Example 2.6.1. *Continuous-time signal examples*

- *CT complex exponential :* $x(t) = e^{j\pi t}$ *is periodic with a fundamental time period of* $T_0 = 2$

  We can see that $x(t) = x(t+2)$ *since* $x(t) = e^{j\pi t}$ *and* $x(t+2) = e^{j\pi(t+2)} = e^{j\pi t}e^{j2\pi}$*. Since* $e^{j2\pi} = 1$*,* $x(t) = x(t+2)$*.*

- *CT complex exponential :* $x(t) = e^{j2t}$ *is periodic with a fundamental time period of* $T_0 = \pi$

- *CT complex exponential :* $x(t) = e^{j\omega_0 t}$ *is periodic with a fundamental time period of* $T_0 = \frac{2\pi}{\omega_0}$ *for any* ω_0

- *CT sinusoid :* $x(t) = \sin(\omega_0 t + \phi)$ *for any* ω_0, ϕ *is periodic with fundamental time period of* $T_0 = \frac{2\pi}{\omega_0}$

A DT signal $x[n]$ is said to be periodic if there exists a positive (non-zero) *integer* N_0 such that

$$x[n] = x[n+lN_0]\ \forall \text{ integer } n \text{ and } \forall \text{ integer } l \quad (2.12)$$

The smallest such positive (non-zero) *integer* is called the fundamental time period.

An example of a periodic DT signal with $N_0 = 5$ is shown below

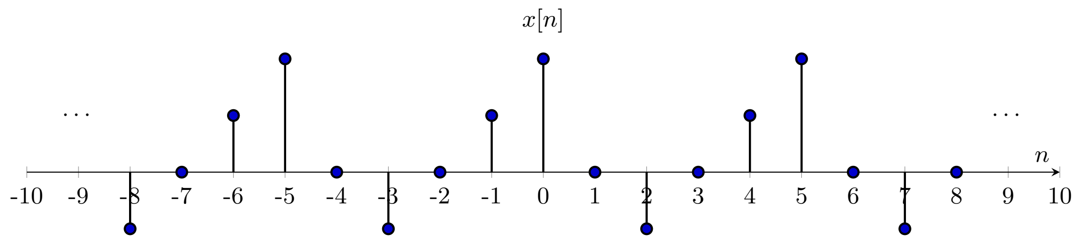


Figure 2.17: Example of a DT periodic signal with fundamental time period 5.

The fundamental time period of a DT signal must be an integer

It does not make sense for the fundamental time period of a discrete-time signal to be a non-integer value. When asked to determine if a DT signal is periodic or not, see if you can find a fundamental time period that is an intger. If you are unable to find an integer value for the fundamental time period, it means that the DT signal is not periodic.

Example 2.6.2. *Discrete-time signal examples*

- *DT complex exponential :* $x[n] = e^{j\pi n}$ *is periodic with a fundamental time period of* $N_0 = 2$
- *DT complex exponential :* $x[n] = e^{j2n}$ *is not periodic*
- *DT complex exponential : Clearly* $x[n] = e^{j\Omega_0 n}$ *is periodic only if* Ω_0 *is a rational multiple of* 2π*.*
- *DT sinusoid :* $x[n] = \sin(\Omega_0 t + \phi)$ *for any* Ω_0, ϕ *is periodic only if* ω_0 *is a rational multiple of* 2π*.*

Can a periodic signal be an energy type signal?

Every periodic signal has infinite energy. Some periodic signals have finite power and the power can be computed by averaging over one time period only. If $x(t)$ is a periodic CT signal with time period T_0,

$$Power : P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt \tag{2.13}$$

Similarly,if $x(t)$ is a periodic DT signal with time period N_0,

$$Power : P_x = \frac{1}{N_0}\sum_{n_0}^{n_0+N_0-1} |x[n]|^2 \tag{2.14}$$

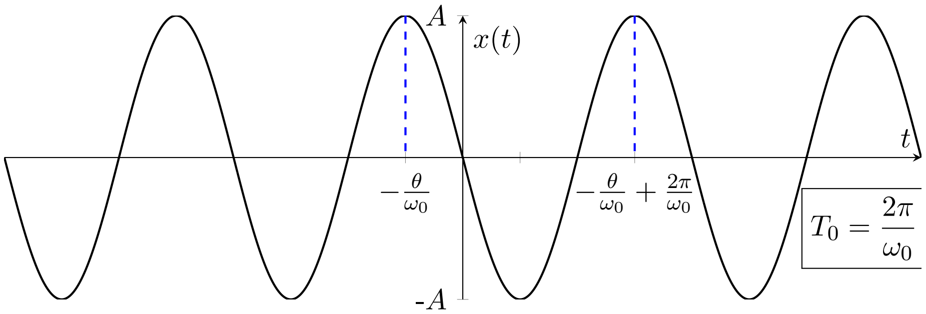


Figure 2.18: Plot of $x(t) = A\cos(\omega_0 t + \theta)$

Example 2.6.3. *Determine whether $x(t) = A\cos(\omega_0 t + \theta)$ is a power or energy signal.*

Solution:

E_x is unbounded. So, it is not an energy signal. Let us now compute the power.

$$
\begin{aligned}
P_x &= \frac{1}{T_0}\int_0^{\frac{2\pi}{\omega_0}} A^2\cos^2(\omega_0 t + \theta)dt \\
&= \frac{1}{T_0}\int_{\frac{-\theta}{\omega_0}}^{\frac{-\theta+2\pi}{\omega_0}} A^2\cos^2(\omega_0 t + \theta)dt \\
&= \frac{A^2}{T_0}\int_{\frac{-\theta}{\omega_0}}^{\frac{-\theta+2\pi}{\omega_0}} \frac{1+\cos(2\omega_0 t + 2\theta)}{2}dt \\
&= \frac{A^2}{T_0}\left[\frac{t}{2} + \frac{\sin(2\omega_0 t + 2\theta)}{4\omega_0}\right]_{\frac{-\theta}{\omega_0}}^{\frac{-\theta+2\pi}{\omega_0}} \\
&= \frac{A^2}{T_0}\left[\frac{1}{2}\left(\frac{-\theta+2\pi}{\omega_0} - \frac{-\theta}{\omega_0}\right)\right] \\
&= \frac{1}{2}\frac{A^2}{T_0}\frac{2\pi}{\omega_0} \\
&= \frac{A^2}{2}
\end{aligned}
$$

So, $x(t)$ is a power Signal.

Example 2.6.4. *What is the power of the signal $x(t) = e^{j\omega_0 t}$? [$T_0 = \frac{2\pi}{\omega_0}$]*

Solution: Given that $T_0 = \frac{2\pi}{\omega_0} \Rightarrow e^{j\omega_0 t} = e^{j\omega_0(t+\frac{2\pi}{\omega_0})} = e^{j(\omega_0 t + 2\pi)}$. We know that $e^{j\theta} = e^{j(\theta+2\pi)} \Rightarrow x(t)$ is periodic with a period of $\frac{2\pi}{\omega_0}$. Therefore,

$$
\begin{aligned}
P_x &= \frac{1}{T_0}\int_0^{T_0} |e^{j\omega_0 t}|^2 dt \\
&= \frac{1}{T_0}\int_0^{T_0} 1dt \qquad \left(|e^{j\theta}| = 1 \ \forall \ \theta \Rightarrow |e^{j\omega_0 t}| = 1\right) \\
&= 1
\end{aligned}
$$

2.7 Basic Operations on Signals

In this section, we consider various operations/transformations that we can perform on signals. In a signal $x(t)$, t is the independent variable and x is the dependent variable. We study operations that modify the dependent variable x first and then we study operations that modify the independent variable.

Operations performed on the Dependent Variable for CT signals - (video, Python notebook1 Python notebook2)

2.7.1 Amplitude Scaling

Consider an ideal amplifier. If the signal $x(t)$ is input to the amplifier, the output will simply be a scaled version of the input and hence, the output $y(t)$ will be given by

$$y(t) = c\, x(t) \tag{2.15}$$

Although I gave an amplifier as an example to motivate scaling the amplitude of a signal, in general, we are interested in both attenutation and amplification and hence c may be < 1 or > 1. In fact, in communication theory, there is a good reason to even encounter complex c. In general, we should be prepared to see complex gains c.

2.7.2 Addition, Subtraction, Multiplication and Division of two signals

Take two signals $x_1(t)$ and $x_2(t)$. We can perform several operations such as adding, subtracting, multiplying and dividing $x_1(t)$ and $x_2(t)$. In these cases, these operations should be performed for each value of t. Similarly, for DT signals, these operations should be performed for every value n. Notice that the signals should be properly defined over the range of values of t or n that we would like to perform these operations.

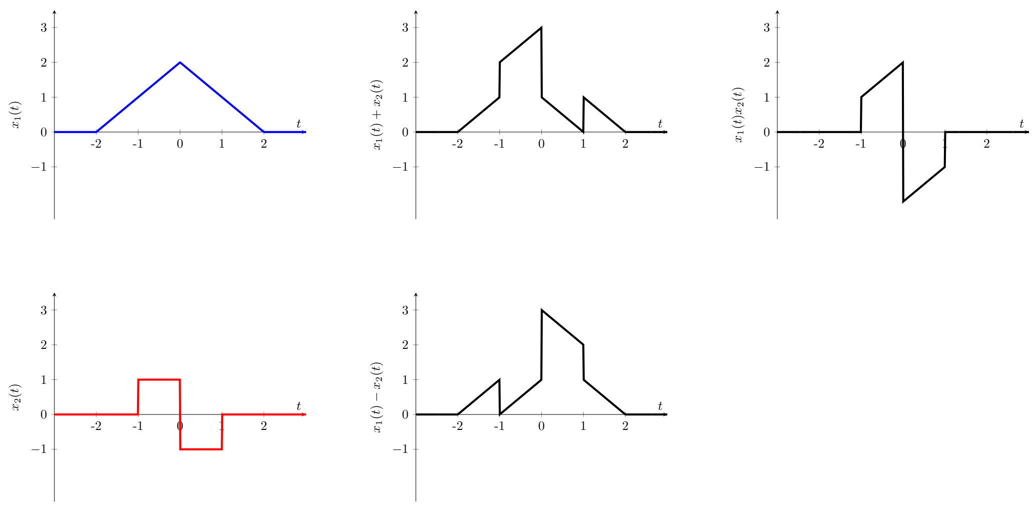


Figure 2.19: Figures showing addition, subtraction and multiplication of signals. The figures show $x_1(t), x_2(t), x_1(t)+x_2(t), x_1(t)-x_2(t), x_1(t)x_2(t)$

2.7.3 Derivative of a signal

If $x(t)$ is a signal, then the derivative of $x(t)$ results in another signal $y(t)$ given by

$$y(t) = \frac{d}{dt}x(t).$$

As an example, the voltage across an inductor of inductance L and the current through an inductor $i(t)$ are related according to $v(t) = L\frac{di(t)}{dt}$.

2.7.4 Integral from $-\infty$ to t of a signal

Suppose $x(t)$ is a CT signal. Sometimes we are interested in defining a new signal $y(t)$ which represents the integral of the signal $x(t)$ from $-\infty$ to the current time t. This is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Notice the notation here where the integrand is expressed as $x(\tau)$ instead of $x(t)$. τ is used as a dummy variable so that the final result of the integral will be a function of t. This notation is standard; if it is not clear, please talk to me. As an example, the voltage across an capacitor of capacitance C and the current through the capacitor $i(t)$ are related according to $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$.

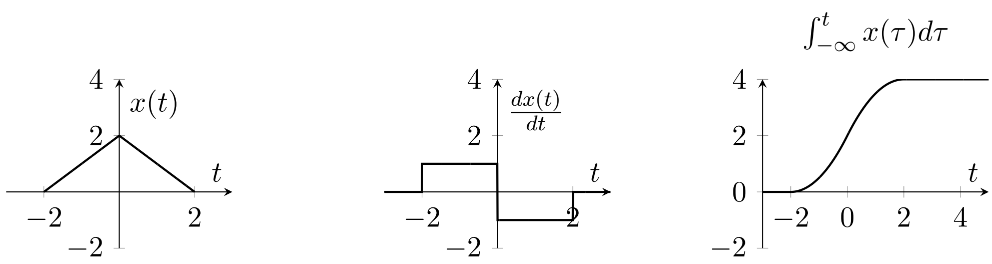


Figure 2.20: Plot of $x(t)$, $\frac{dx(t)}{dt}$ and $\int_{-\infty}^{t} x(\tau)d\tau$.

Transformations/Operations performed on the Dependent Variable for DT signals DT signals can be scaled, added, subtracted, multiplied and divided similar to CT signals. Conceptually, this is not very different from that for CT signals. However, differentiation and integration are not operations that are well-defined for DT signals. The equivalent operations are given by the difference operator and the summation operator

2.7.5 Difference operator for DT signals

Consider the signal

$$y[n] = x[n] - x[n-1] \tag{2.16}$$

This plays the equivalent role of derivative of a CT signal

2.7.6 Summation operator

Consider the signal

$$y[n] = \sum_{m=-\infty}^{n} x[m] \tag{2.17}$$

This plays the equivalent role of integral of a CT signal.

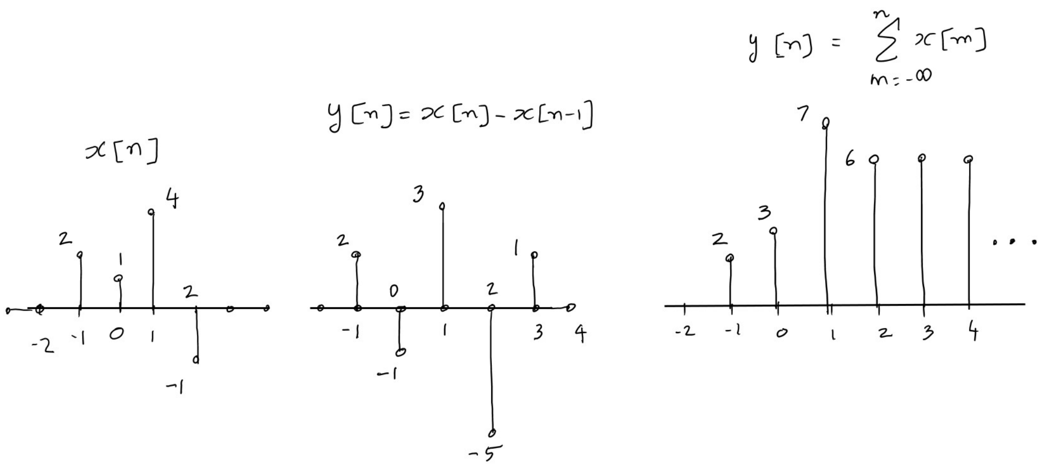


Figure 2.21: Example showing backward difference and summation of a signal

Transformations/Operations performed on the Independent Variable

2.7.7 Time scaling for CT signals - (video)

Given a CT signal $x(t)$, a *time-scaled* version of $x(t)$ is given by

$$y(t) = x(at) \tag{2.18}$$

To get a feel for the relationship between $x(t)$ and $y(t)$, consider an example with $a = 2$. We can see that

$$\begin{aligned} &\vdots && \vdots \\ y(-1) &= x(-2) \\ y(0) &= x(0) \\ y(1/2) &= x(1) \\ y(3/2) &= x(3) \\ &\vdots && \vdots \end{aligned}$$

Graphically, this corresponds to shrinking the time axis while keeping the origin in tact. This is depicted in Figure 2.22

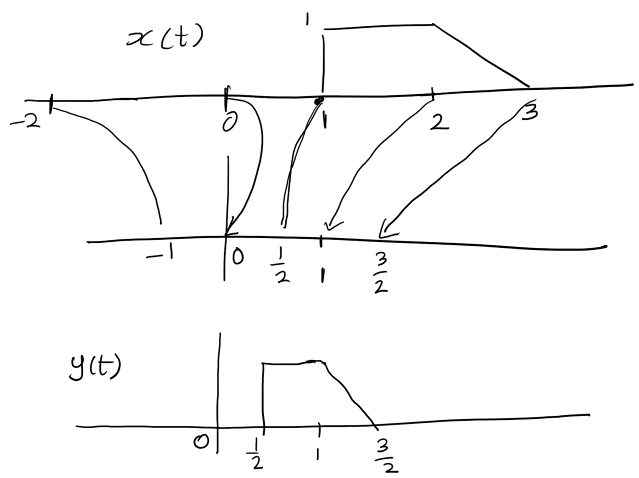


Figure 2.22: Example showing scaling of the time axis by $a = 2$

For any $a > 1$, it can be seen that this corresponds to squeezing (or shrinking, compressing) the signal along the time axis by a factor a. when you play a video in 2x the speed on YouTube, the signal you are watching and listening to is $x(2t)$ if the original video (or, audio) signal is $x(t)$.

Similarly, when $0 < a < 1$, it can be seen that this corresponds to expanding the time axis by a factor $1/a$. An example is shown in Fig. 2.23 for $a = 1/2$. Naturally, this corresponds to playing your music or video at half the speed.

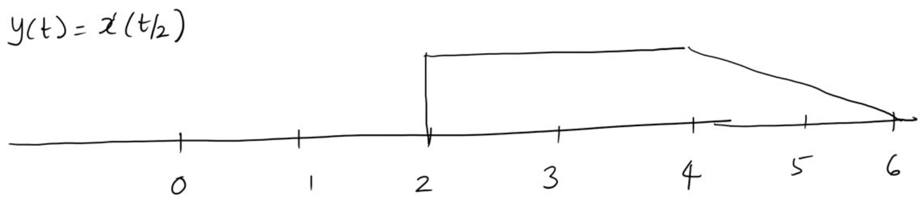


Figure 2.23: Example showing scaling of the time axis by $a = 1/2$

Keep the origin fixed when scaling the time axis

Notice that $y(0) = x(0)$ and hence, $t = 0$ should be held fixed when shrinking or expanding the time axis.

2.7.8 Reflection about the Y-axis

When $a = -1$, $y(t) = x(-t)$ corresponds to the signal $x(t)$ flipped about the Y-axis. This would correspond to play the audio or video in reverse (with a delay).

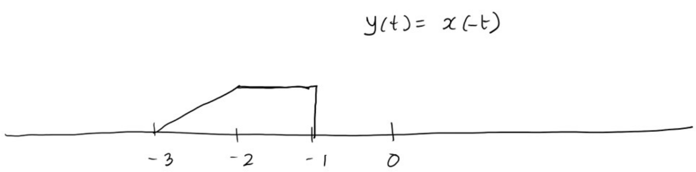


Figure 2.24: Example showing flip of the signal about the y-axis, i.e. $a = -1$

2.7.9 Time scaling for DT signals

The idea of scaling the time axis is conceptually similiar for DT signals; however, since DT signals are defined only for integer values of n, when defining $y[n] = x[an]$, such a signal would make sense only when an is an integer. When an is not an integer, some caution must be exercised and some covention must be followed to make sure $y[n]$ is unambiguously defined. Suppose we have a DT signal $x[n]$, and let us say, we define a new signal $y[n] = x[n/2]$. When n is odd, $n/2$ is not an integer and hence, $y[n]$ is not well-defined for these values. We will use a convention by which we will treat $y[n]$ to be zero. In general, for any integer value of n for which an is not an integer, we will set $y[n] = 0$. When $a < 1$, all values in the original signal appear in $y[n]$ with some zeros inserted, i.e., the time axis is extended and zeros are inserted wherever $y[n]$ was not defined.

When $a > 1$, the transformation $y[n] = x[an]$ corresponds to shrinking the time axis. Unlike for CT signals, when the discrete-time axis is shrunk, some values n will not get mapped to integer values of n after the transformation and hence, the value taken by the signal at these values of n will be lost. Thus this transformation loses information. Notice that in the CT case, this does not happen.

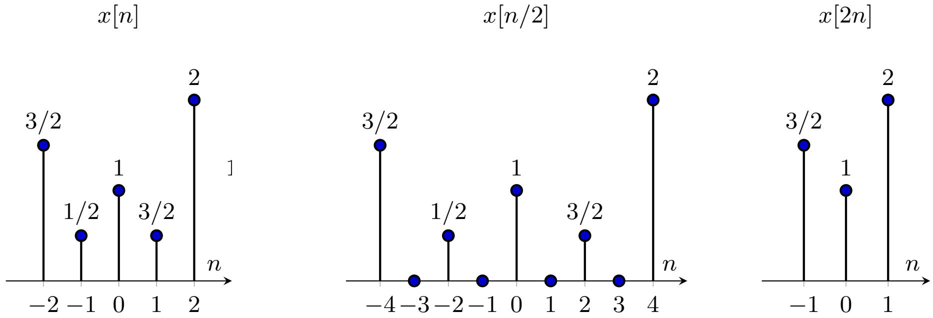


Figure 2.25: Example of time scaling of a DT signal. Example shows $x[n]$, $x[n/2]$, and $x[2n]$

2.7.10 Time Shifting of CT signals

Given a signal $x(t)$, the signal $y(t) = x(t - t_0)$ corresponds to shifting the signal $x(t)$ in time by t_0 seconds to the *right*. If $t_0 > 0$ then $x(t - t_0)$ is indeed a shift to the right, but if $t_0 < 0$ $x(t - t_0)$ in effect becomes a shift to the left by $|t_0|$. We will also refer to this as shift to the right by $-t_0$. Get used to this strange terminology.

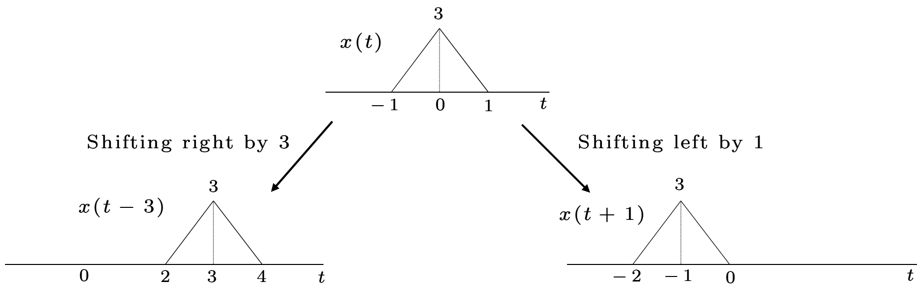


Figure 2.26: Examples of time shifting of CT signals

Example: Cellular Communication System In cellular communication systems, the signal transmitted by a base station arrives at the handset via multiple reflections. Each one of these paths has a slightly different delay since the length of the paths are different. In effect, the signal received at the handset is a linear combination of time shifted versions of $x(t)$.

Example: Radar/Lidar/Bats Provide a high level description of how

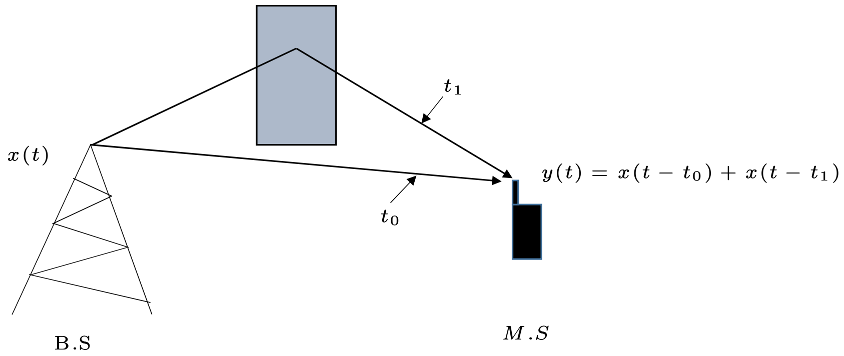
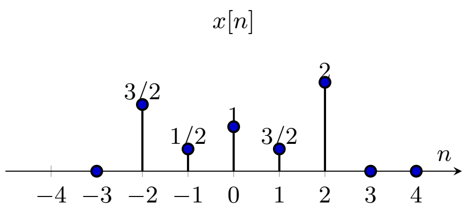


Figure 2.27: The signal received at a handset is a linear combination of two time shifted versions of the signal transmitted by the base station.

2.7.11 Time Shifting of DT signals

For a DT signal $x[n]$, the signal $y[n] = x[n - n_0]$ corresponds to shifting $x[n]$ by n_0 units to the right and since DT signals are defined only for integer values of n, only integer values of n_0 make sense. In an advanced DSP class, you may be able to make sense of non-integer shifts, but in this class, we will restrict ourselves to integer shifts.



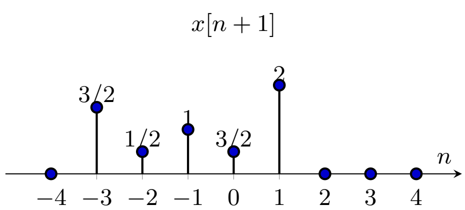


Figure 2.28: Examples of time shifting of DT signals. The signal $x[n + 1]$ is a shift of the signal $x[n]$ to the left by 1 unit or 1 time step.

2.7.12 Multiple Operations on a signal/combination of transformations - (video)

Computing $y(t) = x(at - b)$

Given a signal $x(t)$, the signal $y(t) = x(at - b)$ is obtained by performing two operations on $x(t)$ or passing $x(t)$ through two transformations - scaling and shifting. The scaling and shifting can be performed in any order; however, the scaling and shift parameters must be chosen carefully. The possibilities are shown below. The first option is more natural and in some books, they would say that the preferred order is to first shift by b to the right and then scale by a.

$$x(t) \xrightarrow[t \leftarrow t-b]{\text{Time shift right by } b} x(t-b) \xrightarrow[t \leftarrow at]{\text{Scale time by } a} x(at-b)$$

$$x(t) \xrightarrow[t \leftarrow at]{\text{Scale time axis by } a} x(at) \xrightarrow[t \leftarrow t-\frac{b}{a}]{\text{Shift right by } \frac{b}{a}} x\left(a\left(t-\frac{b}{a}\right)\right) = x(at-b)$$

Figure 2.29: Two different sequence of operations to obtain $x(at - b)$ from $x(t)$

Computing $y(t) = x\left(\frac{t-b}{a}\right)$

Given a signal $x(t)$, the signal $y(t) = x\left(\frac{t-b}{a}\right)$ can be obtained by performing two operations on $x(t)$ or by passing $x(t)$ through two transformations - scaling and shifting. Again, the scaling and shifting can be performed in any order as shown below. In this case, the easier option is to scale by $1/a$ first and then to shift by b to the right.

$$x(t) \xrightarrow[t \leftarrow \frac{t}{a}]{\text{Scale time axis by } \frac{1}{a}} x\left(\frac{t}{a}\right) \xrightarrow[t \leftarrow t-b]{\text{Time shift right by } b} x\left(\frac{t-b}{a}\right)$$

$$x(t) \xrightarrow[t \leftarrow t-\frac{b}{a}]{\text{Time shift right by } \frac{b}{a}} x\left(t-\frac{b}{a}\right) \xrightarrow[t \leftarrow \frac{t}{a}]{\text{Scale time axis by } \frac{1}{a}} x\left(\frac{t}{a}-\frac{b}{a}\right) = x\left(\frac{t-b}{a}\right)$$

Figure 2.30: Two different sequence of operations to obtain $x\left(\frac{t-b}{a}\right)$ from $x(t)$

Computing $y(t) = x(-at - b)$

Given a signal $x(t)$, the signal $y(t) = x(-at-b)$ is obtained by three operations - a shifting, scaling and reflection. The natural order would be shift right by b first and then the scaling and reflection can be done in any order.

$$x(t) \xrightarrow[t \leftarrow t-b]{\text{Time shift right by } b} x(t-b) \xrightarrow[t \leftarrow at]{\text{Scale time by } a} x(at-b) \xrightarrow[t \leftarrow -t]{\text{Reflect about } Y\text{-axis}} x(-at-b)$$

$$x(t) \xrightarrow[t \leftarrow -t]{\text{Reflect about } Y\text{-axis}} x(-t) \xrightarrow[t \leftarrow t+b]{\text{Shift } \textit{left} \text{ by } b} x(-(t+b)) = x(-t-b) \xrightarrow[t \leftarrow at]{\text{Scale time axis by } a} x(-at-b)$$

Figure 2.31: Two different sequence of operations to obtain $y(t) = x(-at-b)$

While there is a natural order to perform these transformations, I highly suggest that you do not blindly memorize them. The main thing to remember from this section is that when performing multiple transformations, scaling the time axis corresponds to replacing t by at and shifting by b units to the right corresponds to replacing t by $t-b$.

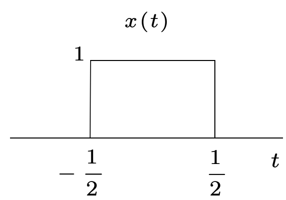
Transformations correspond only to modifying t not the entire argument

The transformations do not correspond to multiplying the argument of the signal by a or shifting the argument by b. To explain this further,if we wanted to compute $y(t) = x(2t-3)$, we start with $x(t)$ and shift it by 3 to get $x(t-3)$ and at this stage, scaling the time axis by 2 correponds to replacing t by $2t$ and it does not correspond to replacing the entire argument $t-3$ by $2(t-3)$.

Wrong arrow directions

The arrow directions in the following section of the notes are in the opposite direction. I would like to use $t \leftarrow 3t$ instead of $t \rightarrow 3t$. This needs to be fixed.

Example 2.7.1. *Given a signal $x(t)$ as shown below:*



Perform the following and plot $y(t)$:-

1. $y(t) = x(3t+2)$
2. $y(t) = x(2 - \frac{t}{2})$

Solution:

1. *The operations are performed as shown below-*

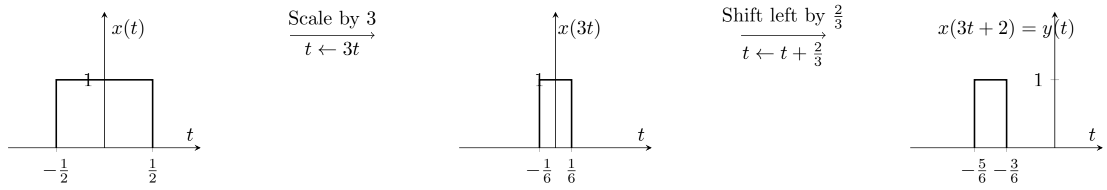


Figure 2.32: Transformations for obtaining $x(3t+2)$ from $x(t)$

2. The operations are performed as shown below-

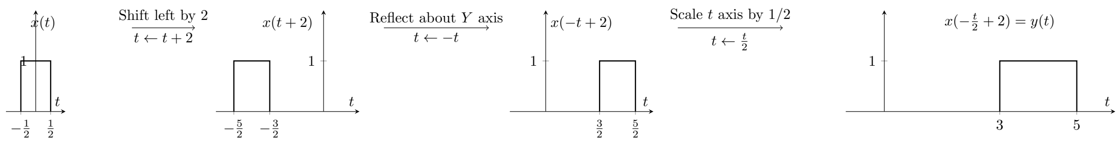


Figure 2.33: Transformations for obtaining $y(t) = x(2 - \frac{t}{2})$ from $x(t)$

When $t = 0,\ y(0) = x(2-0) = x(2) = 0$
and $t = 4,\ y(4) = x(2-2) = x(0) = 1$

Example 2.7.2. *Given a signal $x(t)$ as shown below. Plot $y(t) = x(2t-4)$.*

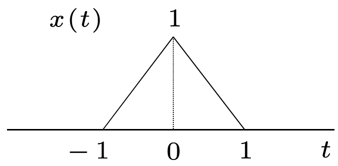


Figure 2.34: Plot of $x(t)$

Solution:The operations are performed as shown below-

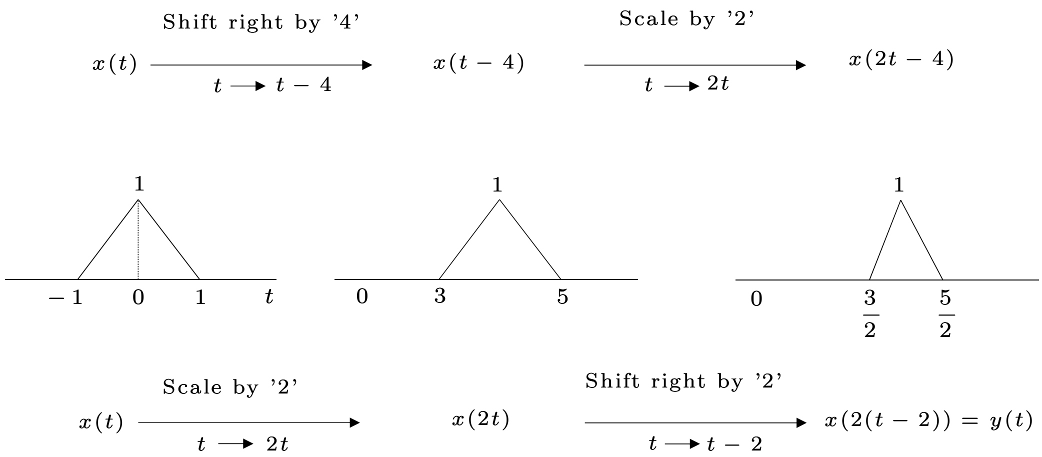


Figure 2.35: Transformations for obtaining $x(2t-4)$ from $x(t)$

2.7.13 Practice Problems

Given a signal $x(t)$ as shown in Fig 2.36, compute and plot $y(t)$ when

1. $y(t) = x(3t-2)$
2. $y(t) = x(-3t+2)$
3. $y(t) = x\left(\frac{3t+4}{5}\right)$

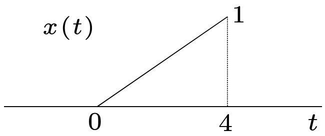


Figure 2.36:

Solution:

1. The operations are performed as shown below-

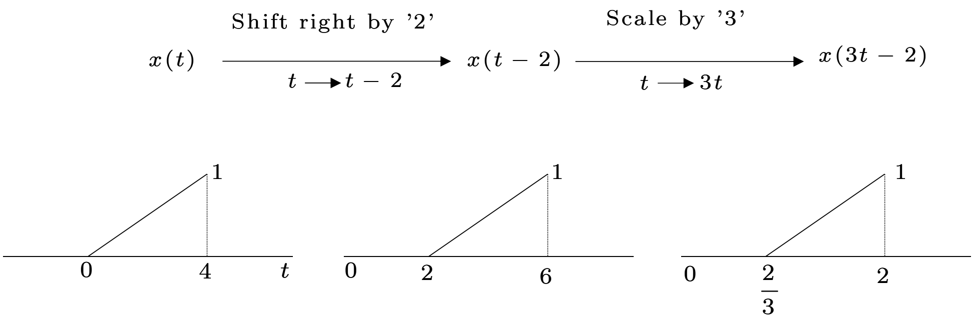


Figure 2.37:

2. The operations are performed as shown below-

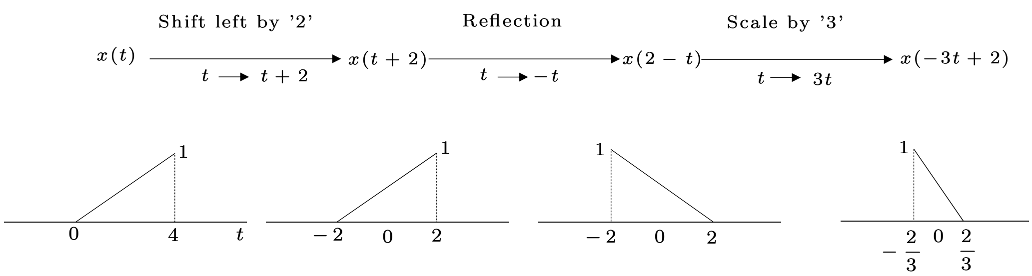


Figure 2.38:

3. The operations are performed as shown below-

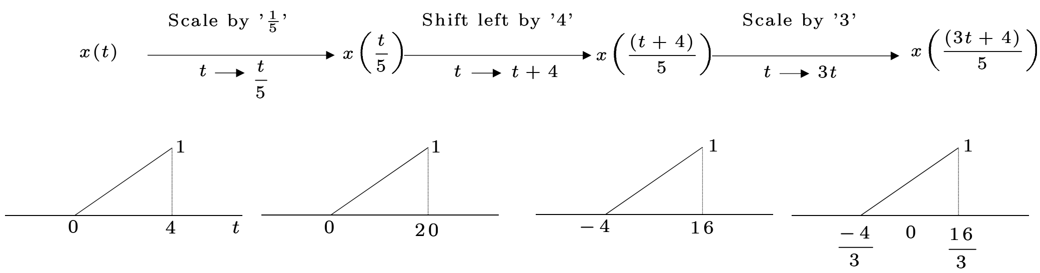


Figure 2.39:

Example 2.7.3. *Let $x(t)$ be a signal with $x(t) = 0$ when $t < 3$ and consider the following signals derived from $x(t)$. For each of these signals find the value of t for which the derived signal is guaranteed to be zero.*

1. $x(1-t) + x(2-t)$
2. $x(1-t)x(2-t)$
3. $x(3t)$
4. $x\left(\frac{t}{3}\right)$

Solution:
The signal $x(t)$ is-

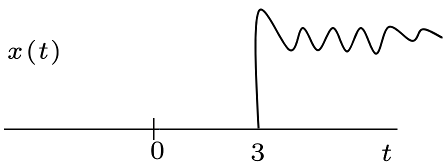


Figure 2.40: Plot of $x(t)$.

The signals $x(1-t)$,$x(2-t)$,$x(3t)$ and $x(t/3)$ are as shown below-

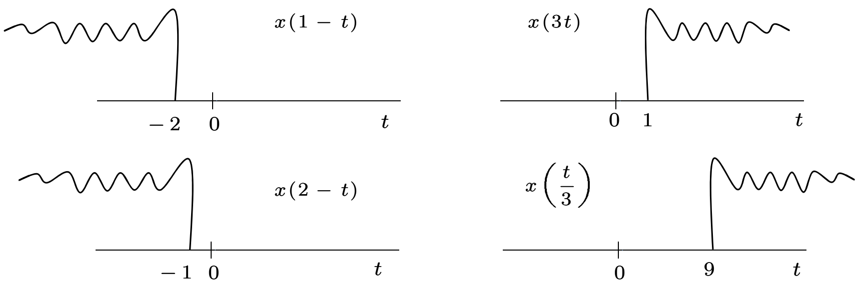


Figure 2.41: Plots of $x(1-t)$, $x(2-t)$, $x(3t)$, and $x\left(\frac{t}{3}\right)$.

1. *For $t > -1$ the signal is zero.*
2. *For $t > -2$ the signal is zero*

3. *For* $t > 1$ *the signal is zero*

4. *For* $t > 9$ *the signal is zero*

2.7.14 Transformation of signals defined piecewise - (video)

We have already seen that sometimes signals will be defined piecewise in this class. When applying transformations of the independent variable (time) to these signals, a common mistake is to apply the transformation to the expression for the function without applying the transformation to the t in the specification of the interval ranges. Consider the following example and pay attention to the t in the definition of the interval too.

Example 2.7.4. *Consider the signal $x(t)$ defined as follows-*

$$x(t) = \begin{cases} 2t & \text{if } t > 0 \\ -t & \text{if } t < 0 \\ 0 & \text{if } t = 0 \end{cases} \tag{2.19}$$

Find $x(-t)$,$\frac{1}{2}\left[x(t) + x(-t)\right]$ and $x(1-2t)$

Solution: Given $x(t)$, we can obtain $x(-t)$ by replacing t by $-t$, but this has to be done everywhere t appears including in the definition of the intervals.

$$x(-t) = \begin{cases} 2(-t) & \text{if } \boxed{-t > 0} \\ -(-t) & \text{if } -t < 0 \\ 0 & \text{if } -t = 0 \end{cases} \tag{2.20}$$

On simplifying,

$$x(-t) = \begin{cases} -2t & \text{if } t < 0 \\ t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases} \tag{2.21}$$

So,the corresponding plot is as follows-

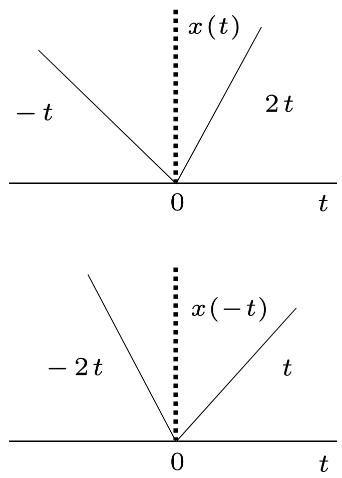


Figure 2.42: Plots of $x(t)$ and $x(-t)$.

Now,

$$\frac{1}{2}\big[x(t)+x(-t)\big] = \begin{cases} -\frac{3}{2}t & \textit{if } t<0 \\ -\frac{3}{2}t & \textit{if } t>0 \\ 0 & \textit{if } t=0 \end{cases} \tag{2.22}$$

Next,

$$x(1-2t) = \begin{cases} 2(1-2t) & \textit{if } 1-2t>0 \Rightarrow t<\frac{1}{2} \\ -(1-2t) & \textit{if } 1-2t<0 \Rightarrow t>\frac{1}{2} \\ 0 & \textit{if } 1-2t=0 \Rightarrow t=\frac{1}{2} \end{cases} \tag{2.23}$$

Generalizing the above, suppose

$$x(t) = \begin{cases} g_1(t) & \text{if } h_1(t)>0 \\ g_2(t) & \text{if } h_2(t)>0 \\ . & . \\ . & . \\ g_N(t) & \text{if } h_N(t)>0 \\ g_{N+1}(t) & \text{if } h_{N+1}(t)>0 \\ . & . \\ . & . \\ g_{N+M}(t) & \text{if } h_{N+M}(t)=0 \end{cases} \tag{2.24}$$

What is $x(f(t))$?

$$x(f(t)) = \begin{cases} g_1(f(t)) & \text{if } h_1(f(t))>0 \\ g_2(f(t)) & \text{if } h_2(f(t))>0 \\ . & . \\ . & . \\ g_N(f(t)) & \text{if } h_N(f(t))>0 \\ g_{N+1}(f(t)) & \text{if } h_{N+1}(f(t))>0 \\ . & . \\ . & . \\ g_{N+M}(f(t)) & \text{if } h_{N+M}(f(t))=0 \end{cases} \tag{2.25}$$

2.8 Symmetry in real signals: even and odd signals - (video)

A *real* CT signal $x(t)$ is said to be

$$\begin{aligned} \text{Even if: } x(t) &= x(-t) \\ \text{Odd if: } x(t) &= -x(-t) \end{aligned}$$

What this means that if $x(t)$ is even, a sketch of $x(t)$ is symmetric about the Y-axis, i.e., flipping $x(t)$ about the Y-axis will result in the same signal $x(t)$. If $x(t)$ is odd, then flipping $x(t)$ about the Y-axis will result in $-x(t)$, or $x(t)$ flipped about the X-axis.

Examples of even signals include $x(t) = \cos(10\pi t)$ or $x(t) = t^2$ and examples of odd signals include $x(t) = \sin(10\pi t)$ or $x(t) = t$. The following figures show an example of an even signal and an odd signal.

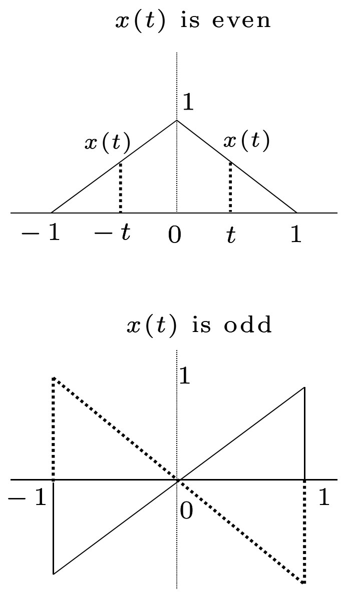


Figure 2.43: Even and odd signals.

Example 2.8.1. *Is* $x(t) = \cos(10\pi t + \pi/50)$ *an even signal?*

Example 2.8.2. *Is* $\cos(10\pi t)u(t)$ *an even signal?*

Any real signal $x(t)$ can be written as the sum of an even signal and odd signal, i.e., $x(t)$ can be written as

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t)$ is an even signal and $x_o(t)$ is an odd signal. $x_e(t)$ and $x_o(t)$ are referred to as the even part and odd part of $x(t)$, respectively. To, see why $x(t)$ can always be written as the sum of an even part and an odd part, notice that $x(t)$ can be written as

$$x(t) = \underbrace{\frac{1}{2}\left[x(t) + x(-t)\right]}_{x_e(t)} + \underbrace{\frac{1}{2}\left[x(t) - x(-t)\right]}_{x_o(t)}.$$

If we set $x_e(t) = \frac{1}{2}\big[x(t) + x(-t)\big]$ and $x_o(t) = \frac{1}{2}\big[x(t) - x(-t)\big]$, we can see that $x_e(t)$ and $x_o(t)$ are even and odd, respectively.

$$x_e(t) = \frac{1}{2}\big[x(t) + x(-t)\big] \Rightarrow x_e(-t) = \frac{1}{2}\big[x(-t) + x(t)\big]$$

and,

$$x_o(t) = \frac{1}{2}\big[x(t) - x(-t)\big] \Rightarrow x_o(-t) = \frac{1}{2}\big[x(-t) - x(t)\big]$$

Even odd decomposition

Summarizing this, any real signal $x(t)$ can be written as the sum of an even signal and an odd signal. The even and odd parts of $x(t)$ are given below

$$x(t) = x_e(t) + x_o(t) \tag{2.26}$$

$$x_e(t) = \frac{1}{2}\big[x(t) + x(-t)\big] \tag{2.27}$$

$$x_o(t) = \frac{1}{2}\big[x(t) - x(-t)\big] \tag{2.28}$$

Example 2.8.3. *Find the even and odd parts of the signal $x(t)$ as shown:-*

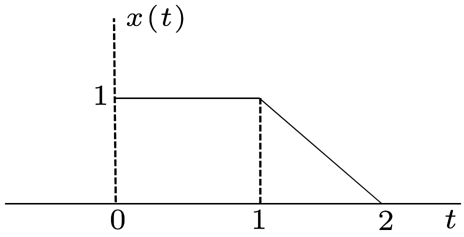


Figure 2.44:

Solution: We know that,

$$x_e(t) = \frac{1}{2}\big[x(t) + x(-t)\big] \qquad \textit{and} \qquad x_o(t) = \frac{1}{2}\big[x(t) - x(-t)\big]$$

Therefore,

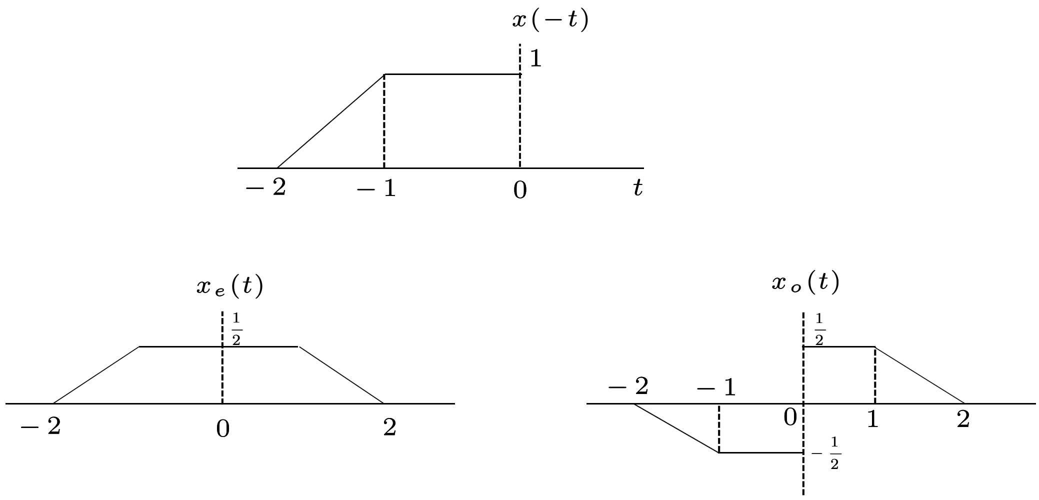


Figure 2.45:

Example 2.8.4. *Find the even and odd parts of the signal* $x(t) = \cos(t) + \sin(t) + \cos(t)\ \sin(t)$.

Solution:

$$\begin{aligned} x(-t) &= \cos(-t) + \sin(-t) + \cos(-t)\ \sin(-t) \\ &= \cos(t) - \sin(t) - \cos(t)\sin(t) \end{aligned} \tag{2.29}$$

So, $x_e(t) = \cos(t)$ *and* $x_o(t) = \sin(t) + \cos(t)\sin(t)$

This can be extended to DT signals. A *real* DT signal $x[n]$ is said to be

$$\begin{aligned} \text{Even if: } x[n] &= x[-n] \\ \text{Odd if: } x[n] &= -x[-n] \end{aligned}$$

Even odd decomposition for DT signals

Any real DT signal $x(t)$ can be written as the sum of an even signal and an odd signal or broken down into an even part and an odd part. The even and odd parts are given by

$$x[n] = x_e[n] + x_o[n] \tag{2.30}$$

$$x_e[n] = \frac{1}{2}\big[x[n] + x[-n]\big] \tag{2.31}$$

$$x_o[n] = \frac{1}{2}\big[x[n] - x[-n]\big] \tag{2.32}$$

2.8.1 Properties of even and odd signals - (video)

1. Addition and subtraction of signals

   - Even signal ± even signal = even signal
   - Odd signal ± odd Signal = odd signal
   - Even signal ± odd Signal = neither even nor odd

2. Multiplication of two signals
   - Even signal $\times$ even signal $=$ even signal
   - Odd signal $\times$ odd signal $=$ even signal
   - Even signal $\times$ odd signal $=$ odd signal
3. Integrals
   - If $x(t)$ is odd then $\int_{-A}^{A} x(t)dt$=0. For example,$\int_{-1}^{1} \sin^3(t)\ dt = 0$
   - If $x(t)$ is even then $\int_{-A}^{A} x(t)dt$=2$\int_{0}^{A} x(t)dt$

2.8.2 Symmetry in complex signals - conjugate symmetry - video

This section can wait until we get to Fourier transforms.

> Typo in the video
>
> There is a small typo in the video at 2.08. The correct expression is $-x(t) = x^*(-t)$

The role of even and odd signals for complex signals is played by what are called conjugate-symmetric signals and conjugate anti-symmetric or skew-symmetric signals. Suppose $x(t)$ is a complex signal written as $x(t) = a(t) + jb(t) = r(t)e^{j\theta(t)}$. $x(t)$ is said to be conjugate symmetric if

$$\text{Conjugate symmetric if } x(t) = x^*(-t) \tag{2.33}$$

$$\text{Conjugate anti-symmetric if } x(t) = -x^*(-t) \Rightarrow -x(t) = x^*(-t). \tag{2.34}$$

At this point, this may seem unmotivated as to why we should care about these properties or why symmetry should be defined with conjugates in mind, i.e., why not define a signal as being symmetric if $x(t) = x(-t)$? We will have to wait until we cover Fourier transforms to see why this definition makes sense and how the Fourier transforms of conjugate symmetric signals has similar properties to that of the Fourier transforms of even signals.

Since $x(t) = a(t)+jb(t)$, $x^*(-t) = a(-t)-jb(-t)$. Then, if $x(t)$ is conjugate-symmetric, this implies that

$$a(t) = a(-t) \Rightarrow a(t) \text{ is even}$$
$$\text{and } b(t) = -b(-t) \Rightarrow b(t) \text{ is odd}$$

If $x(t)$ is expressed in polar form as $x(t) = r(t)e^{j\theta(t)}$ then $x^*(-t) = r(-t)e^{-j\theta(-t)}$. If $x(t)$ is conjugate-symmetric, then

$$r(t) = r(-t) \Rightarrow r(t) \text{ is even}$$
$$\text{and } \theta(t) = -\theta(-t) \Rightarrow \theta(t) \text{ is odd}$$

In summary,

1. If $x(t)$ is conjugate-symmetric, its real part is an even signal, its imaginary part is an odd signal, its magnitude is an even signal and its phase is an odd signal.
2. Can you derive the similar relationships when $x(t)$ is conjugate anti-symmetric?

2.9 Commonly Encountered Signals

2.9.1 Real continuous-time exponential signals - (video)

A *real* CT exponential signal is given by

$$x(t) = Be^{at} \tag{2.35}$$

where a is *real*. If $a < 0$, it is called a damped exponential and if $a > 0$, it is called a growing exponential. Notice that the signal $x(t)$ is defined to Be^{at} for all values of t. Signals that extend for all values of t are indeed impractical since we are often interested only in things that happen over a finite time duration. However, signals that extend forever are very useful mathematical abstractions.

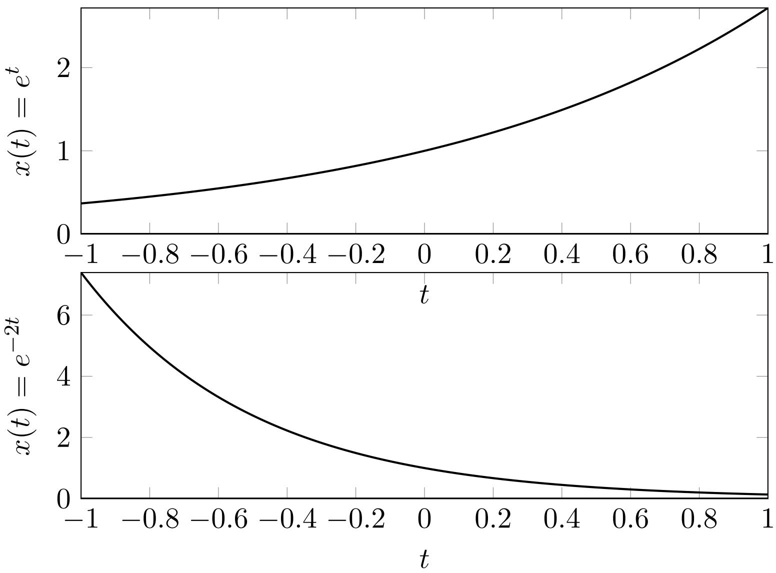


Figure 2.46: Plots of signals $x(t) = e^t$ and $x(t) = e^{-2t}$

You are more likely to encounter signals that are defined to be a CT exponential over specific range of values of t, for example, $x(t) = Be^{at}$ only for $t \geq 0$ and $x(t) = 0$ for $t < 0$. An example of where you might encounter such a signal is when a capacitor discharges after being charged to a certain voltage. The voltage across the capacitor at time t is given by

$$V(t) = V(0)e^{-t/RC}. \tag{2.36}$$

This corresponds to $B = V(0)$ and $a = 1/RC$.

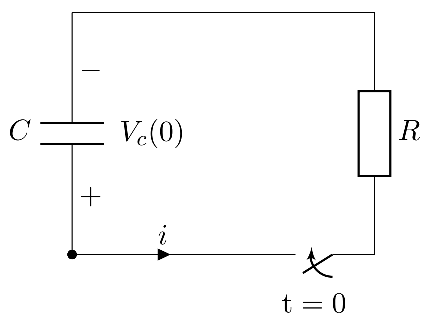


Figure 2.47: The voltage across the capacitor is a decaying exponential signal

2.9.2 Real discrete-time exponential signal

A *real* DT exponential signal is given by

$$x[n] = Br^n$$

where r is *real*. Sometimes such signals naturally occur as DT signals. When discussing the spread of the pandemic, you have heard about the term R_0 in recent times. If $x[n]$ represents the number of infected patients every day and if we assume that infected people infect other people with a certain probability, if this is left unchecked, the number of infected people would increase exponentially. A more uplifting example is that if you invest \$$B$ in the bank and if the bank gives you i% interest, the money will grow exponentially due to compound interest. The money in the bank at the end of the nth year will be given by

$$x[n] = B\left(1 + \frac{i}{100}\right)^n.$$

This corresponds to $r = \left(1 + \frac{i}{100}\right)$. It might be a bit confusing to see why this signal is called an exponential signal even though e does not appear directly in $x[n]$. However, a little manipulation will show that this is indeed an exponentially increasing signal. If we rewrite r as $e^{\ln r}$, we can see that $x[n] = Be^{(\ln r)n}$ where now $\ln r$ plays the role of a in the case of CT signals.

Sometimes, the DT real exponential signal is obtained as a sampled version of a CT real exponential signal. If the signal $x(t) = Be^{at}$ is sampled every T_s seconds, to get $x[n] = x(nT_s)$, then

$$x[n] = Be^{anT_s} = Be^{aT_sn} = Br^n$$

where $r = e^{aT_s}$.

Sketch an example

2.9.3 Continuous-time sinusoids

A CT sinusoid is given by

$$x(t) = A\sin(\omega t + \theta) \text{ or } A\cos(\omega t + \theta) \tag{2.37}$$

where, A is the amplitude, ω is the angular frequency in rad/s and $\omega = 2\pi f$, where f is the frequency in Hertz, θ is phase shift. As an example $x(t) = 2\cos(4\pi t + \frac{\pi}{4})$ is plotted in Fig. 2.48

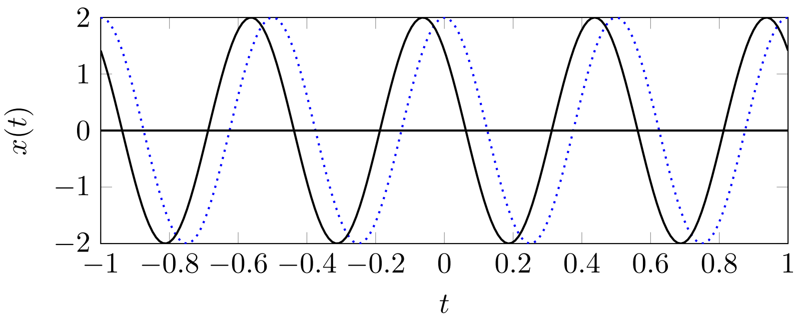


Figure 2.48: The solid line is a plot of $x(t) = 2\cos(4\pi t + \frac{\pi}{4})$.

Notice that $x(t)$ is a periodic signal, i.e. $x(t+T) = A\cos(\omega t + \omega T + \theta)$ for any $T = k\frac{2\pi}{\omega}$, where k is *any* integer. $\frac{2\pi}{\omega}$ is called the fundamental period.

2.9.4 Discrete-time sinusoids

A discrete-time sinusoid is given by

$$x[n] = A\cos(\Omega n + \theta), \tag{2.38}$$

where A is called the amplitude, Ω is the frequency and θ is a phase shift. DT sinusoids are typically obtained by sampling continuous-time sinusoidal signals $x(t) = A\cos(\omega t + \theta)$. Assuming the sampling time as T_s, its discrete-time version is

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \theta)$$

By letting $\Omega = \omega T_s$, we can see that $x[n]$ can be written as

$$x[n] = A\cos(\Omega n + \theta). \tag{2.39}$$

Unlike CT sinusoids, DT sinusoids are not always periodic and this has to do with the fact that for DT sinusoids, only integer values make sense for the fundamental time period. Let us consider $x[n+N] = A\cos(\Omega n+\Omega N+\theta)$. For this to be same as $x[n] = A\cos(\Omega_n+\theta)$, we need $\Omega N = 2\pi m$, for some integers m, N. This is sometimes feasible but not always. The following two examples will illustrate this. If this is feasible, the smallest positive integer value of N for which this is feasible is called the fundamental time period.

Example 2.9.1. *Is the signal $x[n] = 5\cos(0.6\pi n)$ periodic? If so, what is the time period? Solution:*

$$\begin{aligned} x[n+N] &= 5\cos(0.6\pi n + 0.6\pi N) = 5\cos(0.6\pi n) \\ \Rightarrow 0.6\pi N &= 2\pi m \\ \Rightarrow N &= \frac{2m}{0.6} = \frac{20m}{6} = \frac{10m}{3} \\ \Rightarrow m &= 3, N = 10 \end{aligned}$$

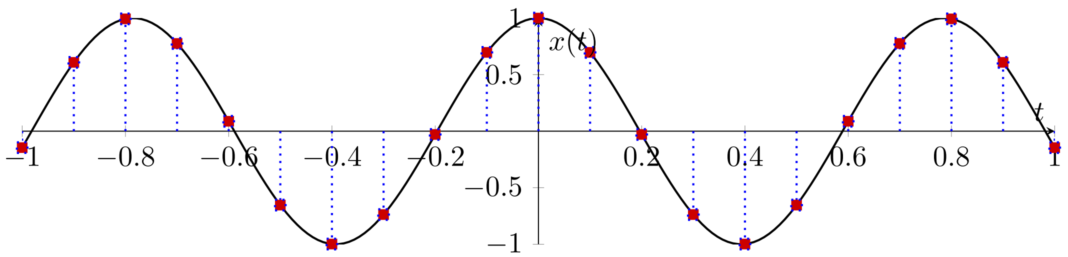
Example 2.9.2. *Determine if $x[n] = A\cos(6n)$ is periodic or not.*

Solution:

$$6N = 2\pi m$$

$$\Rightarrow 6N = 2\pi m$$
$$\Rightarrow \pi = \frac{3N}{m}$$
$$\Rightarrow \textit{Not periodic}$$

Remark. *It is important to realize that while the CT signal $x(t) = A\cos(6t)$ is periodic, the DT signal $x[n] = A\cos(6n)$ obtained by sampling $x(t)$ is not periodic.*



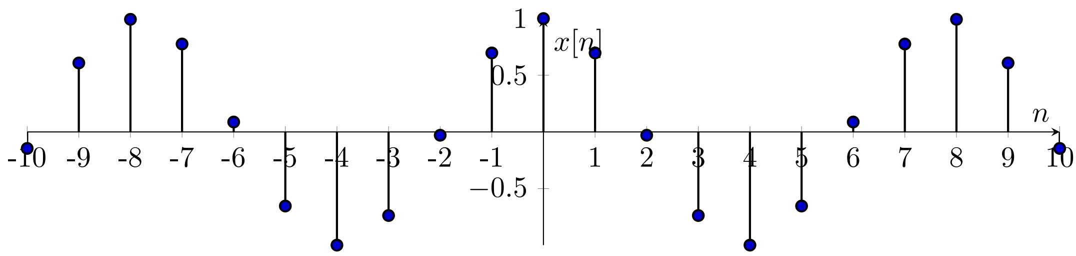


Figure 2.49: The solid line in the top figure is a plot of $x(t) = \cos(8t)$. This signal is periodic. The dotted lines correspond to the time instants where the signal is sampled. The stem plot in the bottom figure is a plot of the sampled signal $x[n] = \cos(8nT_s)$ for $T_s = 0.1$. The discrete-time signal is symmetric, but it *not periodic*.

2.9.5 Complex exponential signals - (video)

Consider the signal $x(t) = e^{st}$ where $s = \sigma + j\omega$ is some complex number. Such a signal is called a *Complex* exponential signal and s is called the complex frequency. The real part and imaginary part of $x(t)$ are each real functions of a real variable t and can be obtained as follows. Notice that $x(t)$ can be written as

$$\begin{aligned} x(t) &= e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} \\ &= e^{\sigma t}\left(\cos(\omega t) + j\sin(\omega t)\right) \\ &= \underbrace{e^{\sigma t}\cos(\omega t)}_{\text{Real part}} + j\underbrace{e^{\sigma t}\sin(\omega t)}_{\text{imaginary part}} \end{aligned}$$

Notice that

$$\begin{aligned} |x(t)| &= e^{\sigma t} \\ \angle x(t) &= \omega t \end{aligned}$$

The real part of $x(t)$, imaginary part of $x(t)$, magnitude of $x(t)$ and phase of $x(t)$ are signals themselves. These are plotted in Fig. 2.50 for $\sigma = -1$ and $\omega = 2\pi$. The dotted blue lines in the figures show $|x(t)|$ and it can be seen that it defines the envelope of the real and imaginary parts of $x(t)$.

It is useful to get insight into what happens to $x(t) = e^{st}$ as $t \to \infty$. It can be seen that if $\sigma < 0$, $x(t) \to 0$ as $t \to \infty$ and if $\sigma > 0$, $x(t)$ becomes undefined at $t \to \infty$ with the $|x(t)| \to \infty$ as $t \to \infty$. Thus, the real part of s determines whether the signal $x(t)$ is bounded or unbounded as $t \to \infty$.

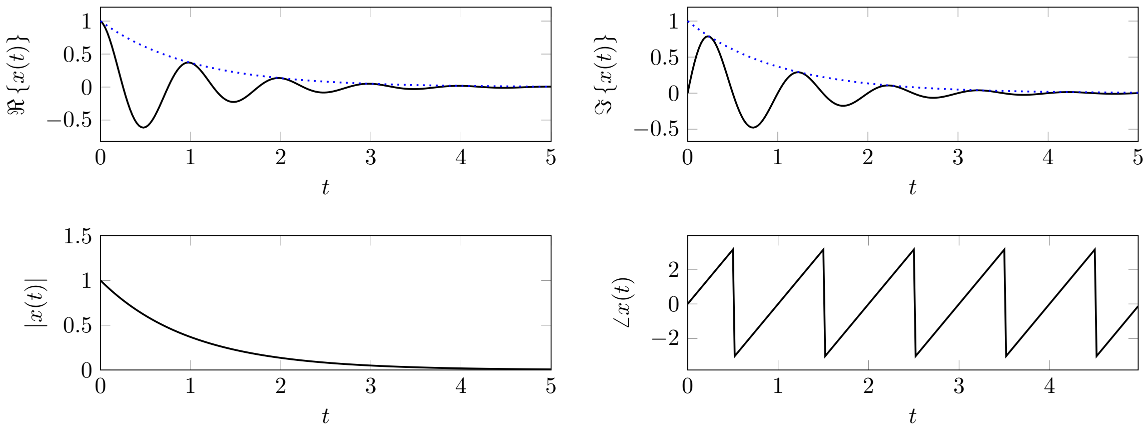


Figure 2.50: Plots of $\Re x(t)$ and $\Im x(t)$ versus t and $|x(t)|$ and $\angle x(t)$ versus t.

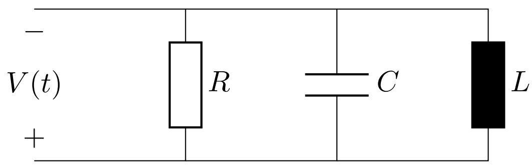


Figure 2.51: An LRC circuit

As a concerete example of where you might encounter complex exponentials in your ECE curriculum, consider the following LRC circuit. If the voltage across the capacitor at time 0 is denoted by $V(0)$ and $V'(0) = \sigma V(0)$, then the voltage across the capacitor as a function of time is given by

$$V(t) = V(0)e^{-t/2RC}\cos(\omega_0 t)$$

This is $V(0)\Re\left[e^{(\sigma+j\omega)t}\right]$, with $\sigma = \frac{-1}{2RC}$. For the RLC circuit in Fig. 2.51, ω_0 is $\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$.

More importantly, complex exponential signals form a key role in the Laplace and Fourier transforms. Roughly speaking, a lot of signals can be written as linear combinations of complex exponential signals and this decomposition turns out to be one of the most insightful tools in the analysis of what are called linear time invariant systems.

2.9.6 Discrete-time complex exponential signals

A Discrete-time *complex* exponential is a DT signal given by

$$x[n] = Bz^n \tag{2.40}$$

where z is *complex*. Since z is complex, it can be written as $z = re^{j\Omega}$ and hence $x[n] = Br^n e^{j\Omega n}$. Similar to CT signals, we can think of the real part, imaginary part, magnitude and phase of $x[n]$ as signals and plot them as a function of n.

2.9.7 Unit Step Signal (Function) - (video)

The unit step signal is used to conveniently model phenomena that remains inactive until a certain time and then jumps to a constant value at a particular time instant, such as when a switch is flipped. Consider the voltage across the two terminals in Fig. 2.52. The voltage remains at zero until time $t = 0$ and then jumps to 1v at $t = 0$ as soon as the switch is closed. In this case, the voltage would be a unit step signal. Formally, the continuous time unit step function is defined as follows:

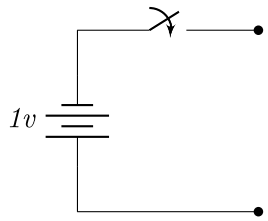


Figure 2.52: The output voltage waveform is a unit step signal

$$u(t) := \begin{cases} 1 & \text{if } t > 0 \\ 1/2 & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{2.41}$$

and is plotted in Fig. 2.53. In reality, physical phenomena do not abruptly jump at one time instant and they usually increase sharply over a small time interval. Consider the signal $u_\Delta(t)$ shown in Fig. 2.53. $u(t)$ can be thought of as the $\lim_{\Delta\to 0} u_\Delta(t)$.

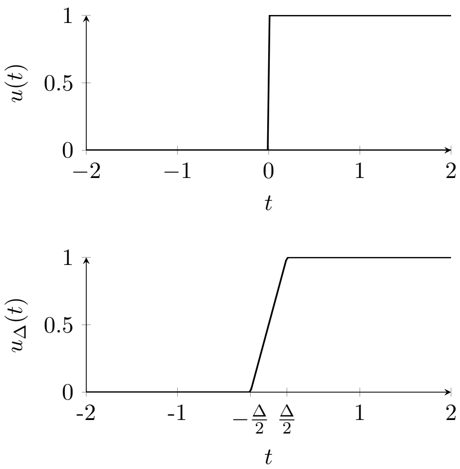


Figure 2.53: Continuous-time unit step signal $u(t)$ is obtained as the limit of $u_\Delta(t)$ as $\Delta \to 0$.

Effect of multipliying by $u(t-t_0)$ What is the relationship between the signal $x(t)$ and $y(t) = x(t)u(t-t_0)$? The latter signal is a succint representation for the signal given by

$$y(t) = \begin{cases} 0, & t < t_0 \\ x(t), & t > t_0 \end{cases}$$

Students are often not careful about the difference between these two signals. Consider two signals $2\cos(4\pi t)$ and $2\cos(4\pi t)u(t)$. These two signals are depicted in Fig. 2.54. Notice the effect of multiplying by $u(t)$. This results in what are called right sided signals.

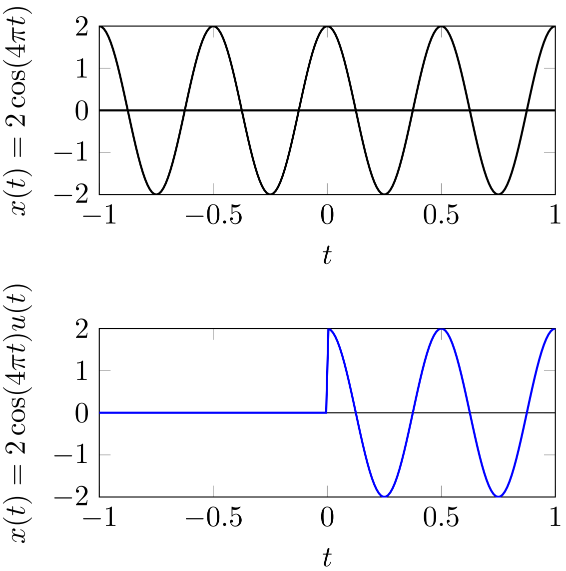


Figure 2.54: Difference between the signals $x(t) = 2\cos(4\pi t)$ and $x(t) = 2\cos(4\pi t)u(t)$

Think about the answer to the following questions

1. Plot $x(t) = e^{-t}$ and $x(t) = e^{-t}u(t)$ for $-1 \le t < 1$.
2. What happens to a signal $x(t)$ if it is multiplied by $u(-t)$?
3. What happens to a signal $x(t)$ if it is multiplied by $u(t-2)$?
4. What happens to a signal $x(t)$ if it is multiplied by $u(t-t_1) - u(t-t_2)$?
5. What is the difference between $x(t)u(t-2)$ and $x(t-2)u(t-2)$?

2.9.8 DT Unit step function - (video)

The discrete time unit step function is defined as

$$u[n] := \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases} \tag{2.42}$$

and is plotted in Fig. 2.55. The DT unit step signal is similar in spirit to the CT unit step signal. Mathematically, the only difference is that since $U[n]$ is a DT signal, there is no issue of discontinuity at $n = 0$ and hence, $U[0]$ is defined to 1.

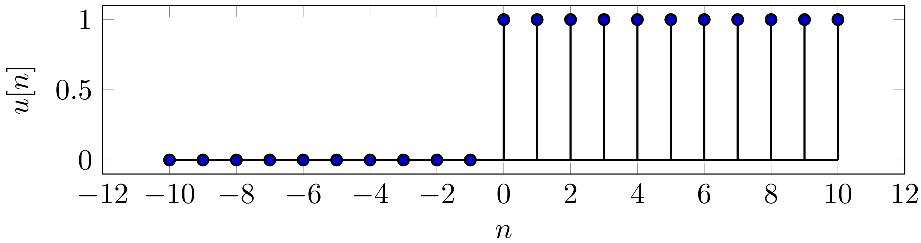


Figure 2.55: Discrete-time unit step function

2.9.9 CT and DT rectangular signals

A rectangular signal is a signal that can be used to describe phenomena that are active only over a finite time window. The unit rectangular signal is defined as follows

$$\text{rect}(t) := \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 1/2 & \text{if } t = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2.43}$$

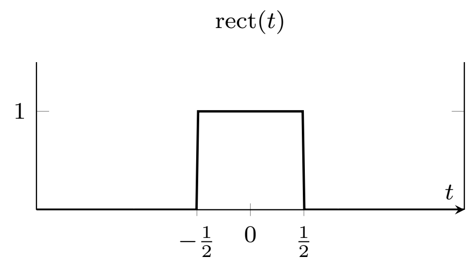


Figure 2.56: Continuous-time rectangular signal

The general form of the rectangular function both in continuous and discrete form is given by $\text{rect}_T(t)$ or $\text{rect}_N[n]$ shown in the following figure.

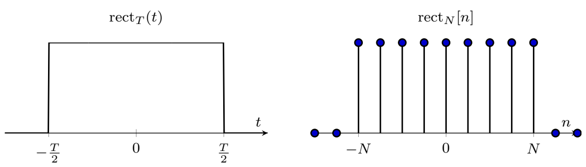


Figure 2.57: Continuous-time and Discrete-time rectangular signals

The rectangular signal can be used to define signals piecewise as shown in the examples below.

Example 2.9.3. *Consider the signal*

$$x(t) = \begin{cases} \sqrt{t}, & \textit{if } 1 \leq t \leq 3 \\ 0, & \textit{otherwise.} \end{cases}$$

This signal can be succintly written as $x(t) = \sqrt{t}\ rect((t-2)/2)$

Example 2.9.4. *Consider the signal*

$$x(t) = \begin{cases} \sqrt{t}, & \textit{if } \ 1 \leq t \leq 3 \\ \sin(2\pi t) & \textit{if } \ 3 < t \leq 4 \\ 0, & \textit{otherwise.} \end{cases}$$

$x(t)$ *can be succintly written as* $x(t) = \sqrt{t}\ rect((t-2)/2) + \sin(2\pi t) rect\big(t - \frac{7}{2}\big)$

Example 2.9.5. *Consider the signal*

$$x(t) = \begin{cases} f_1(t), & \text{if } \; t_1 \leq t \leq t_2 \\ f_2(t), & \text{if } \; t_3 \leq t \leq t_4 \\ 0, & \text{otherwise.} \end{cases}$$

How can you express this succintly using rect signals?

2.9.10 CT and DT ramp signals

The continuous time ramp function is given by-

$$ramp(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{2.44}$$

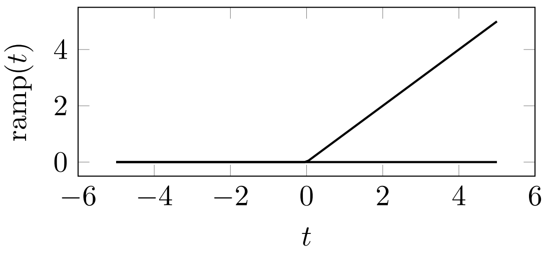


Figure 2.58: Continuous-time ramp signal

The discrete time ramp function is given by-

$$ramp[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \tag{2.45}$$

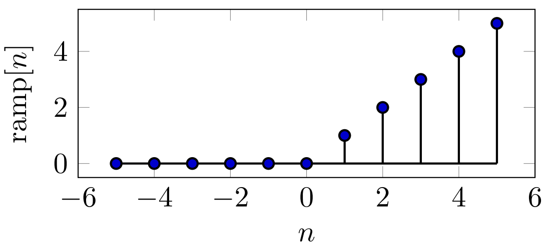


Figure 2.59: Discrete-time ramp signal

2.9.11 Sinc function

This section can be skipped until we get to Fourier transforms. A function that will be used often in the later chapters is the sinc function defined as

$$\operatorname{sinc}(t) := \begin{cases} \frac{\sin \pi t}{\pi t}, & t \neq 0; \\ 1, & t = 0. \end{cases}$$

A plot of the signal is shown in Fig. 2.60. Notice that the sinc signal (or, function) crosses zero for every integer value of t ($\neq 0$).

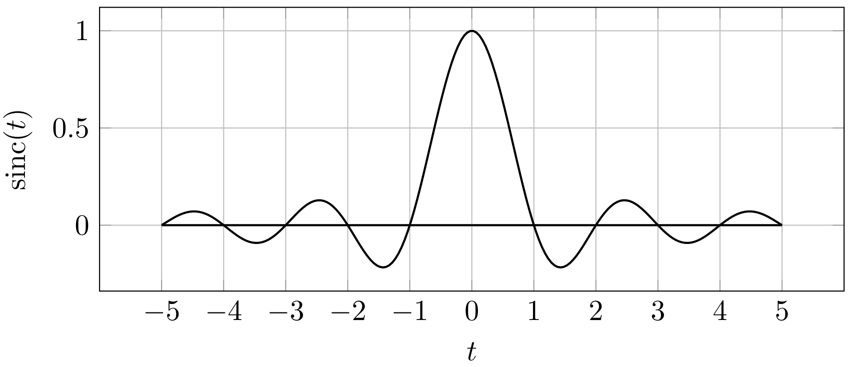


Figure 2.60: Sinc function $x(t) = \operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$

Example 2.9.6. *Express the signal $x(t) = \frac{\sin at}{at}$ in terms of the sinc function*

$$x(t) = \frac{\sin at}{at} = \frac{\sin \pi\left(\frac{at}{\pi}\right)}{\pi \frac{at}{\pi}} = sinc\left(\frac{at}{\pi}\right)$$

Example 2.9.7. *Express the signal $x(t) = \frac{\sin at}{bt}$ in terms of the sinc function.*

$$x(t) = \frac{\sin at}{bt} = \frac{a}{b}\frac{\sin \pi\left(\frac{at}{\pi}\right)}{\pi \frac{at}{\pi}} = \frac{a}{b} sinc\left(\frac{at}{\pi}\right)$$

2.9.12 Discrete time Impulse or Delta function - (video)

The unit impulse function is also known as the Kronecker delta function. It is used to model phenomena that occur only at one point in time. It is defined as below and shown in Fig. 2.61.

$$\delta[n] := \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.46}$$

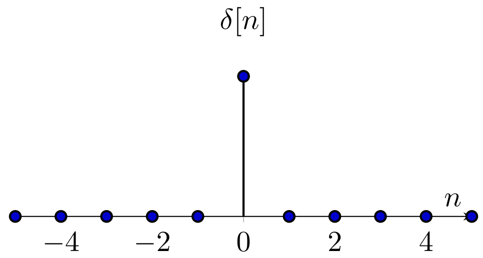


Figure 2.61: Discrete-time unit impulse signal or the Kronecker delta function $\delta[n]$.

From the definition of $\delta[n]$, we can see that

$$\delta[n]x[n] = \begin{cases} x[0] & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we see that

$$x[n]\delta[n] = x[0]\delta[n] \tag{2.47}$$

$$\sum_{n=-\infty}^{-\infty} x[n]\delta[n] = x[0] \tag{2.48}$$

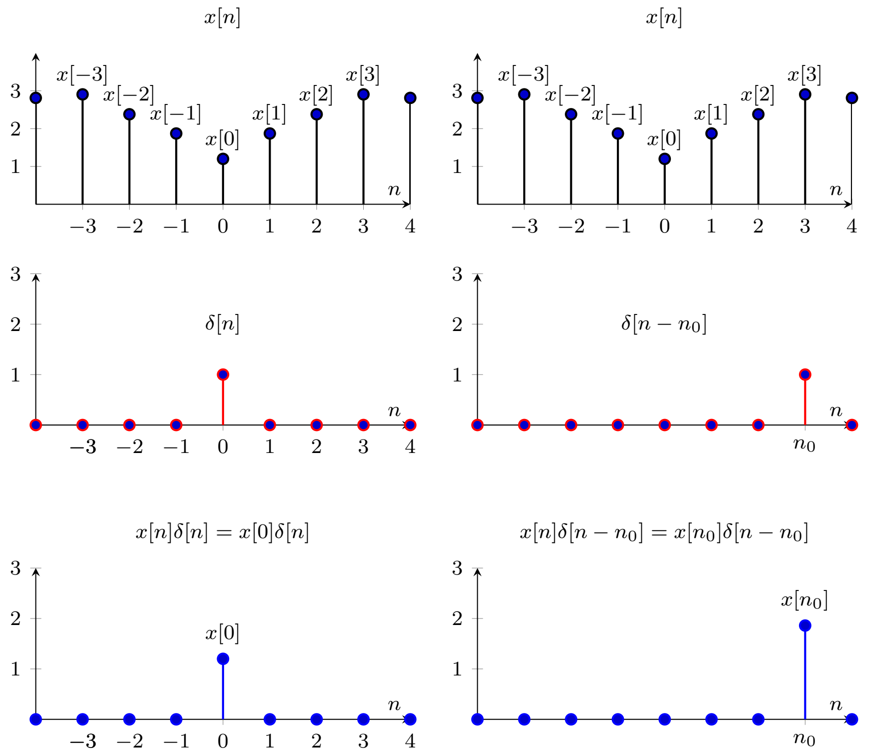


Figure 2.62: Figure showing that $x[n]\delta[n] = x[0]\delta[n]$ (left panel) and $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$.

Two main properties of the DT impulse or Kronecker delta signal

Generalizing this we get,

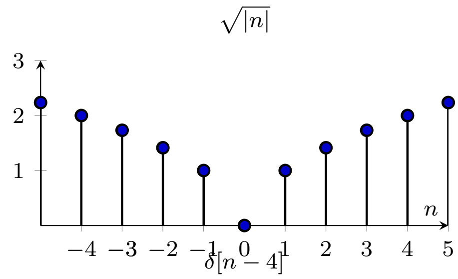
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0] \tag{2.49}$$

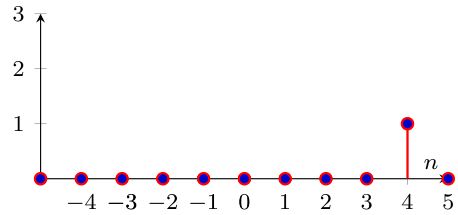
$$\sum_{n=-\infty}^{-\infty} x[n]\delta[n-n_0] = x[n_0] \tag{2.50}$$

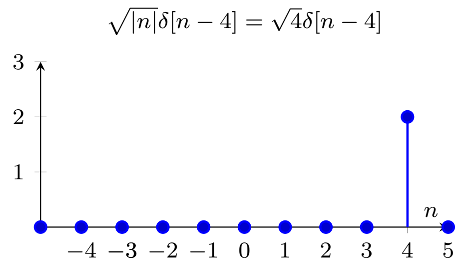
The second result is called the sifting property. Notice the spelling - it is not shifting property

Example 2.9.8.

$$\sqrt{|n|}\delta[n-4] = \sqrt{|4|}\delta[n-4] = 2\delta[n-4]$$







Example 2.9.9. *Evaluate* $\sum_{n=-\infty}^{-\infty} \cos\left(\frac{n\pi}{8}\right)\delta[n-2]$.

$$\sum_{n=-\infty}^{-\infty} \cos\left(\frac{n\pi}{8}\right)\delta[n-2] = \cos\left(\frac{n\pi}{8}\right)\Big|_{n=2} = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Expressing DT signals using $\delta[n]$ and shifted versions

Example 2.9.10.

$$x[n] = \begin{cases} 1, & \text{if } n = -3 \\ 2, & \text{if } n = -1 \\ 2, & \text{if } n = 0 \\ -1 & \text{if } n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

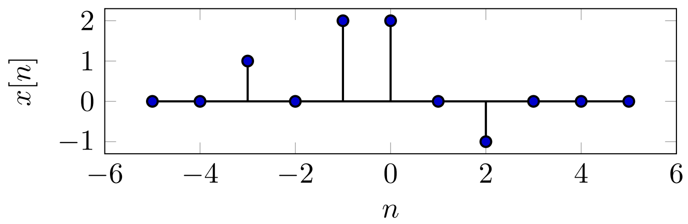


Figure 2.63: Plot of the signal $x[n]$

This signal can be succintly written as $x[n] = 1\delta[n+3] + 2\delta[n+1] + 2\delta[n] + (-1)\delta[n-2]$.

From this example, we can generalize and see that any DT signal $x[n]$ can be written as

$$x[n] = \ldots x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] \tag{2.51}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{2.52}$$

We should get used to this notation as it is much easier to express DT signals this way.

2.9.13 Continuous-time Impulse signal or Dirac-Delta function - (video)

Intuition: A continuous-time delta function or delta signal is a signal that can be used to model phenomena that occur for a very short period of time, but whose effect may last for a non-trivial amount of time. To derive some physical intuition as to how to define the CT equivalent of an impulse function, consider a golfer hitting a golf ball at time $t = 0$ (or your favorite sport where a ball is hit or kicked from rest). Let us ignore the effect of gravity for the purpose of this exercise and let us think about the signal which represents the force acting on the ball as a function of time. Until time $t = 0$, the force acting on the golf ball is zero. The golf club makes contact with the ball for a fraction of a second and so some force acts on the ball for a small time interval around $t = 0$ and once the club loses contact with the ball, no force acts on the ball. How would a plot of force on the ball as a function of time look in the case when the club makes contact for an infitesimally small amount of time? It is tempting to say that this function should be zero everywhere and non-zero at $t = 0$ and look like the one in Fig. 2.65.

Figure 2.64: A CT impulse can be used to model the force acting on a golf ball.

$$\delta(t) = \begin{cases} 1, & \text{if } t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Defining the CT impulse this way does *not* work

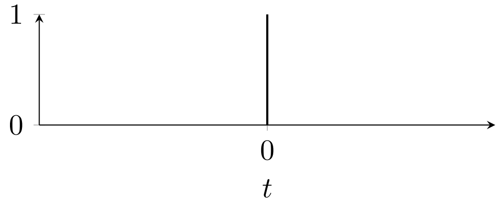


Figure 2.65: How not to define the continuous-time unit impulse signal

However, such a function fails to capture an important aspect of the situation - notice that the momentum of the ball jumps from 0 to a finite value within the time that the force acted on the ball. The impulse-momentum relationship in physics tells us that the integral of the force that acted on the ball should equal the (change in) momentum. Since the integral of the force is also the area under the force curve as a function of time, the force must act on the ball for an infinitesimal amount of time but the area under the force versus time plot must be non-zero! A function which satisfies this property is the Dirac-delta function or the CT impulse signal.

What properties must the delta function have? Mathematically, we would like to define a continuous-time signal that has similar properties as that of the DT delta function in (2.49) and (2.50), i.e., we would like to define a function $\delta(t)$ such that for any signal $x(t)$, the following two conditions are true

$$\begin{aligned} x(t)\delta(t) &= x(0)\delta(t) \\ \int_{-\infty}^{\infty} x(t)\delta(t)dt &= x(0). \end{aligned}$$

It is a bit tempting to follow the definition of the discrete-time delta function and define the continuous-time function as shown in Fig. 2.65. ⚠But this does not work. This will be non-zero only for one value of t and hence, this would not satisfy $\int_{-\infty}^{-\infty} x(t)\ \delta(t)dt = x(0)$.

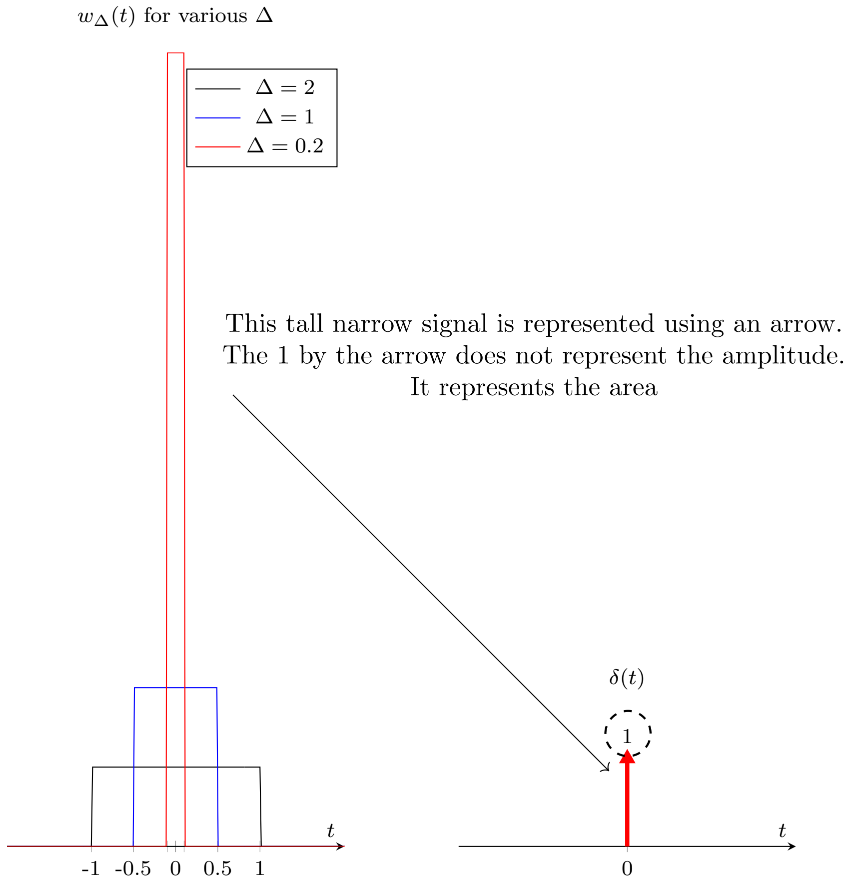


Figure 2.66: The CT delta function is obtained as the limit of $\lim_{\Delta\to\infty} w_\Delta(t)$

Formal definition: To formalize our intuition from the golf example, let us consider the family of rectangular signals $w_\Delta(t)$ defined as

$$w_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \le t < \frac{\Delta}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Here Δ is a parameter and notice that for every Δ, the area under the signal $w_\Delta(t)$ is 1. The signal $w_\Delta(t)$ is plotted in Fig. 2.66 for $\Delta = 2, 1, 0.2$. Notice that as Δ becomes small, it represents a signal that is non-zero for a small time interval, but the area is still 1. We define the CT impulse function $\delta(t)$ as

$$\delta(t) := \lim_{\Delta\to 0} w_\Delta(t)$$

This definition satisfies both properties of the DT delta function that we set out to mimic.

$$\begin{aligned} \delta(t)x(t) &= x(0)\delta(t) \quad \text{(function of time)} \\ \int_{-\infty}^{-\infty} x(t)\delta(t)dt &= x(0) \quad \text{(scalar)} \end{aligned}$$

To understand the second property and to understand the effect of $w_\Delta(t)$ on any other signal $x(t)$ as $\Delta \to 0$, let us consider,

$$\lim_{\Delta\to 0}\int_{-\infty}^{-\infty} x(t)w_\Delta(t)dt = \lim_{\Delta\to 0}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} x(t)\frac{1}{\Delta}dt = x(0)\underbrace{\lim_{\Delta\to 0}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta}dt}_{1} = x(0).$$

$\delta(t)$ is a generalized function

While our definition of $\delta(t)$ allowed us to define a signal that has similar properties to that of the discrete-time delta function, note that the CT impulse function satisfies

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \uparrow & \text{otherwise} \end{cases}$$

At $t = 0$, we cannot assign any value to $\delta(t)$ and hence, it is not defined at $t = 0$. If it were a function, it should be defined for every value in its domain. Since $\delta(t)$ is defined everywhere except at $t = 0$, it is called a generalized function and also as a singularity function. This is why it is drawn with an arrow instead of merely drawing a line.

In spite of several reminders, students often confuse the impulse function for the function which is zero everywhere and takes the value 1 at $t = 0$. The 1 by the side of the arrow does not denote that its amplitude is 1. It denotes the fact that the area under the arrow is 1. They are *not* the same. If this is still unclear, talk to me. At the least, remind yourself that an arrow is used to represent the function for a reason.

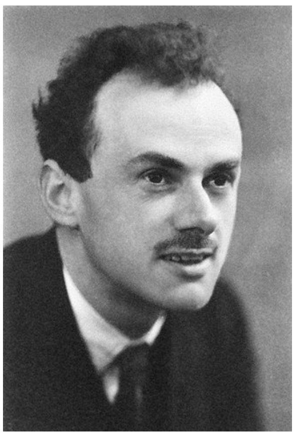


Figure 2.67: Paul Dirac

The idea of defining the impulse signal is due to Paul Dirac (Wikipedia page, Podcast about Paul Dirac), one of the most significant physicists of the 20th century. Dirac was an electrical engineer who then became a physicist. The development of the Dirac-delta function is extremely useful in engineering and it was perhaps his interest in engineering and physics that led him to develop something that is not formally well-defined as a function, but is nevertheless extremely useful and amenable to computation.

If the impulse signal is shifted by t_0, then we get the signal $\delta(t-t_0)$. Such a signal can be used to model phenomena that occur only for a small time around t_0. We will use the terminology, "an impulse occurring at time t_0" to refer to $\delta(t-t_0)$.

Two key properties of the CT Dirac-delta signal

These two properties about the continuous-time delta function or the Dirac delta function are very useful

$$x(t)\;\delta(t-t_0) = x(t_0)\;\delta(t-t_0) \tag{2.53}$$

$$\int_{-\infty}^{\infty} x(t)\;\delta(t-t_0)\;dt = x(t_0) \tag{2.54}$$

As shown in the figure below $\delta(at) = \frac{1}{a}\delta(t)$.

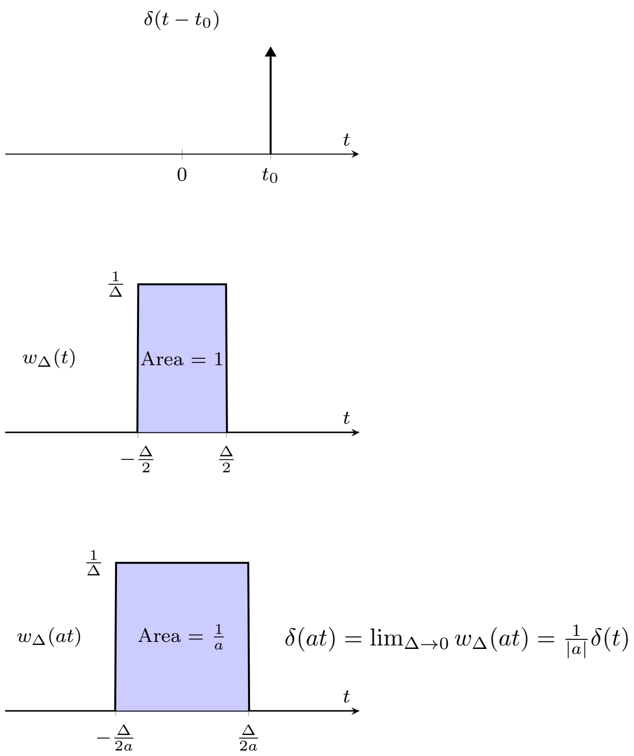


Figure 2.68: Figures showing $\delta(t-t_0)$ and $\delta(at)$

Finally, we will refer to an amplitude-scaled version of $\delta(t)$, namely $a\delta(t)$ as the impulse function with *strength* a. So, when I say "a delta function occurring at $t=2$ with strength 3", I mean the signal $3\delta(t-2)$.

Example 2.9.11.

$$(t^3+3)\delta(t) = 3\delta(t)$$

Example 2.9.12.

$$\left(\sin\left(t^2-\frac{\pi}{2}\right)\right)\delta(t) = \sin\left(\frac{-\pi}{2}\right)\delta(t) = -\delta(t)$$

Example 2.9.13.

$$\int_{-\infty}^{\infty} \frac{\omega^2+1}{\omega^2+9}\ \delta(\omega-1)\ d\omega = \frac{2}{10} = \frac{1}{5}$$

2.10 Relationship between elementary signals

Several of the elementary signals are related to each other and can often be derived from one another. We present a few examples here.

2.10.1 Relation between unit step signal and rectangular signal:

The rectangular signal can be expressed in the form of shifted unit step functions as show below-

$$rect(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$$

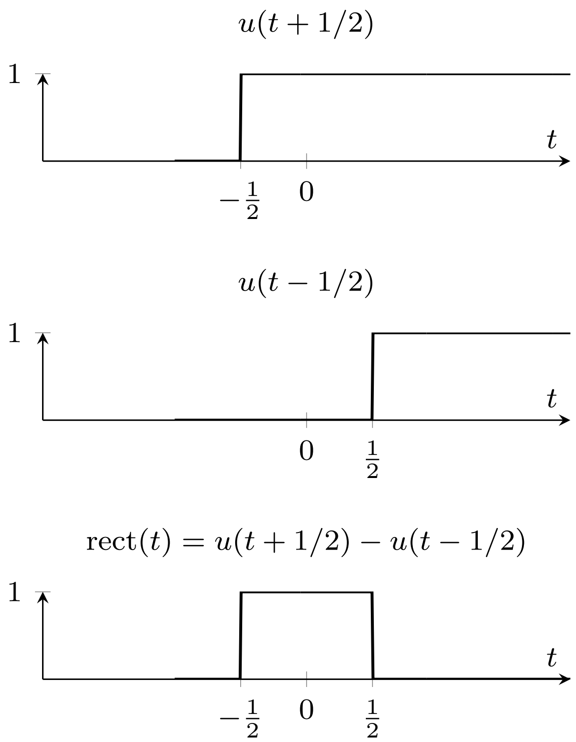


Figure 2.69: The rectangular signal can be obtained as $u(t + 1/2) - u(t - 1/2)$

In the same way, in discrete time-

$$rect_N[n] = u[n + N] - u[n - (N + 1)]$$

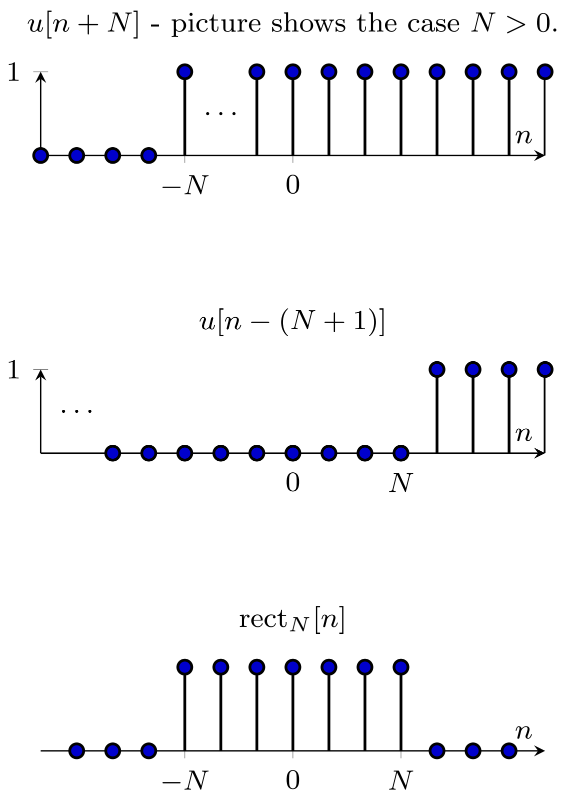


Figure 2.70: Discrete-time rectangular signal can be obtained as the difference of two shifted unit step functions.

2.10.2 Relation between ramp function and the unit step signal:

The ramp signal can be expressed in the form of unit step signals as show below-

$$rampt(t) = \int_{-\infty}^{t} u(\tau)d\tau \tag{2.55}$$

In the same way, in discrete-time,

$$ramp[n] = \sum_{-\infty}^{n} u[m] \tag{2.56}$$

2.10.3 Relationship between $\delta(t)$ and $u(t)$

$$\int_{-\infty}^{t} \delta(\gamma)d\gamma = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{otherwise} \end{cases}$$

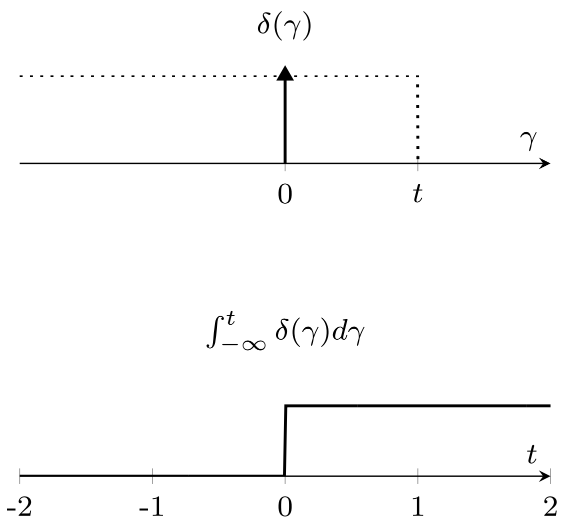


Figure 2.71: Figure showing $u(t) = \int_{-\infty}^{t} \delta(\gamma) d\gamma$.

Let us consider the signal $\int_{-\infty}^{t} \delta(\gamma)\ d\gamma$. If $t < 0$, this signal takes the value 0 and if $t > 0$, the integral includes the delta function and hence, the integral evaluates to 1. Thus,

$$\int_{-\infty}^{t} \delta(\gamma) d\gamma = u(t)$$

$$\Rightarrow \frac{d}{dt} u(t) = \delta(t)$$

We know that,

$$\int x(t)\delta(t) dt = x(0)$$

$$\text{So, is } \int_{-\infty}^{\infty} x(t) \frac{d}{dt} u(t) dt = x(0)\ ?$$

Let's check it. Integrating by parts we get,

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t) \frac{d}{dt} u(t) dt &= u(t)x(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t) \frac{d}{dt} x(t) dt \\
&= x(\infty) - \int_{0}^{\infty} \frac{d}{dt} x(t) dt \\
&= x(\infty) - x(t)\Big|_{0}^{\infty} \\
&= x(0)
\end{aligned}$$

Now, we have done something that is inconsistent with what you were told in calculus. We just took the derivative of the unit-step function which is discontinuous. Weren't you told in calculus that discontinuous functions are not differentiable? Indeed, they are not in the strict sense if you want the derivative to be a function. Since $\delta(t)$ is not a function (it is not defined for $t = 0$), it true that the unit-step is not differentiable according to the definition in your calculus class. However, we will wear our engineering hat and say that it valuable to be able to express derivatives of the unit-step using a generalized function such as the $\delta(t)$ function. If we can take the derivative of the unit step function, can we take derivatives of other discontinuous functions? Indeed, this idea can be generalized as we will see in the next section.

The unit step function can be thought of a the limit of the function $u_\Delta(t)$ defined below as $\Delta \to 0$.

$$u_\Delta(t) := \begin{cases} 1, & \text{if } t > \Delta/2 \\ t + \Delta/2, & \text{if } -\Delta/2 \le t \le \Delta/2 \\ 0, & \text{otherwise.} \end{cases} \tag{2.57}$$

With this definition, we can see $w_\Delta(t) = \frac{du_\Delta(t)}{dt}$.

2.11 Derivatives of discontinuous signals

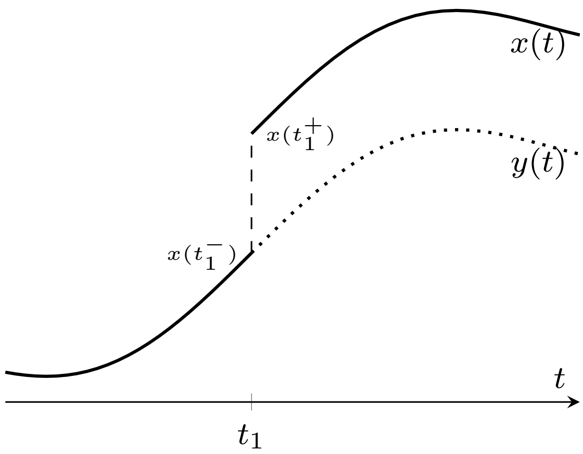


Figure 2.72: Signal with discontinuity at $t = t_1$.

For any function or signal $x(t)$ such that $x(t)$ is discontinuous at t_1 as shown in the figure, let $y(t)$ be the signal that is similar to $x(t)$ without the discontinuity - this is shown in the dotted line in the figure. Then,

$$x(t) = y(t) + \left(x^{t_1^+} - x^{t_1^-}\right)u(t - t_1)$$

$$\frac{dx(t)}{dt} = \frac{dy(t)}{dt} + (x(t_1^+) - x(t_1^-))\delta(t - t_1)$$

So generalizing it,if t_1, t_2, ... are the instants of time at which $x(t)$ is discontinuous then,

$$\frac{d}{dt}x(t) = x^1(t) + \sum_i (x(t_i^+) - x(t_i^-))\delta(t - t_i)$$

where $x^1(t)$ denotes the derivative of $x(t)$ at all points except at the discontinuities $t_1, t_2, \ldots,$.

Example 2.11.1. *Let* $x(t) = e^{-t}u(t)$*. What is* $\frac{dx(t)}{dt}$ *?*

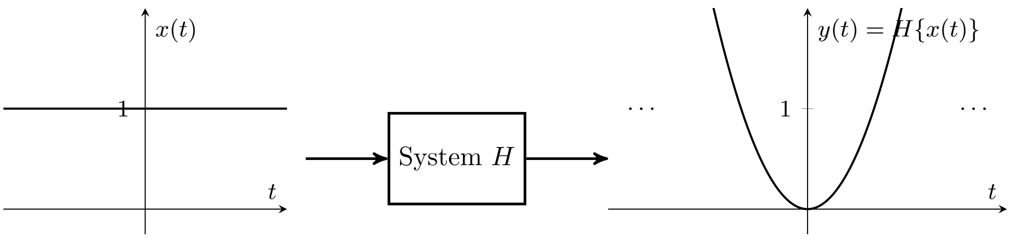
Using the product rule

$$\frac{d}{dt}\big(e^{-t}u(t)\big) = e^{-t}\frac{du(t)}{dt} + \frac{de^{-t}}{dt}u(t) = e^{-t}\delta(t) - e^{-t}u(t) = \delta(t) - e^{-t}u(t)$$

Here, we simplified $e^{-t}\delta(t)$ *to* $\delta(t)$ *using the fact that for any signal* $z(t)$*,* $z(t)\delta(t) = z(0)\delta(t) = e^{0}\delta(t)$*.*

Chapter 3

Systems



3.1 What is a system and how do we describe them mathematically?

A system can be thought of as a series of transformations that transform an input signal into an output signal. It can be represented as a box with an input signal and an output signal as shown in Fig. 3.1. The transformation performed by the system can be represented using an operator (such as H) that operates on $x(t)$ to produce $y(t)$. We denote this by saying $y(t) = H\{x(t)\}$. A system could either represent nature acting on signals and transforming them or human-built devices processing signals to produce other signals. We will model both these as systems and the following examples will clarify this idea.

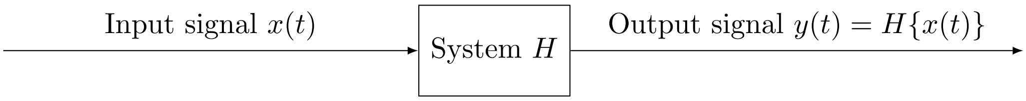


Figure 3.1: A system can be abstractly thought of as a black box that transforms an input signal into an output signal

Systems are described mathematically by providing a mathematical relationship between the input signal $x(t)$ and the output signal $y(t)$. The precise form of this relationship varies with the system. An algebraic relationship between shifted versions of $x(t)$ and $y(t)$ and a differential equation relating $x(t)$ and $y(t)$ are common ways to specify this relationship. The following examples will illustrate this.

3.2 Theme examples

Before jumping into the theory of systems, let us look at some examples of systems where the material covered in this class can be useful for gaining an understanding. It is unrealistic that one undergraduate course suffices to fully understand the main components of any interesting system; but the theory developed here will form the basis on which we can build to understand some critical components of the following systems.

3.2.1 Cellular communications

In cellular communication, a base station transmits a signal $x(t)$ which arrives at the antenna at the handset after undergoing many transformations when passing through the wireless channel. The transmitted electromagnetic signal is usually radiated in many directions and it arrives at the receiver via multiple reflections. The received signal $y(t)$ is the sum of multiple reflected paths. Consider an example where there are two reflected paths and let the attenuation associated with the two paths be h_1 and h_2 and let the time delays associated with these paths be τ_1 and τ_2. Then,

$$\boxed{y(t) = h_1 x(t - \tau_1) + h_2 x(t - \tau_2)}$$

We can treat the effect of the channel as passing $x(t)$ through a system that denotes the channel and $y(t)$ is the output of this system. This system refers to what nature does to the transmitted signal $x(t)$. Since $y(t)$ is a distorted version of $x(t)$, typically, $y(t)$ will processed at the receiver by passing it through a receiver processing block. This block can be modeled as a system (human made) and the input to this block is the signal $y(t)$ and the output is an estimate of $x(t)$, given by $\hat{x}(t)$, which is then played through the speaker if we were originally using the phone for talking on the phone.

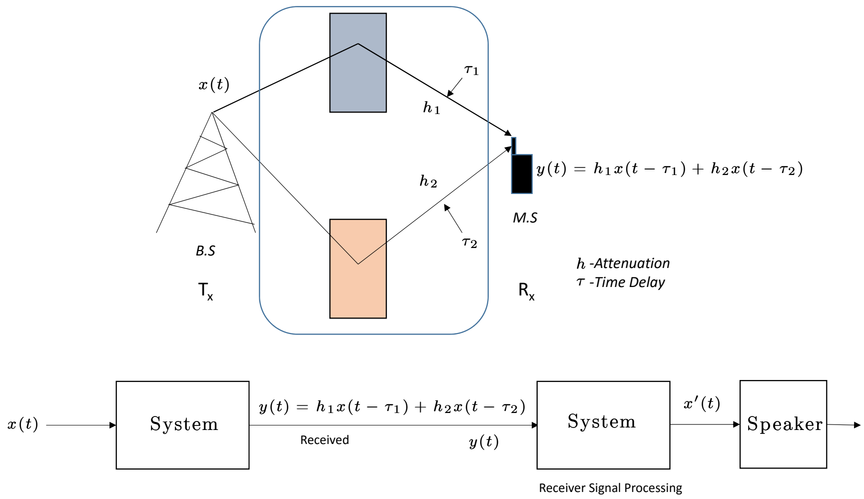


Figure 3.2: An example of how a wireless communication system can be modeled as a cascade of two systems

What are some things that we would like to learn about systems? First and foremost, we would like to model the physical process of a signal propagating through a wireless channel using a model that is simple and amenable to insightful analysis. This process is called **system modeling**. Once we choose a model, we might want to learn the parameters of the system model (h_1, h_2, τ_1, τ_2 in the wireless channel example). This is called **parameter identification**. Then, we might want to 'understand' how the system has affected $x(t)$. This is called **system analysis**. At the receiver, we want to build a system that allows us to 'undo' the effects of the channel and helps us to recover a good approximation to $x(t)$ (often it is not possible to recover $x(t)$ exactly and we have to settle for recovering a good approximation). This is called **signal processing or data mining** since we try to extract useful information about $x(t)$ from $y(t)$. In the next example, we will see that often we may want to **control** the input to the system to achieve a desired output.

Example 3.2.1. *Consider the wireless channel example described above. If* $x(t) = rect(t - 1/2)$ *and* $y(t) = h_1 x(t-\tau_1) + h_2 x(t-\tau_2)$ *where* $h_1 = \frac{1}{2}$*,* $\tau_1 = \frac{1}{10}$*sec,* $h_2 = \frac{1}{4}$*,* $\tau_2 = \frac{2}{10}$*sec, compute* $y(t)$*.*

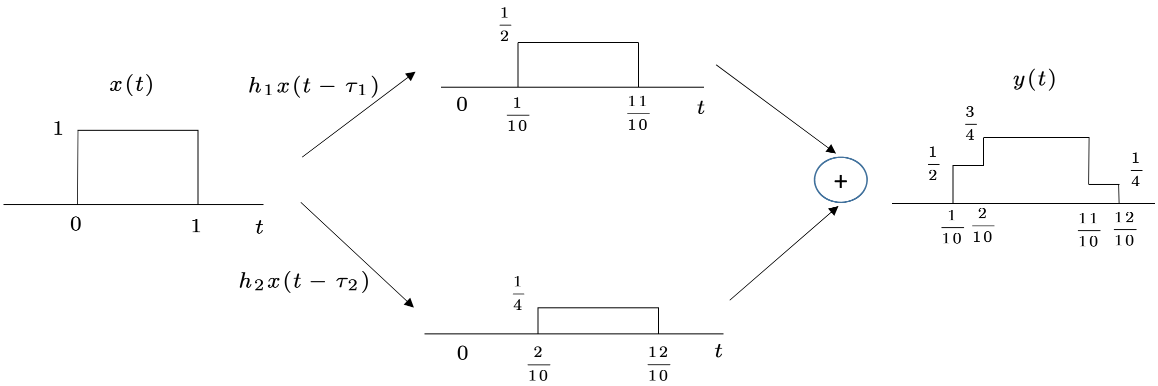


Figure 3.3: Figure showing the relationship between $y(t)$ and $x(t)$ for a 2-path wireless channel when $x(t) = rect(t - 1/2)$.

3.2.2 Autonomous vehicles

Consider the act of driving a car with mass m. We apply a certain force $x(t)$ to the gas pedal as a function of t and the car moves obeying the laws of mechanics and let $y(t)$ be the position of the car as a function of t. We can think of the car as a system with input $x(t)$ and output $y(t)$. How can we get a mathematical relationship between the input and the output? To illustrate the main idea, we will consider a simple model where the car moves along a straight line. If the force applied on the gas pedal is $x(t)$, the actual force applied to the car is $cx(t)$. The velocity of the car at time is given by $\frac{dy(t)}{dt}$ and the acceleration is given by $\frac{d^2y(t)}{dt^2}$. If ρ is the coefficient of friction, then the force due to friction is proportional to the velocity and is given by $\rho\frac{dy(t)}{dt}$. Therefore, the input force minus the force due to friction should equal the mass times the acceleration and hence, we have

$$\begin{aligned} cx(t) - \rho\frac{d}{dt}y(t) &= m\frac{d^2}{dt^2}y(t) \\ m\frac{d^2}{dt^2}y(t) + \rho\frac{d}{dt}y(t) &= cx(t) \end{aligned}$$

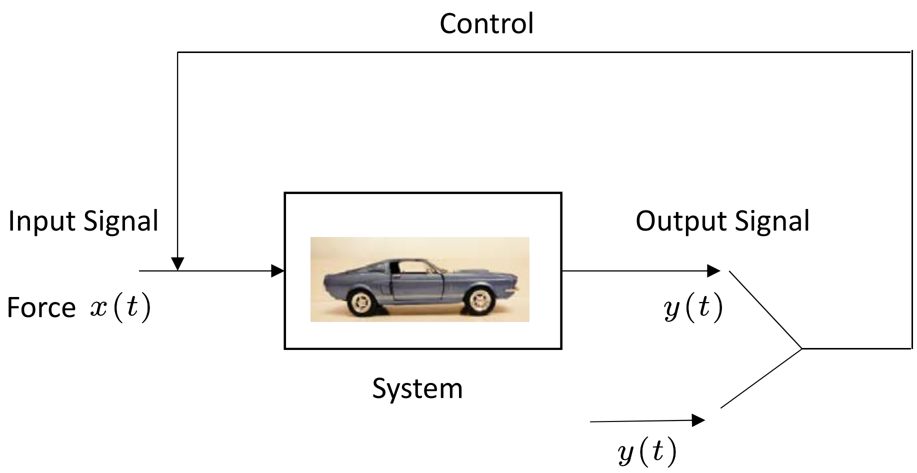


Figure 3.4: Modeling driving a car as a system

Think about an autonomous driving system, where there is autonomous control that is applied to the input to make the car perform a certain maneuver. For e.g., we would like to come to a stop 5 miles away from the starting point exactly 10 minutes after the start of driving. This means that we need to find the input $x(t)$ such that $y(t)\big|_{t=10mins} = 5$ miles, and $\frac{d}{dt}y(t)\big|_{(t=10mins)} = 0$. In closed loop control, the car has the ability to measure its position and velocity at time t and a control algorithm changes $x(t)$ such that the desired maneuver is obtained. This is typically done by computing the error between the desired trajectory $x_d(t)$ and the car's trajectory $x(t)$.

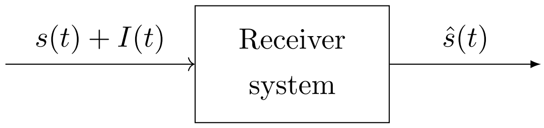
An important take away from this examples is that in many physical systems, the input-output relationship in a system may be specified through a differential equation. For DT systems, the input-output relationship may be specified through a difference equation.

3.2.3 Filtering interference/noise in communication systems

Communication systems are prone to interference from other users/systems. Suppose we transmit a signal $s(t)$, we might receive the signal

$$y(t) = s(t) + I(t) \tag{3.1}$$

at the receiver, where $I(t)$ refers to the interference signal. A typical task for the receiver is to remove or minimize the effects of $I(t)$ without distorting the desired signal $s(t)$ significantly. In order to be able to do that, we should find some domain in which $s(t)$ and $I(t)$ are separable. It is often the case that $s(t)$ and $I(t)$ occupy different parts of the spectrum. In such cases, we can build a system to remove the effects of $I(t)$ through a process called filtering, i.e., we build a system that will block the frequencies occupied by $I(t)$.



In this case, the input to the system is the signal which is the sum of the desired signal and the interference and the output from the system is an estimate of the desired signal $s(t)$.

3.2.4 Some example from biomedical engineering

3.2.5 Object recognition using convolutional neural networks

Explain how convolution neural networks can be viewed as system. Emphasize this is a non-linear system. Explain where convolution is used.

3.3 Computing the output of a system for a given input

Example 3.3.1. *Let H be a system with input-output relationship given by*

$$y[n] = x[n] + \frac{1}{2}x[n-1].$$

Compute the output $y[n]$ when $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$$\begin{aligned} y[n] &= (\delta[n] + \delta[n-1] + \delta[n-2]) + \frac{1}{2}(\delta[n-1] + \delta[n-2] + \delta[n-3]) \\ &= \delta[n] + \frac{3}{2}\delta[n-1] + \frac{3}{2}\delta[n-2] + \delta[n-3] \end{aligned}$$

Example 3.3.2. *Let H be a system with input-output relationship given by*

$$y[n] = \frac{1}{2}y[n-1] + x[n].$$

Compute the output $y[n]$ when $x[n] = \delta[n]$ given $y[-1] = 0$. Notice the difference between the system in this example and the one in the previous example. This system corresponds to having feedback whereas the previous system corresponds to an open loop system.

3.4 System properties

In this section, we will study several properties of systems.

3.4.1 Stability - Bounded Input Bounded Output (BIBO) Stability

The word stability has several meanings in the English language and hence we may have many preconceived notions for what it might mean for a system to be stable and indeed, there are multiple such meaningful notions. In this class, we will consider one such a notion called bounded input bounded output stability.

Before we discuss the notion of stability, we will first discuss what it means for a signal to be bounded. Roughly speaking, a signal is said to be bounded if the magnitude of the signal does not exceed a bound for all t. If the signal is real, then this corresponds to saying that we can draw two horizontal lines - one at $y = M_x$ and the other at $y = -M_x$ such that the amplitude of the signal always lies between these two bounds. If the signal is complex, then it means that we can draw a horizontal line at $y = M_x$ such that the *magnitude* always lies between 0 and M_x. Formally,

Definition 3.4.1. *A CT signal $x(t)$ is said to be bounded if*

$$|x(t)| \leq M_x < \infty, \ \forall t \tag{3.2}$$

and a DT signal $x[n]$ is said to be bounded if

$$|x[n]| \leq M_x < \infty, \ \forall n \tag{3.3}$$

An example of a bounded CT signal is shown below

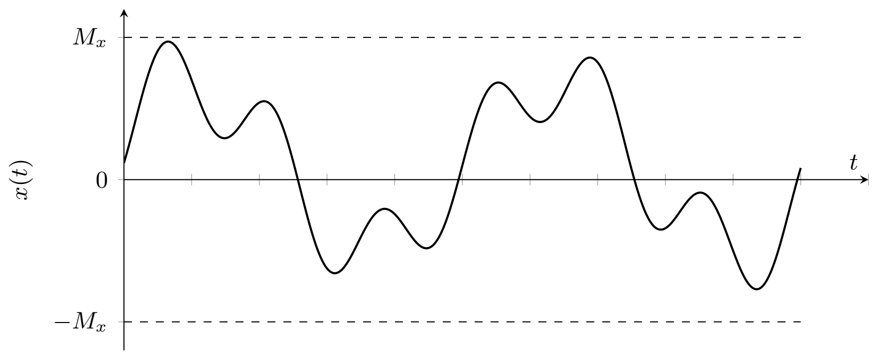


Figure 3.5: Example of a bounded signal.

A common mistake I have seen is that some students think a signal is bounded if it is bounded in duration, i.e., it is non-zero only between time instants t_1 and t_2. ⚠Boundedness has to do with the *amplitude or magnitude* being bounded, not the duration.

Here are a few examples of bounded signals

Example 3.4.2. *The signal $x(t) = e^{-t}u(t)$ is bounded and $M_x = 1$.*

Here we note that $|x(t)| \leq 1 \ \ \forall\, t$ and $M_x = 1$

Example 3.4.3. *The signal $x(t) = \cos(t)$ is also bounded. Here also, $M_x = 1$*

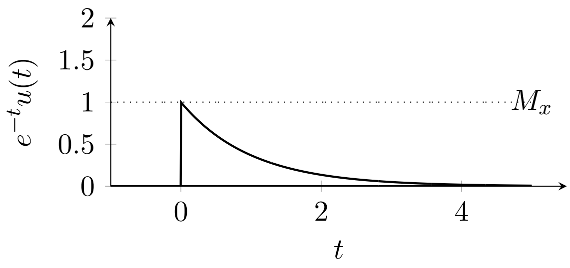


Figure 3.6: Plot of $x(t) = e^{-t}u(t)$. This is a bounded signal with a bound of $M_x = 1$.

Here are some examples of unbounded signals

Example 3.4.4. *The signal* $x(t) = e^{-t}$ *is unbounded. Note that this signal is different from* $e^{-t}u(t)$*. This is an unbounded signal since* $x(t) \to \infty$ *as* $t \to -\infty$*.*

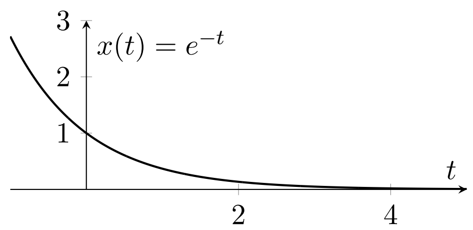


Figure 3.7: Plot of $x(t) = e^{-t}$. This is an unbounded signal since $x(t) \to \infty$ as $t \to -\infty$.

Example 3.4.5. *The signal* $x(t) = t$ *is unbounded. The signal grows unbounded both as* $t \to \infty$ *and* $t \to -\infty$*.*

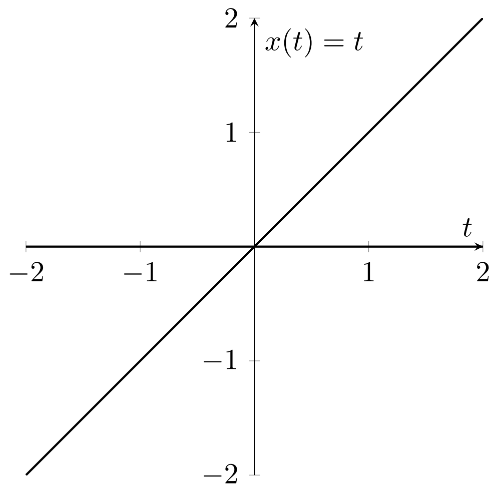


Figure 3.8: Plot of $x(t) = t$. This is an unbounded signal since $x(t) \to \infty$ as $t \to -\infty$ and as $t \to \infty$.

Definition 3.4.6. *A system is Bounded-Input Bounded-Output (BIBO) stable if* every *bounded input signal produces a bounded output signal.*

⚠Being bounded is a property of a signal, whereas being stable is a property of a system. It does not make sense to talk about a stable input or a bounded system. I have noticed many students use these terms. It would be valuable to keep these mind.

When asked to determine if a system is bounded, I suggest we first try to observe the input-output relationship and build intuition about the system.

We now present a few examples

Example 3.4.7. *Consider the system given by $y(t) = (x(t))^2$. This is an example of a stable system. Intuitively, the output of the system is the square of the input. As long as the input signal is bounded by M_x, the output will remain bounded by $M_y = M_x^2$. Thus, every bounded input signal will produce a bounded output signal.*

Example 3.4.8. *Consider the system given by $y(t) = t^2x(t)$. This is an unstable system. Intuitively, the output of the system is the input multiplied by t^2, i.e., the system acts as an amplifier who's gain changes with time. As $t \to \infty$, we see that the output can grow unbounded even when the input stays bounded because the term t^2 can become arbitrarily large. For example, when $x(t) = 1, \forall t$, the output is $y(t) = t^2$. In this case, the input signal is bounded but the output signal is not. Hence, the system is unstable.*

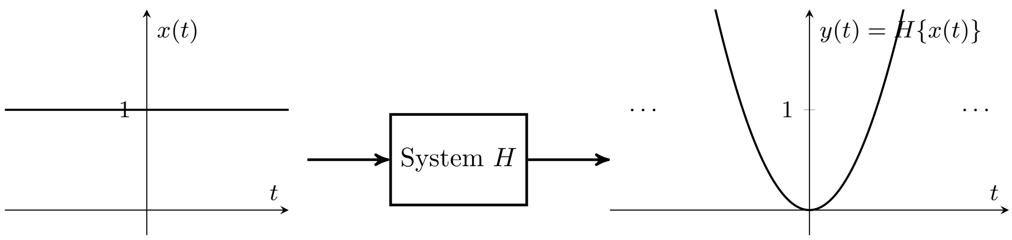


Figure 3.9: Example of a unstable system. Figure shows a bounded input signal that produces an unbounded output signal.

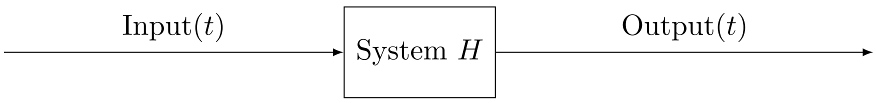
Example 3.4.9. *The system given by* $y(t) = e^{x(t)}$ *is stable. If* $|x(t)| \leq M_x$ *then* $|y(t)| \leq e^{M_x}$*. Notice that even though* e^{M_x} *can be a large number, it is finite, nevertheless.*

Example 3.4.10. *Consider the system* $y(t) = e^t\ x(t)$*. This system is unstable. Can you think of an input signal that is bounded for which the output will be unbounded ?*

Example 3.4.11. *Consider a DT system given by* $y[n] = \sum_{m=-\infty}^{n} x[m]$*. This system is unstable. Can you think of a bounded input signal that will result in an unbounded output signal?*

3.4.2 Linearity

Consider a system denoted by H.



Additivity or Superposition property:

The system is said to satisfy addivity or the superposition property if the output signal corresponding to a sum of two input signals is the same as the sum of the output signals corresponding to each of the two input signals. To clarify this, let $y_1(t) = H\{x_1(t)\}$ be the output of a CT system when the input is $x_1(t)$ and let $y_2(t) = H\{x_2(t)\}$ be the output of the system when the input is $x_2(t)$. Suppose we constructed a third signal called $x_3(t)$ which is the sum of $x_1(t)$ and $x_2(t)$, i.e., $x_3(t) = x_1(t) + x_2(t)$ and let $y_3(t) = H\{x_3(t)\}$ be the output of the system corresponding to the input signal $x_3(t)$. If $y_3(t)$ is the same as $y_1(t) + y_2(t)$, then we say that the system is additive, i.e., we say that a CT system H is additive if

$$H\{x_1(t) + x_2(t)\} = H\{x_1(t)\} + H\{x_2(t)\}$$

$$x_1(t) \xrightarrow{H} y_1(t) \tag{3.4}$$

$$x_2(t) \xrightarrow{H} y_2(t) \tag{3.5}$$

$$x_3(t) = x_1(t) + x_2(t) \xrightarrow{H} y_3(t) \stackrel{?}{=} y_1(t) + y_2(t)$$

Homogenity

A system is said to be homogenous if scaling the input by any *complex* number a, scales the output also by the same complex number a. In other words, if the output corresponding to the input $x(t)$ is given by $y(t) = H\{x(t)\}$, then the output corresponding to $ax(t)$ is $ay(t)$. That is,

$$H\{ax(t)\} = aH\{x(t)\}, \ \forall t, \text{ and}, \ \forall a$$

The notion of addivity and homogenity carry over for DT systems naturally. It would be a good exercise for a student to write down what it means for a DT system to be additive and homogenous. The correct answer is

Additivity: A DT system is additive if - $H\{x_1[n] + x_2[n]\} = H\{x_1[n]\} + H\{x_2[n]\}$

Homogenity: A DT system is homogenous if - $H\{ax[n]\} = aH\{x[n]\}$ for any complex number a

Example 3.4.12. *Consider the system H given by $y(t) = tx(t)$. Is the system additive? Is it homogenous?*

$$\begin{aligned} x_1(t) &\longrightarrow y_1(t) = tx_1(t) \qquad (3.6) \\ x_2(t) &\longrightarrow y_2(t) = tx_2(t) \\ x_3(t) = x_1(t) + x_2(t) &\longrightarrow y_3(t) = tx_3(t) = t(x_1(t) + x_2(t)) = tx_1(t) + tx_2(t) \end{aligned}$$

Since $y_3(t) = y_1(t) + y_2(t)$, the system satisfies additive property.

The output corresponding to an input of $ax(t)$ is given by $tax(t)$ which is the same as $atx(t)$, which is $ay(t)$. Hence, the system is also homogenous.

Example 3.4.13. *Consider the CT system $y(t) = x^2(t)$. Is the system additive ? Is it Homogenous? Consider two inputs $x_1(t)$ and $x_2(t)$. The outputs corresponding to these two inputs are $y_1(t) = x_1^2(t)$ and $y_2(t) = x_2^2(t)$, respectively. Now, consider a third input $x_3(t) = x_1(t) + x_2(t)$. What will be the output $y_3(t)$ corresponding to this input? $y_3(t) = x_3^2(t)$. To determine if the system is additive, we need to determine if $y_3(t) = y_1(t) + y_2(t)$.*

$$y_3(t) = x_3^2(t) = (x_1(t) + x_2(t))^2 \neq x_1^2(t) + x_2^2(t)$$

Hence, $y_3(t) \neq y_1(t) + y_2(t)$ and hence, the system is not additive.

The system is not homogenous since the output corresponding to $ax_1(t)$ is $a^2y_1(t)$ and not $ay_1(t)$.

Example 3.4.14. *Examples of systems that are additive but not homogenous are rare. But here is an example. Consider the system $y[n] = \Re\{x[n]\}$, where $\Re$ refers to the real part. This system is additive but not homogenous. When the input $x[n]$ is multiplied by a complex number a, the system does not satisfy the additive property.*

Definition 3.4.15. *A system is linear if it is both Additive and Homogeneous.*

If $x_1(t)$ and $x_2(t)$ are two CT signals, the signal $x_3(t) = a_1x_1(t)+a_2x_2(t)$ is called a *linear combination* of the two signals $x_1(t)$ and $x_2(t)$. Similarly, if $x_1[n]$ and $x_2[n]$ are two DT signals, the signal $x_3[n] = a_1x_1[n]+a_2x_2[n]$ is called a *linear combination* of the two signals $x_1[n]$ and $x_2[n]$. A system is linear if the output corresponding to a linear combination of input signals is the linear combination of the individual outputs with the *same* coefficients in the linear combination. That is,

- A CT system H is linear if - $H\{a_1x_1(t) + a_2x_2(t)\} = a_1H\{x_1(t)\} + a_2H\{x_2(t)\}$
- A DT system H is linear if - $H\{a_1x_1[n] + a_2x_2[n]\} = a_1H\{x_1[n]\} + a_2H\{x_2[n]\}$

The idea of linearity for two signals can be extended to the case when we take a linear combination of N input signals. Then, a system denoted by H is linear,

$$\begin{aligned}
x_1(t) &\longrightarrow y_1(t) \\
x_2(t) &\longrightarrow y_2(t) \\
\vdots \quad &\ \vdots \quad \vdots \\
x_N(t) &\longrightarrow y_N(t) \\
\Rightarrow a_1x_1(t) + a_2x_2(t) + \ldots + a_Nx_N(t) &\longrightarrow a_1y_1(t) + a_2y_2(t) + \ldots + a_Ny_N(t)
\end{aligned}$$

Linearity

Thus,

$$x(t) = \sum_{i=1}^{N} a_i x_i(t) \longrightarrow y(t) = \sum_{i=1}^{N} a_i y_i(t) \tag{3.7}$$

or, equivalently,

$$H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\} = \sum_{i=1}^{N} a_i H\{x_i(t)\} \tag{3.8}$$

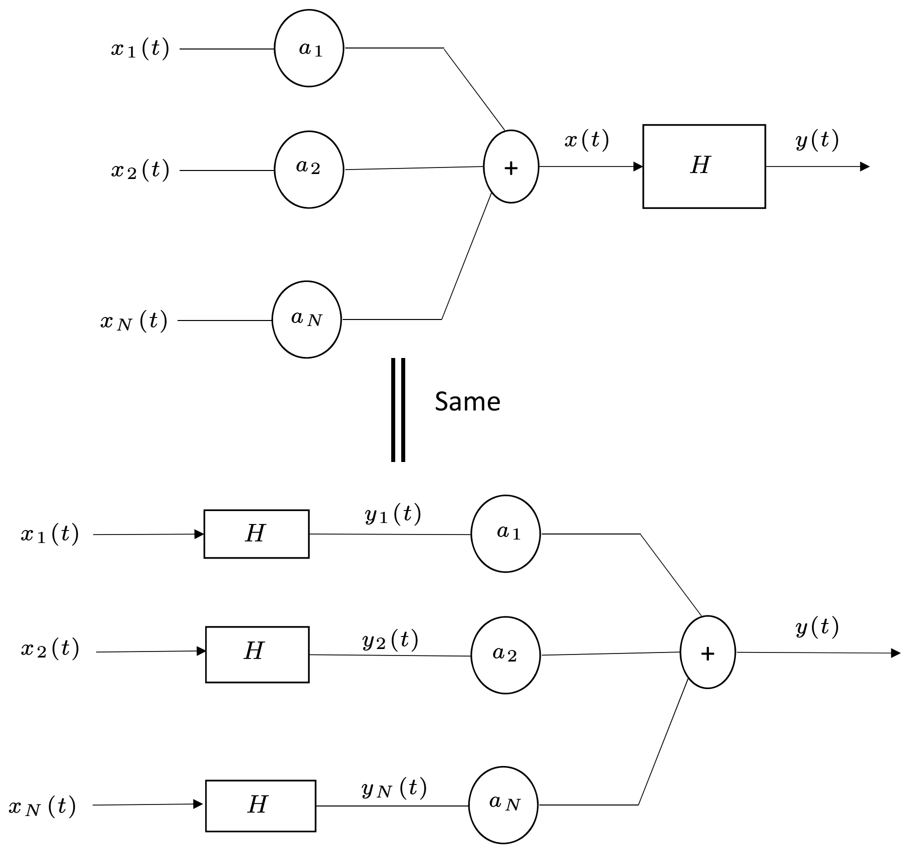


Figure 3.10: Schematic showing what it means for a system to be linear.

The system defined by $y(t) = [x(t)]^2$ is not linear. This example along with prior exposure to linear functions in calculus gives the wrong idea of linearity to many students. The following examples should dispel this confusion

Example 3.4.16. *Consider the system $y(t) = 1 + x(t)$. Is this system linear? This system is* not linear *! Let $y_1(t)$ and $y_2(t)$ denote the output corresponding to two inputs $x_1(t)$ and $x_2(t)$. The output corresponding to an input of $a_1x_1(t) + a_2x_2(t)$ is $1 + a_1x_1(t) + a_2x_2(t)$, which is not the same as $a_1y_1(t) + a_2y_2(t)$!*

Example 3.4.17. *Consider a linear system H. It is known that the output of the system corresponding to an input $e^{j2\pi t}$ is $e^{j3\pi t}$ and that the output of the system corresponding to an input of $e^{-j2\pi t}$ is $e^{-j3\pi t}$. What will be the output of the system when the input is a) $\sin 2\pi t$? and b) $\sin\left(2\pi(t - \frac{1}{\pi}\right)$. Another way to state the problem is given,*

$$x_1(t) = e^{j2\pi t} \longrightarrow y_1(t) = e^{j3\pi t}$$

$$x_2(t) = e^{-j2\pi t} \longrightarrow y_2(t) = e^{-j3\pi t}$$

Find,

$$a)\ x_3(t) = \sin(2\pi t) \quad \longrightarrow \quad ?$$
$$b)\ x_4(t) = \sin\left(2\pi(t - \frac{1}{\pi})\right) \quad \longrightarrow \quad ?$$

Solution: Using Euler's formula, we can write $x_3(t) = \sin(2\pi t)$ *as a linear combination of* $x_1(t)$ *and* $x_2(t)$ *as follows. The coefficients in the linear combination are* $a_1 = \frac{1}{2j}$, $a_2 = \frac{-1}{2j}$. *Since we know that the system is linear, the output corresponding to* $x_3(t)$ *should be* $a_1 y_1(t) + a_2 y_2(t)$ *and hence,* $y_3(t)$ *is given by*

$$\sin(2\pi t) = \frac{1}{2j}e^{j2\pi t} + \frac{-1}{2j}e^{-j2\pi t} \longrightarrow \frac{1}{2j}e^{j3\pi t} - \frac{1}{2j}e^{-j3\pi t} = \sin(3\pi t)$$

To compute the output corresponding to $x_4(t) = \sin\left(2\pi(t - \frac{1}{\pi})\right)$, *we should again try to write* $x_4(t)$ *as a linear combination of* $x_1(t)$ *and* $x_2(t)$. *Again, using Euler's formula, we can write*

$$\sin\left(2\pi\left(t - \frac{1}{\pi}\right)\right) = \frac{1}{2j}e^{j2\pi t}e^{-2j} - \frac{1}{2j}e^{-j2\pi t}e^{2j} = \frac{e^{-2j}}{2j}e^{j2\pi t} + \frac{-e^{2j}}{2j}e^{-j2\pi t}$$

$$x_4(t) = \sin\left(2\pi\left((t - \frac{1}{\pi}\right)\right) \longrightarrow \qquad ?$$

3.4.3 Time Invariance

We said that a system is a box that transforms an input signal into an output signal. While performing such transformations, the system typically has internal *parameters* that are used in the transformation process. For example, if the system is an ideal amplifier, the gain associated with the amplifier is something that is not dependent on the input, but is an inherent parameter of the system. Intuitively, a system is said to be time invariant if the parameters of the system or the input-output relationship defined by the system is not dependent on time, for example, the gain of the amplifier does not change with time. In the amplifier example, the system is time variant if the amplifier gain changes with time or the system behaves like an ideal amplifier at times and like a non-ideal amplifier at other times.

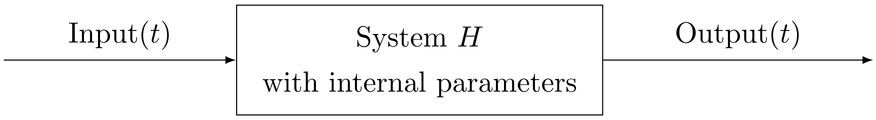


Figure 3.11: A system is time invariant if the internal parameters of the system do not change with time.

This intuitive notion is formalized in this definition

Definition 3.4.18. *Consider a system H and let $y(t) = H\{x(t)\}$ be the output of the system when the input is $x(t)$. The system is time invariant if the output of the system corresponding to $x(t-t_0)$ is $y(t-t_0) \forall t_0, t$.*

That is, shifting the input signal by t_0 simply shifts the output signal by the same amount t_0 and does no other transformation to $y(t)$. This is depicted in the picture below. Here S^{t_0} denotes a system that simply shifts the input by t_0. If we take $x(t)$ and first shift it by passing it through the system S^{t_0} and then pass it through the system H, then the final output will be $H\{x(t-t_0)\}$. If we first pass $x(t)$ through H and then pass it through S^{t_0}, the output will be $y(t-t_0)$ for all t_0 and t where $y(t) = H\{x(t)\}$. A system is time invariant if

$$y(t-t_0) = H\{x(t-t_0)\}, \ \forall x(t), and \ \forall t_0.$$

Time invariance

A CT system is time invariant if

$$HS^{t_0}\{x(t)\} = S^{t_0}H\{x(t)\}, \ \forall x(t), and \ \forall t_0. \tag{3.9}$$

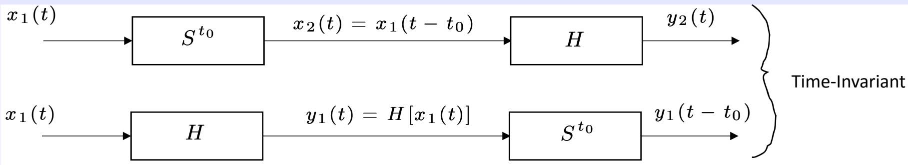


Figure 3.12: A system H is time invariant if $H \ S^{t_0}x(t) = S^{t_0}Hx(t)$.

The definition of time invariance extends naturally to DT systems as well. In this case, we would define a system S^{n_0} that shifts the input signal by an integer value n_0 and say that a DT system is time invariant if

$$HS^{n_0}\{x[n]\} = S^{n_0}H\{x[n]\}, \ \forall x[n], and \ \forall n_0. \tag{3.10}$$

Example 3.4.19. *Consider the system $y(t) = kx(t)$. Is this system time invariant? This system is time invariant. Intuitively, the system is an ideal amplifier with a gain k that does not change with time. To see this from the definition,*

$$\begin{array}{ll} H\{x(t)\} = kx(t), & S^{t_0}H\{x(t)\} = S^{t_0}\{kx(t)\} = kx(t-t_0) \\ S^{t_0}\{x(t)\} = x(t-t_0) & HS^{t_0}\{x(t)\} = H\{x(t-t_0)\} = kx(t-t_0) \end{array}$$

Example 3.4.20. *Consider the system $y(t) = \frac{1}{t}x(t)$. Is this system time invariant? This system is time variant. Intuitively, the system is an ideal amplifier with a gain that changes with time, i.e. at time t, the gain is $1/t$. To see this from the definition,*

$$\begin{array}{ll} H\{x(t)\} = \frac{1}{t}x(t), & S^{t_0}H\{x(t)\} = S^{t_0}\{\frac{1}{t}x(t)\} = \frac{1}{t-t_0}x(t-t_0) \\ S^{t_0}\{x(t)\} = x(t-t_0) & HS^{t_0}\{x(t)\} = H\{x(t-t_0)\} = \frac{1}{t}x(t-t_0) \end{array}$$

The confusing part for some students is why $S^{t_0}\{\frac{1}{t}x(t)\} = \frac{1}{t-t_0}x(t-t_0)$ and why $H\{x(t-t_0)\} = \frac{1}{t}x(t-t_0)$. S^{t_0} is a shift operator which corresponds to a time shift by t_0. Mathematically, this corresponds to replacing t by $t - t_0$ and hence $S^{t_0}\{\frac{1}{t}x(t)\} = \frac{1}{t-t_0}x(t-t_0)$. However, the operation of the system H should be interpreted as follows - the system multiplies the input signal by $1/t$. When considering $HS^{t_0}\{x(t)\}$, the input to H is $x(t-t_0)$ and hence, the output is $\frac{1}{t}x(t-t_0)$.

Example 3.4.21. *Consider the system defined by $y(t) = cos(x(t))$. Is this system time invariant?*

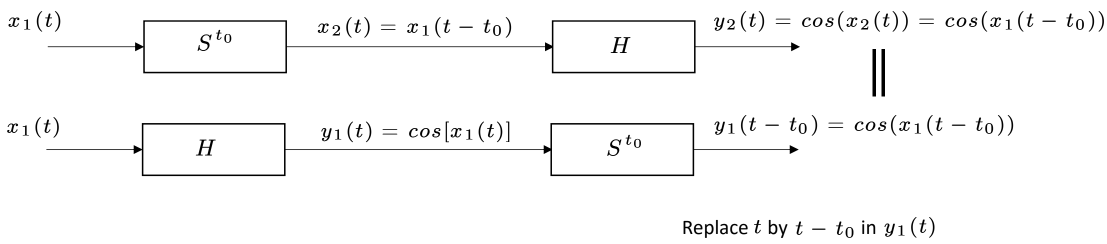


Figure 3.13:

Example 3.4.22. *Consider the system $y(t) = x(2t)$. Is this system time invariant?*

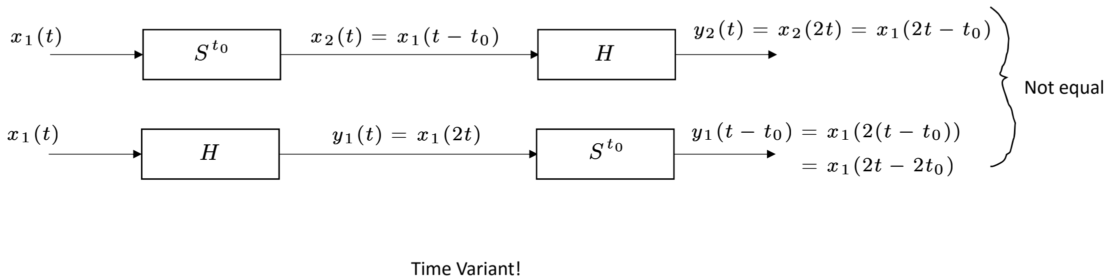


Figure 3.14:

Example 3.4.23. *Consider the system described by:-*

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ x(t) + x(t-2) & \text{if } t \geq 0 \end{cases} \tag{3.11}$$

Is this system Time Invariant?

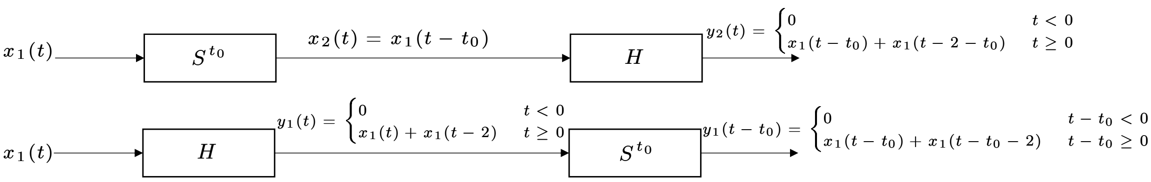


Figure 3.15:

Notice the difference in the intervals over which the signals $y_2(t)$ and $y_1(t-t_0)$ are non-zero. Hence the system is not time-invariant. Can you construct a simple counter example, i.e., $x(t)$ and t_0 such that $H\{x(t-t_0)\} \neq y(t-t_0)$?

3.4.4 Invertibility

Definition 3.4.24. *A CT system H is said to be invertible if the input $x(t)$ can be uniquely recovered or determined from the output $y(t)$. Similarly, a DT system H is said to be invertible if the input $x[n]$ can be uniquely recovered or determined from the output $y[n]$. We should think of the system as a mapping from the set of all possible input signals to the set of all possible output signals. If this mapping is not many-to-one, then the system is invertible.*

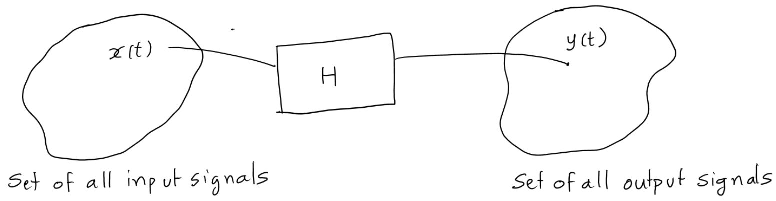


Figure 3.16: A system is invertible if the mapping from the set of input signals to output signals is not many-to-one.

⚠Invertibility does not have anything to do with systems that implement reciprocals of the form $y(t) = \frac{1}{x(t)}$. Invertibility has to do with one-to-one mapping between the input signal and the output signal.

Let us consider a couple of examples

Example 3.4.25. *Consider the system $y(t) = e^{x(t)}$. Is this invertible? Given $y(t)$, we can obtain $x(t)$ by taking $\ln(y(t))$. Hence, this system is invertible.*

Example 3.4.26. *Consider the system $y(t) = x^2(t)$. Is this invertible? Given $y(t)$, we cannot uniquely identify what $x(t)$ is since for every t, $\pm x(t)$ results in the same $y(t)$. Hence, this system is not invertible.*

Example 3.4.27. *Consider the system*

$$y(t) = \begin{cases} 1, & t < 0, \\ x(t), & \textit{otherwise.} \end{cases}$$

This system is not invertible because we cannot recover the values of $x(t)$ for any $t < 0$. All $x(t)$ get mapped to $y(t) = 1$ for $t < 0$.

3.4.5 Memoryless systems and systems with memory:

Definition 3.4.28. *A CT system is said to have memory if the output at time t, namely $y(t)$, depends on the values taken by the input signal at at least one other value of t, i.e. $y(t)$ depends on $x(t')$ for at least one $t' \neq t$. Conversely, a system is said to be memoryless or instantaneous if $y(t)$ depends on the input signal value only at time t or equivalently, $y(t)$ does not depend on $x(t')$ for* any *$t' \neq t$. The definition can be extended to DT systems in the same way*

Example 3.4.29. *Examples of a system that has memory-*

1. $y(t) = x(t-1) + (x(t-2))^2$

   We say that the system has memory because if we have to implement the system, this would require having to buffer $x(t-1)$ and $x(t-2)$ and hence will require memory.

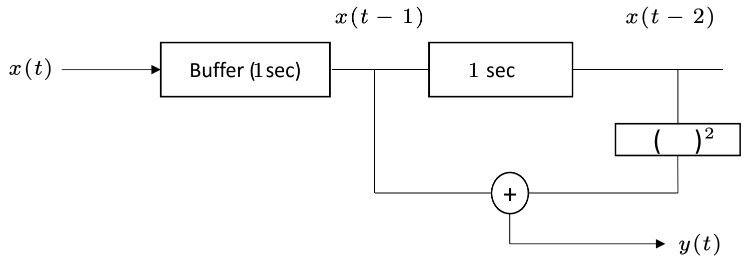


Figure 3.17:

2. $y(t) = 2x(t+1) - 3x(t)$

Example 3.4.30. *Here are some examples of memoryless systems*

1. $y(t) = (x(t))^3$
2. $y(t) = \cos(t+1)x(t)$
3. $y(t) = \sin(t-2)\ x(t)$ *(Be careful here!)*

Example 3.4.31. *Here is an example of a discrete-time system that has memory.*

Consider the system

$$y[n] = y[n-1] + x[n], \quad y[0] = 0, x[0] = 0 \tag{3.12}$$

This one is a bit tricky because the dependence of $y[n]$ on the past values of $x[n]$ is not obvious. But it can be seen that

$$\begin{aligned} y[1] &= x[1] \\ y[2] &= y[1] + x[2] = x[1] + x[2] \end{aligned}$$

3.4.6 Causality

Definition 3.4.32. *A CT system is non-causal (or anticipative) if the output at time t, i.e., $y(t)$ depends on some future value of the input, i.e. on $x(t')$, for $t' > t$. Similarly, a DT system is non-causal(or anticipative) if the output at time n ,i.e $y[n]$ depends on any future value of the input, i.e $x[n']$ for some $n' > n$.*

⚠Notice the spelling of causal. It is not casual. The word causal is related to cause and effect.

Example 3.4.33. *Consider the system $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$. Is this system causal? This system is causal since the output at time n does not depend on any future values of the input, i.e., on any $n' > n$. It only depends on the values taken by x at time $n-1, n-2$ and n, i.e., only on the past values and the present value.*

Example 3.4.34. *Consider the system $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$. Is this system causal? This system is non-causal since the output at time n depend on the value taken by x at time $[n+1]$, i.e., a future value of x.*

Example 3.4.35. *Consider the system $y[n] = x[-n]$. Is this system causal? This system is non-causal since the output at time $n = -1$ depend on the value taken by x at time $n = 1$, i.e., a future value of x.*

Example 3.4.36. *Consider the system $y(t) = \sin(t+1)x(t)$. Is this system causal? This system is causal. Don't get confused by the $t+1$ in the argument of the sine function. The output of the system at time t depends on only the current value of the input x.*

Why should we care about non-causal systems?

You might wonder how it is possible for the output of a system at time t to be dependent on a future value of the signal x? Indeed, if time is the independent variable, this is not possible for any physically realizable system. So, why then should we care about non-causal systems? I will discuss two situations where this makes sense.

When the independent variable is not time: Sometimes we may be interested in signal where the independent variable is not time and, hence, non-causality may not make something physically unrealizable. Consider an image processing example where we wish to reconstruct a lost part of an image. In this case, it might make sense to fill the missing values as a function of the pixel values around the point under consideration. For example, we might take the average pixel values both to the left and to the right of the pixel under consideration. Mathematically, this will correspond to setting the pixel value at location n_1, n_2 to $\hat{x}[n_1, n_2] = \frac{1}{2}(x[n_1-1, n_2] + x[n_1+1, n_2])$. If you fix n_2 and think of this as a signal with independent variable n_1, notice that this signal is non-causal.

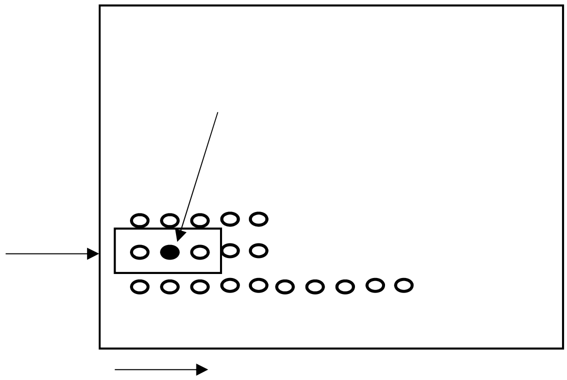


Figure 3.18: An example of a non-causal system which interpolates a missing pixel value by averaging the nearest 2 pixels along the row.

Averaging to remove noise Consider a music file that was recorded sometime in the past and let the sampled signal be given by $x[n]$. Suppose there is some noise that is added to the recording and we want to denoise the noisy signal. One sensible thing to do might be to replace each value value of $x[n]$ by $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$.

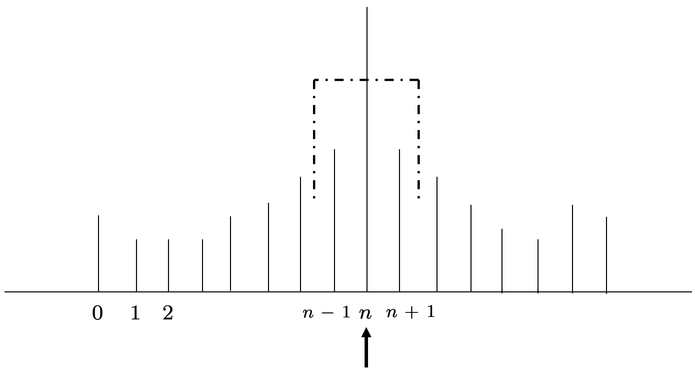


Figure 3.19: An example of a non-causal system where a noisy signal is averaged on both sides of the point under consideration.

According to our definition, such a system is non-causal; however, it is clearly physically realizable. What is the catch? The catch is that we shifted the signal such that time $t = 0$ corresponds to the beginning of the audio file instead of the actual time at which the recording happened. Thus, non-causal systems can be realized if a delay is allowed. In many practical applications, we do not necessarily need things to happen in real time and we can typically tolerate some delay. Then, non-causal systems can be implemented with such a delay. To explain this further, suppose we defined a new signal called $\tilde{x}[n]$ to be $\tilde{x}[n] = x[n+1]$. If we think of $y[n]$ in terms of $\tilde{x}[n]$ instead of $x[n]$, we get $y[n] = \frac{1}{3}(\tilde{x}[n-2] + \tilde{x}[n-1] + \tilde{x}[n])$. So, this is a causal system now.

3.4.7 Tips for solving problems

Here are some tips for solving problems where you are asked to determine if a given system is linear, time invariant, stable, invertible, or causal. First of all, there is no general plug and chugg recipe for solving problems that is guaranteed to work all the time. However, some general problem solving approaches can be useful.

First, try to see if you can build some intuition into what the system is doing instead of blindly trying to apply the mathematical definition. If you have time, pick some simple input signals (rectangular signals, sinusoidal, unit step, impulse etc) and compute the corresponding output signals. Picking the correct input signal that will provide meaningful insight about a system requires expertise and will come with practice. If you are unable to build intuition, you can then try to use the mathematical definition.

I have seen some students apply faulty logic in using the mathematical definition to prove whether a system has some properties. Some students will try one example and depending on what happens with that, they will conclude whether the system has a certain property or not. Trying examples is a good thing, but you should be careful about what you can conclude from the example. To prove that a system is *not* linear, TI, stable, invertible or causal, it suffices to find one example such that the property does not hold; however, if the property holds for the one example, it does not mean that it will hold for every $x(t)$, which is typically what you need to prove that a property holds. For example, to prove that a system is *not* time invariant, it suffices to find one example of $x(t)$ and a shift t_0 such that shifting the input does not result in a mere shift of the output, i.e., $HS^{t_0}x(t) \neq S^{t_0}Hx(t)$. However, it does not suffice to show specific examples of $x(t)$ and t_0 for which $HS^{t_0}x(t) = S^{t_0}Hx(t)$ to conclude that the system is time invariant. Just because this condition is true for *some* $x(t)$ and t_0, it does not mean that it will be true for all $x(t)$ and t_0.

Chapter 4

Time Domain Analysis of Linear Time Invariant Systems

4.1 Impulse response and Convolution

We will first define the impulse reponse of a Linear Time Invariant system and discuss its significance for an LTI system.

Impulse response of an LTI system

Definition 4.1.1. *The response of (i.e., output from) a system H to an unit impulse input signal is called the impulse reponse of the system.*

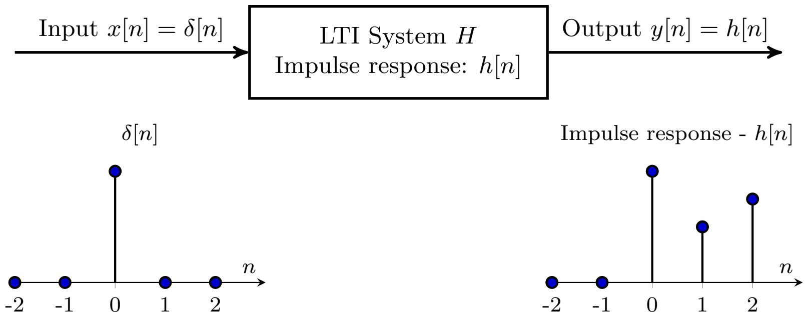


Figure 4.1: The impulse response of a system is the output (response) of the system when the input is the unit impulse signal.

- Impulse response is a signal.
- It is the response of the system to an input which is the impulse signal.
- To find the impulse response, compute $y[n]$ when $x[n] = \delta[n]$.

Some intuition: Here is some intuition for why the impulse response is useful for modeling LTI systems. Consider a large lecture (or music) hall. Suppose a lecturer speaks (or singer sings) the signal $x[n]$ and we wish to determine what exactly will be heard (the signal

$y[n]$) by a person in the audience. Intuitively, the signal $y[n]$ depends on the acoustics of the hall. What we really mean by the acoustics is the profile of echoes - i.e., the number of dominant echoes, their delays, and the attenuation associated with each of the echoes. Once this echo profile is known, for a given $x[n]$, we can shift $x[n]$ and attenuate it to produce the echoes and then sum all the echo components to compute $y[n]$. But how can we determine the echo profile of the room? One way to do this is for the speaker to just clap their hands and for us to listen (with an appropriate measuring instrument) to the echoes of the clap. The time delays and the relative attenuation can be determine by just observing when the echoes are heard and what their relative strengths are. A person clapping their hand is the same as creating an input signal that is $\delta(t)$ or $\delta[n]$ and the echo profile that we observe is the impulse response of the room. Once this is known, we can understand how any signal $x[n]$ or $x(t)$ would be transformed by the room.

Example 4.1.2. *Consider a specific example where there is one direct path from the speaker to the listener and one echo component with attenuation of 0.5 with a delay of one unit. Modeling the system as a DT system, the received signal is related to the transmitted signal according to* $y[n] = x[n] + 0.5x[n-1]$.

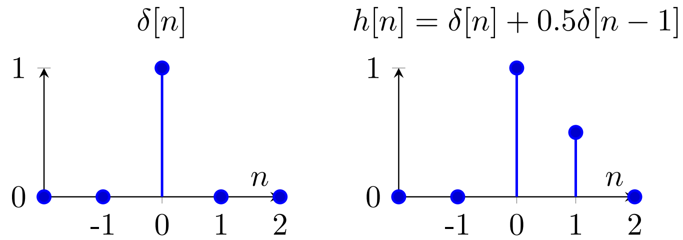


Figure 4.2: The impulse response of a room with one echo component.

In this case, the impulse response of the system is given by $h[n] = \delta[n] + 0.5\delta[n-1]$.

Example 4.1.3. *Let the output of a DT System $y[n]$ be related to its input $x[n]$ according to:*

$$y[n] - 0.5y[n-1] = 2x[n]$$

with initial conditions $y[-1] = 0$. Compute the impulse response of the system $h[n]$.

We first write $y[n]$ in terms of $y[n-1]$ and $x[n]$ as follows

$$y[n] = 0.5y[n-1] + 2x[n]$$

and then we step through n starting from the provided initial conditions to compute $y[n]$ for $n = 0, 1, 2, \dots$ as follows

$$\begin{aligned} y[n] &= 0, \ \textit{for } n < 0 \\ y[0] &= 0.5y[-1] + 2x[0] = 0 + 2 = 2 \\ y[1] &= 0.5y[0] + 2x[1] = 1 + 0 = 1 \\ y[2] &= 0.5y[1] + 2x[2] = 0.5 + 0 = 0.5 \\ \vdots &= \vdots \\ y[n] &= 2\left(\frac{1}{2}\right)^n \textit{for } n \geq 0. \end{aligned}$$

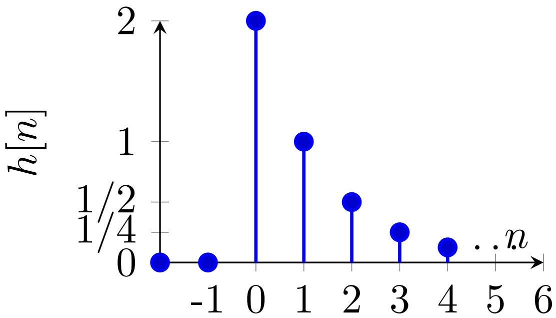


Figure 4.3: Impulse response of the system defined by $y[n] = 0.5y[n-1] + 2x[n]$.

$$\begin{aligned} y[n] &= 2\left(\frac{1}{2}\right)^n n \geq 0 \\ h[n] &= \begin{cases} 2\left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0. \end{cases} \end{aligned} \tag{4.1}$$

4.2 Deriving the DT convolution sum

Why should we care about the impulse response of an LTI system? For LTI systems, the impulse response completely characterizes the system, i.e., we can find the output $H\{x[n]\}$ for any $x[n]$ if the impulse response $h[n]$ is known. We will demonstrate this through an example.

Example 4.2.1. *Consider a linear time invariant system with impulse response* $h[n] = \frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n-3]$*. What will be the output of the system when the input is* $x[n] = -\delta[n+1] + \frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2]$ *?*

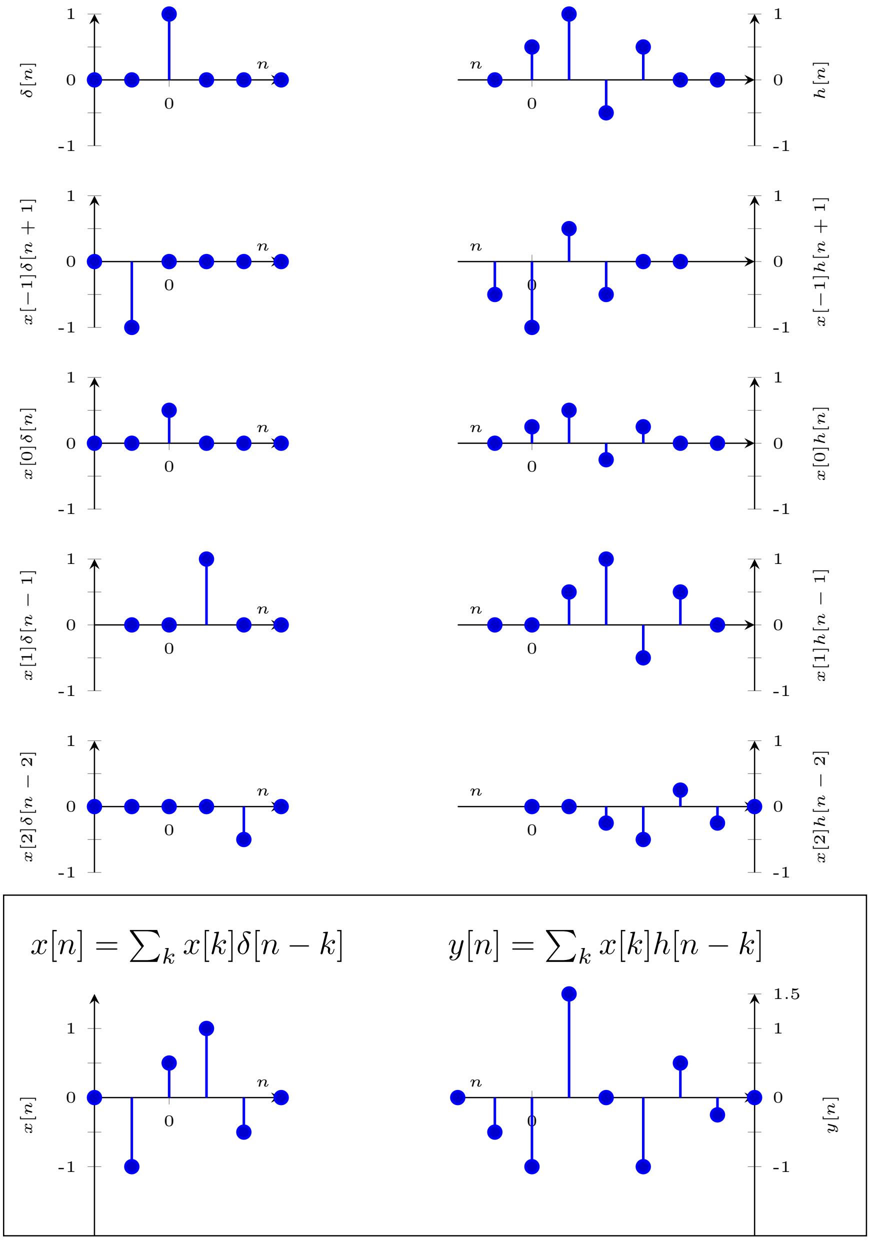


Figure 4.4: Computing the output of an LTI system when the input is $x[n] = -\delta[n+1] + \frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2]$ and the impulse response is $h[n] = \frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n-3]$.

The above example can be generalized to an arbitrary input signal $x[n]$. Any input signal $x[n]$ can be written as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{4.2}$$

where $x[k]$ is the value taken by the signal $x[n]$ at $n = k$

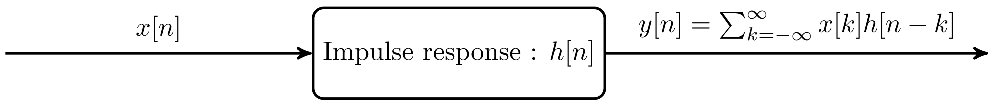
$$y[n] = H\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{4.3}$$

Convolution sum for DT signals

The above equation is known as the convolution sum and is written as:

$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{4.4}$$

In effect, what this means is that if an arbitrary DT signal $x[n]$ is input to a system with impulse response $h[n]$, the output will be $y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$. This is shown pictorially here



We now study a few different ways to perform the computation that will yield $y[n]$ given a $x[n]$ and $h[n]$.

4.3 DT Convolution by taking Weighted Combinations of Shifts of $h[n]$

If we are given a $x[n]$ and $h[n]$, one way to compute $y[n] = x[n] \star h[n]$ is to interpret $h[n-k]$ as the *signal* $h[n]$ shifted to the right by k and therefore, we form a weighted linear combination of shifted versions of $h[n]$ where the weight associated with $h[n-k]$ is $x[k]$. The following example will clarify this

Example 4.3.1. *Let* $x[n] = \frac{1}{2}\delta[n] + 2\delta[n-1]$ *and* $h[n] = u[n] - u[n-3] = \delta[n] + \delta[n-1] + \delta[n-2]$. *Then* $y[n]$ *can be computed as*

$$\begin{aligned} y[n] &= \frac{1}{2}h[n] + 2h[n-1] = \frac{1}{2}(\delta[n] + \delta[n-1] + \delta[n-2]) + 2(\delta[n-1] + \delta[n-2] + \delta[n-3]) \\ &= \frac{1}{2}\delta[n] + \frac{5}{2}\delta[n-1] + \frac{5}{2}\delta[n-2] + 2\delta[n-3] \end{aligned} \tag{4.5}$$

This approach typically works well when $x[n]$ and $h[n]$ are non-zero only for a few values of n.

4.4 DT convolution as Polynomial Multiplication

A third way to compute the convolution sum is to realize that $\sum_k x[k]h[n-k]$ is not a new quantity at all; it is infact, something you have seen in calculus, just not exactly in this form. To elaborate on this, we associate two polynomials with $x[n]$ and $h[n]$ given by

$$\begin{aligned} X(z) &:= \sum_{k=-\infty}^{\infty} x[k]z^k = \ldots + x[-2]z^{-2} + x[-1]z^{-1} + x[0]z^0 + x[1]z^1 + x[2]z^2 + \ldots \\ H(z) &:= \sum_{k=-\infty}^{\infty} h[k]z^k = \ldots + h[-2]z^{-2} + h[-1]z^{-1} + h[0]z^0 + h[1]z^1 + h[2]z^2 + \ldots \end{aligned}$$

Now, let $Y(z) := \sum_{k=-\infty}^{\infty} y[n]z^n = X(z)H(z)$. What will be coefficient of the term corresponding to z^n? The coefficient of z^n, namely $y[n]$ would be the sum of the coefficients of all the terms which when multiplied would result in z^n. That is $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Example 4.4.1. *Let* $x[n] = \frac{1}{2}\delta[n] + 2\delta[n-1]$ *and* $h[n] = u[n] - u[n-3] = \delta[n] + \delta[n-1] + \delta[n-2]$. *Compute the convolution of* $x[n]$ *and* $h[n]$.

$$X(z) = \frac{1}{2}z^0 + 2z^1 \tag{4.6}$$

$$H(z) = z^0 + z^1 + z^2 \tag{4.7}$$

Compute $Y(z) = (\frac{1}{2}z^0 + 2z^1)(z^0 + z^1 + z^2) = \frac{1}{2}z^0 + \frac{5}{2}z^1 + \frac{5}{2}z^2 + 2z^3$. *We can read off the coefficients of* $Y(z)$ *to get* $y[n]$, *i.e.* $y[0] = \frac{1}{2}, y[1] = \frac{5}{2}, y[2] = \frac{5}{2}, y[3] = 2$ *and all the other terms are zeros. Thus,*

$$y[n] = \frac{1}{2}\delta[n] + \frac{5}{2}\delta[n-1] + \frac{5}{2}\delta[n-2] + 2\delta[n-3]$$

4.5 Graphical DT Convolution Procedure

The approach of computing weighted linear combinations of shifted impulse responses or polynomial multiplication may become cumbersome in many cases. In these cases, there is an alternate way to compute the convolution sum. To compute the convolution sum or signal $y[n]$, perform the following steps:

1. Think of $x[n]$ and $h[n]$ as signals $x[k]$ and $h[k]$ respectively, i.e., with the independent variable being k instead of n.

2. Flip $h[k]$ about the Y -axis to obtain the signal $h[-k]$.

3. To compute the signal $y[n]$ for a fixed value of n, shift the signal $h[-k]$ by n units to the right to obtain the signal $h[n-k]$. When n is negative, this amounts to shifting the signal $h[-k]$ to the left, but mathematically it is equivalent to shifting to the right by a negative number.

4. Compute $w_n[k] = x[k]h[n-k]$, i.e., $w_n[k]$ is the product of the signals $x[k]$ and $h[n-k]$.

5. Compute $y[n] = \sum_k w_n[k] = \sum_k x[k]h[n-k]$ by summing the values of $w_n[k]$ for all values of k.

This will give you the value of the signal $y[n]$ for one value of n. Repeat this procedure for every integer value of n, i.e., $n \in$ [...,-3,-2,-1,0,1,2,3,...] to obtain the full signal $y[n]$. In practice, it will be easy to start with large negative values of n and increase n.

Consider a linear time invariant system with impulse response $h[n] = \frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n-3]$. What will be the output of the system when the input is $x[n] = -\delta[n+1] + \frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2]$?

0	0	0	0	-1	0.5	1	-0.5	0	0	0	0

Table 4.1: The signal $x[k]$

4.6 Computing the Convolution Sum Examples

Example 4.6.1. *Let* $x[n] = \frac{1}{2}\delta[n] + 2\delta[n-1]$ *and* $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$. *Compute the convolution of* $x[n]$ *and* $h[n]$.

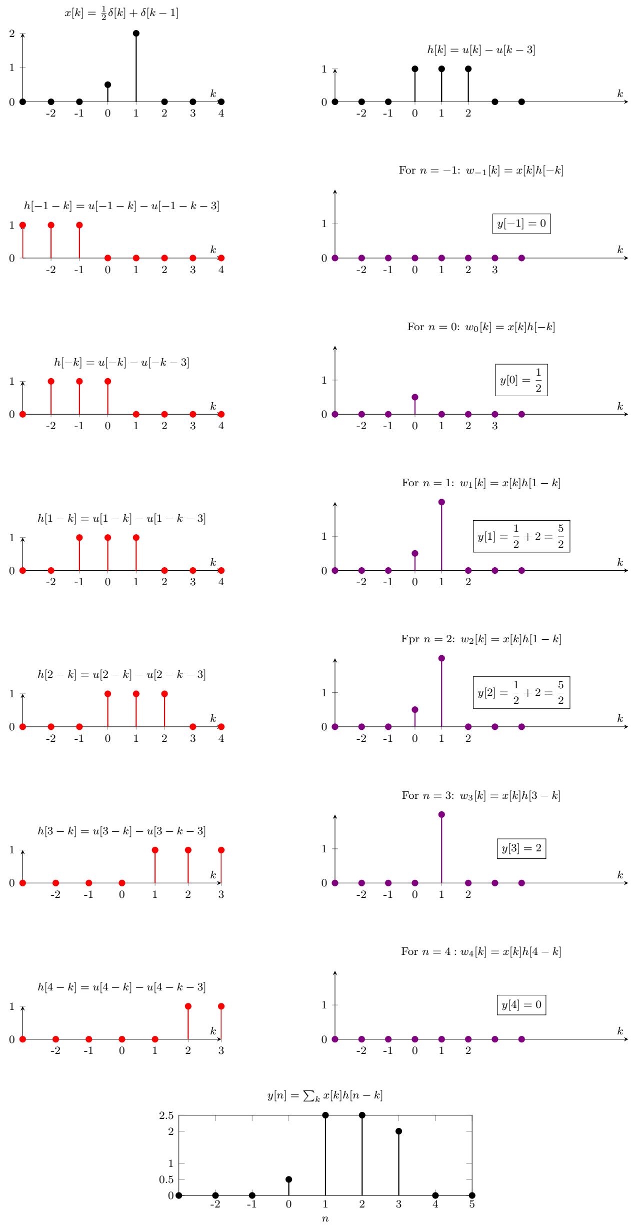
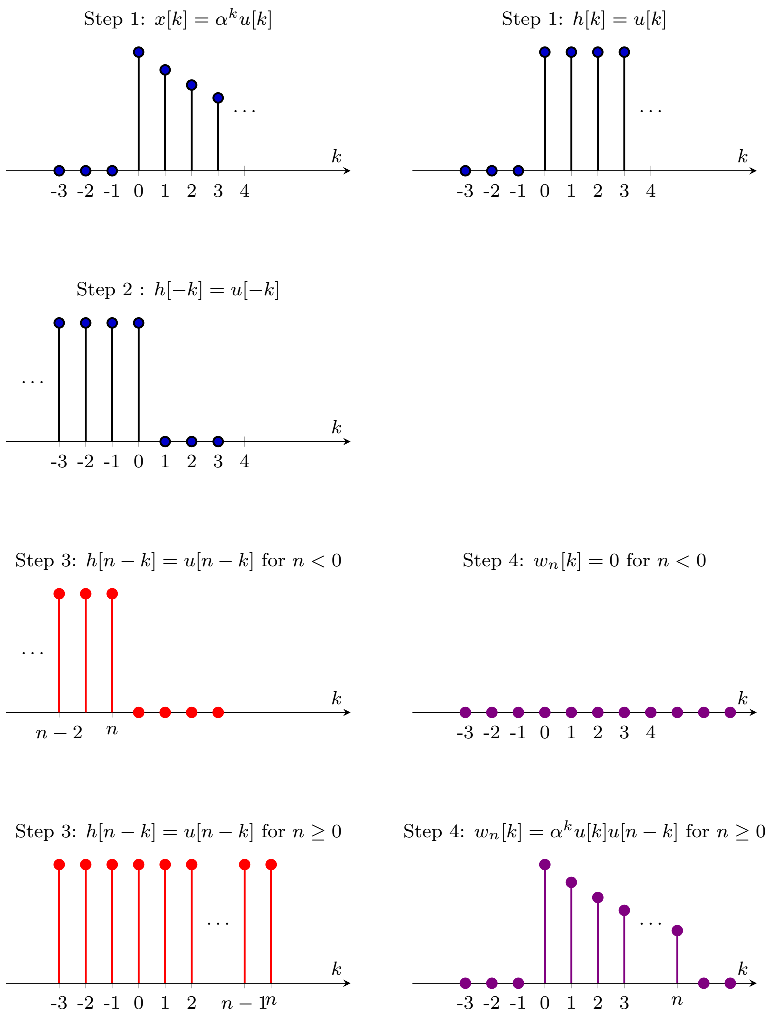


Figure 4.5: Example showing the steps in the convolution of $x[n] = \frac{1}{2}\delta[n] + 2\delta[n-1]$ and $h[n] = u[n] - u[n-3]$

4.7 More DT Convolution Examples

Example 4.7.1. *Let* $x[n] = \alpha^n u[n]$ *and* $h[n] = u[n]$ *Compute* $y[n] = x[n] \star h[n]$

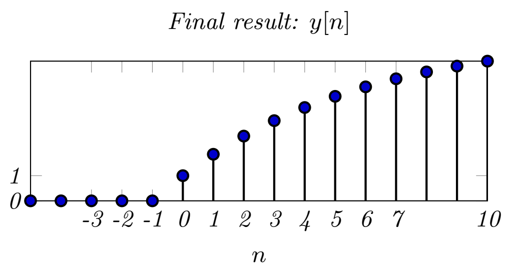
We can solve this by following the five step procedure for performing convolution graphically as follows



Step 5:

- *For* $n < 0$*:* $y[n] = 0$
- *For* $n \geq 0 : y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$.

This can succintly be written as $y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n]$.



4.8 More DT Convolution Examples

Example 4.8.1. *Let* $x[n] = \alpha^n u[n]$ *and* $x[n] = u[n] - u[n-5]$ *Compute* $y[n] = x[n] \star h[n]$

Example 4.8.2. *Let* $x[n] = \alpha^n u[n]$ *and* $h[n] = \beta^n u[n]$. *Compute* $y[n] = x[n] \star h[n]$

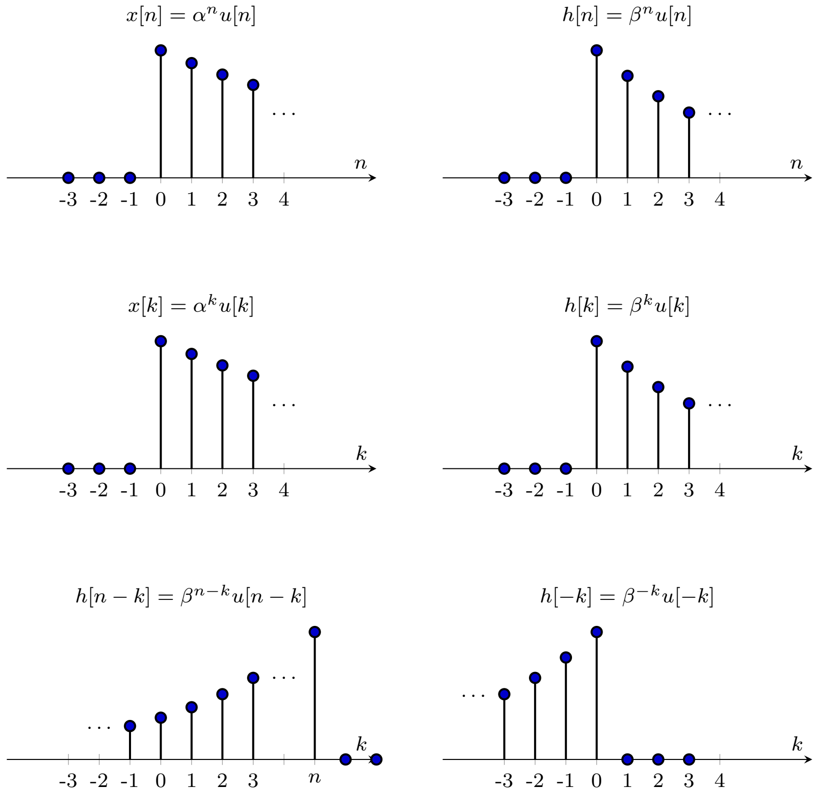


Figure 4.6: Operations involved in computing $(\alpha^n u[n]) \star (\beta^n u[n])$.

Let $w_n[k] = x[k]h[n-k]$. *It can be seen that*

$$w_n[k] = \begin{cases} 0 & \text{if } k < 0 \\ \alpha^k \beta^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n \end{cases}$$

$$\begin{aligned} y[n] &= \sum_k w_n[k] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} \\ &= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \\ &= \frac{\beta^n}{\beta^{n+1}} \frac{\beta^{n+1} - \alpha^{n+1}}{\frac{\beta-\alpha}{\beta}} \\ &= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \end{aligned}$$

$$y[n] = \begin{cases} 0 & \text{if } n < 0 \\ \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} & \text{if } n \ge 0 \end{cases}$$

which can be succintly written as $y[n] = \left(\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}\right) u[n]$.

Example 4.8.3. $x[n] = \alpha^n u[n]$ *and* $h[n] = \beta^n u[n]$ *(Solve from equations only and without pictures)*

Solution: We know that, $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

So, $x[k] = \alpha^k u[k]$, $h[n-k] = \beta^{n-k}u[n-k]$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\beta^{n-k}u[n-k] \\ &= \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k}u[k]u[n-k] \end{aligned}$$

Now,

$$u[k] = \begin{cases} 1 & \textit{if } k \geq 0 \\ 0 & \textit{if } k < 0 \end{cases}$$

$$u[n-k] = \begin{cases} 1 & \textit{if } n-k \geq 0 \\ 0 & \textit{if } n-k < 0 \end{cases}$$

So,

$$u[k]u[n-k] = \begin{cases} 1 & \textit{if } 0 \geq k \geq n \\ 0 & \textit{otherwise} \end{cases}$$

$$\begin{aligned} &\textit{For } n < 0 : y[n] = 0 \\ &\textit{For } n \geq 0 : y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} \\ &\qquad\qquad\qquad = \beta^n \sum_{k=0}^{n} \big(\frac{\alpha}{\beta}\big)^k \\ &\qquad\qquad\qquad = \beta^n \frac{1-\big(\frac{\alpha}{\beta}\big)^{n+1}}{1-\frac{\alpha}{\beta}} \\ &\qquad\qquad\qquad = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}. \end{aligned}$$

Therefore, $y[n] = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}u[n]$.

4.9 Deriving the Convolution Integral

In this section, we will derive the output of a CT LTI system when the input is $x(t)$. An arbitrary input signal $x(t)$ can be approximated as the sum of weighted shifted rectangular signals according to

$$\begin{aligned}
x(t) &\approx ... + x(-T_p)\text{rect}\left(\frac{t+T_p}{T_p}\right) + x(0)\text{rect}(t) + x(T_p)\text{rect}\left(\frac{t-T_p}{T_p}\right) + ... \\
&= \sum_{n=-\infty}^{\infty} x(nT_p)\text{rect}\left(\frac{t-nT_p}{T_p}\right) \\
&= \sum T_p\ x(nT_p)\ \left[\frac{1}{T_p}\text{rect}\left(\frac{t-nT_p}{T_p}\right)\right]
\end{aligned}$$

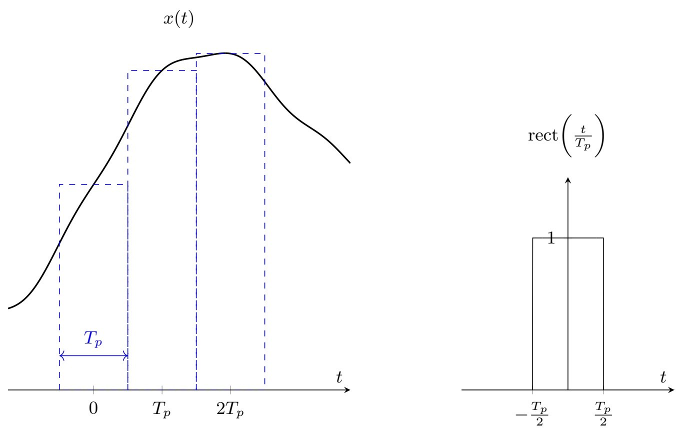


Figure 4.7: A function $x(t)$ can be approximated as linear combination of weighted shifted narrow rectangles

Suppose $h_p(t)$ is the response of the system to an input $\frac{1}{T_p}\text{rect}\left(\frac{t-nT_p}{T_p}\right)$. Then, since the system is linear and time-invariant, $y(t)$ corresponding to $x(t)$ is given by $y(t) = \sum_n T_p\ x(nT_p)\ h_p(t-nT_p)$. When $T_p \to 0$, $\frac{1}{T_p}\text{rect}(\frac{t}{T_p}) \to \delta(t)$. Let the impulse response be $h(t)$. Now, let us look at $y(t) = \sum T_p x(nT_p)h_p(t-nT_p)$ and let $\tau = nT_p$. Then, the output of the LTI system $y(t)$ as $T_p \to 0$ is given by $y(t) = \int d\tau x(\tau)h(t-\tau)$. Thus, we can define the convolution operator for CT signals as

Convolution integral for CT signals

$$y(t) := x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{4.8}$$

4.10 CT Convolution Procedure

Let $h(t)$ be the impulse response of a linear and time-invariant(LTI) system. If the signal $x(t)$ is input to the system, the output signal from the system is given by the convolutional integral

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)$$

This operation is called convolution and we say that the signal $y(t)$ is the convolution of the signal $x(t)$ and the signal $h(t)$ and we denote this by $y(t) = x(t) \star h(t)$. Thus,

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{4.9}$$

Graphical procedure for CT convolution

To compute the signal $y(t)$, perform the following steps:

1. Think of $x(t)$ and $h(t)$ as signals $x(\tau)$ and $h(\tau)$ respectively, i.e., with the independent variable being τ instead of t.

2. Flip $h(\tau)$ about the Y-axis to obtain the signal $h(-\tau)$.

3. To compute the signal $y(t)$ for a fixed value of t, shift the signal $h(-\tau)$ by t units to the right to obtain the signal $h(t-\tau)$. When t is negative, this amounts to shifting the signal $h(-\tau)$ to the left, but mathematically it is equivalent to shifting to the right by a negative number.

4. Compute $w_t(\tau) = x(\tau)h(t-\tau)$, i.e., $w_t(\tau)$ is the product of the signals $x(\tau)$ and $h(t-\tau)$.

5. Compute $y(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ by integrating over τ.

This will give you the value of the signal $y(t)$ for one value of t. In principle, we need to repeat this for every value of t. Since t takes uncountably many values, that is clearly impossible. We need to look for regions in t such that we can write an expression that is valid for all t in that region. The following examples will clarify this.

4.11 Continuous Time Convolution Examples

CT convolution can be best understood from the following examples:-

Example 4.11.1. *Find the convolution of $x(t) = u(t+3)$ and $h(t) = e^{-t}u(t)$ and roughly sketch the result.*

Solution:

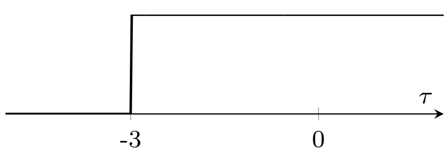
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

We begin by considering h and x as signals with τ being the independent variable

$$x(\tau) = u(\tau+3), \quad h(\tau) = e^{-\tau}u(\tau) \tag{4.10}$$

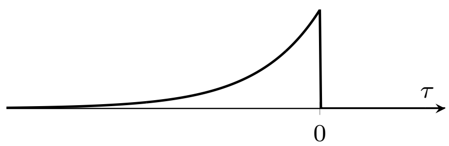
Let $w_t(\tau) = x(\tau)h(t-\tau)$

$x(\tau) = u(\tau+3)$

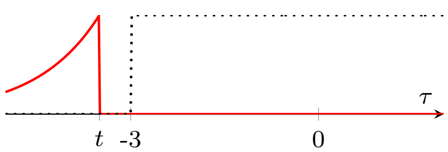


$h(\tau) = e^{-\tau}u(\tau)$

$h(-\tau) = e^{\tau}u(-\tau)$

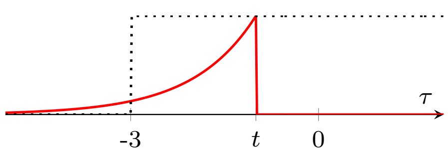


For $t < -3$: $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$

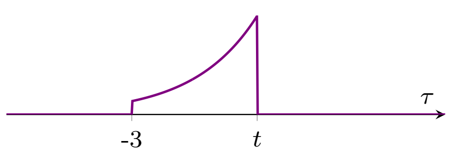


For $t < -3$: $w_t(\tau) = 0$

For $t \geq -3$: $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$



For $t \geq -3$: $w_t(\tau) = e^{-(t-\tau)}[u(t-\tau) - u(\tau+3)]$



- *Interval 1: For $t < -3$, $w_t(\tau) = 0$ for $-\infty < \tau < \infty$.*

- *Interval 2: For $t \geq -3$,*

$$w_t(\tau) = \begin{cases} e^{\tau-t} & \textit{if } -3 \leq \tau \leq t \\ 0 & \textit{otherwise.} \end{cases}$$

Therefore,

$$y(t) = \begin{cases} 0 & \textit{if } t < -3 \\ \int_{-3}^{t} e^{\tau - t} d\tau = 1 - e^{-3-t} & \textit{if } t \geq -3. \end{cases}$$

We can succintly write the answer as $y(t) = (1 - e^{-3-t})u(t+3)$ *and it is sketched below.*

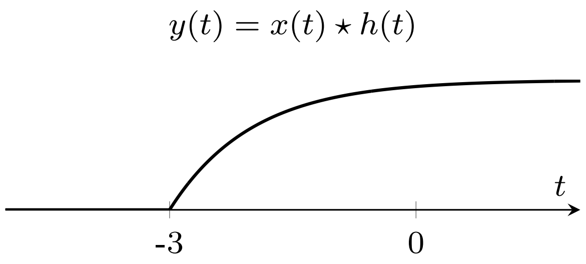
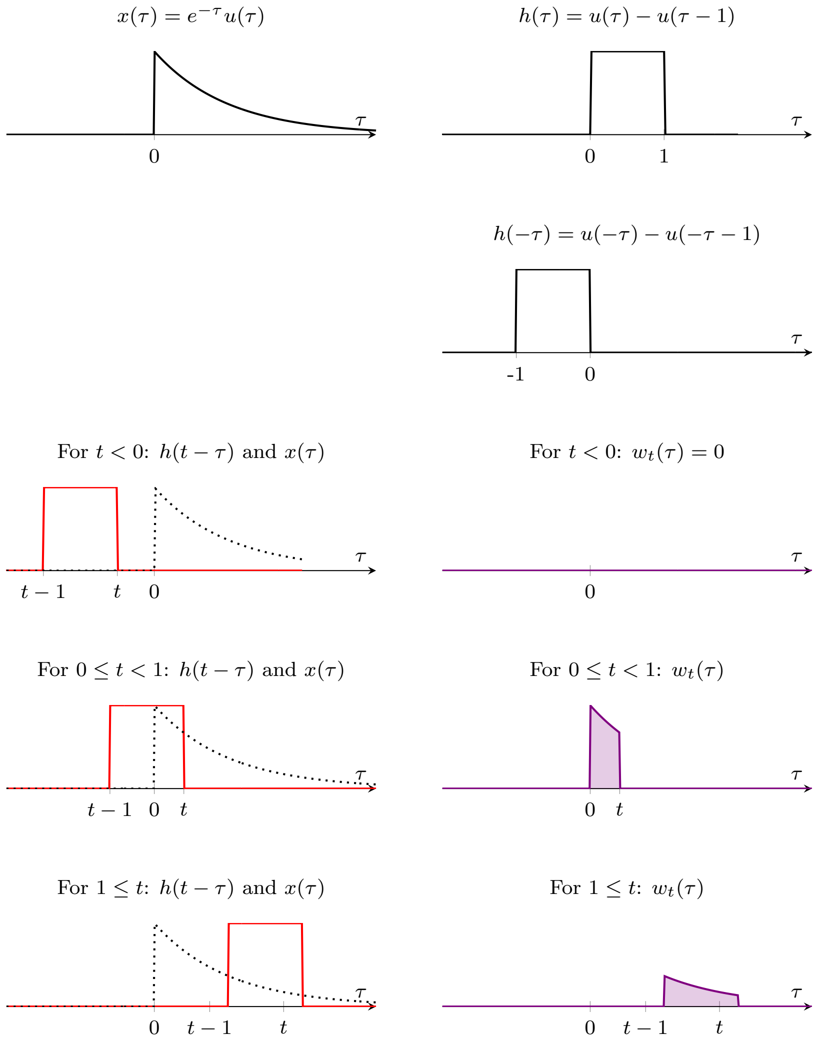


Figure 4.8: Result of $u(t+3) \star e^{-t}u(t)$.

Example 4.11.2. *Let* $h(t) = rect(t - \frac{1}{2})$ *and* $x(t) = e^{-t}u(t)$*. Compute* $y(t) = x(t) \star h(t)$ *and roughly sketch the result*

We begin by sketching the signals $x(\tau)$ *and* $h(t-\tau)$ *for various intervals and determining the intermediate signal* $w_t(\tau)$ *for each of these intervals.*



- *Interval 1:* $t < 0$, $y(t) = 0$
- *Interval 2:* $0 \leq t < 1$, $y(t) = \int_0^t e^{-\tau} d\tau = 1\big(1 - e^{-t}\big)$
- *Interval 3:* $1 \leq t$, $y(t) = \int_{t-1}^t e^{-\tau} d\tau = -e^{-t} + e^{-(t-1)} = e^{-(t-1)}(1 - e^{-1})$

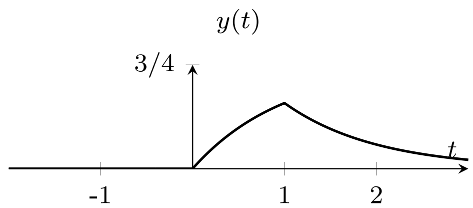
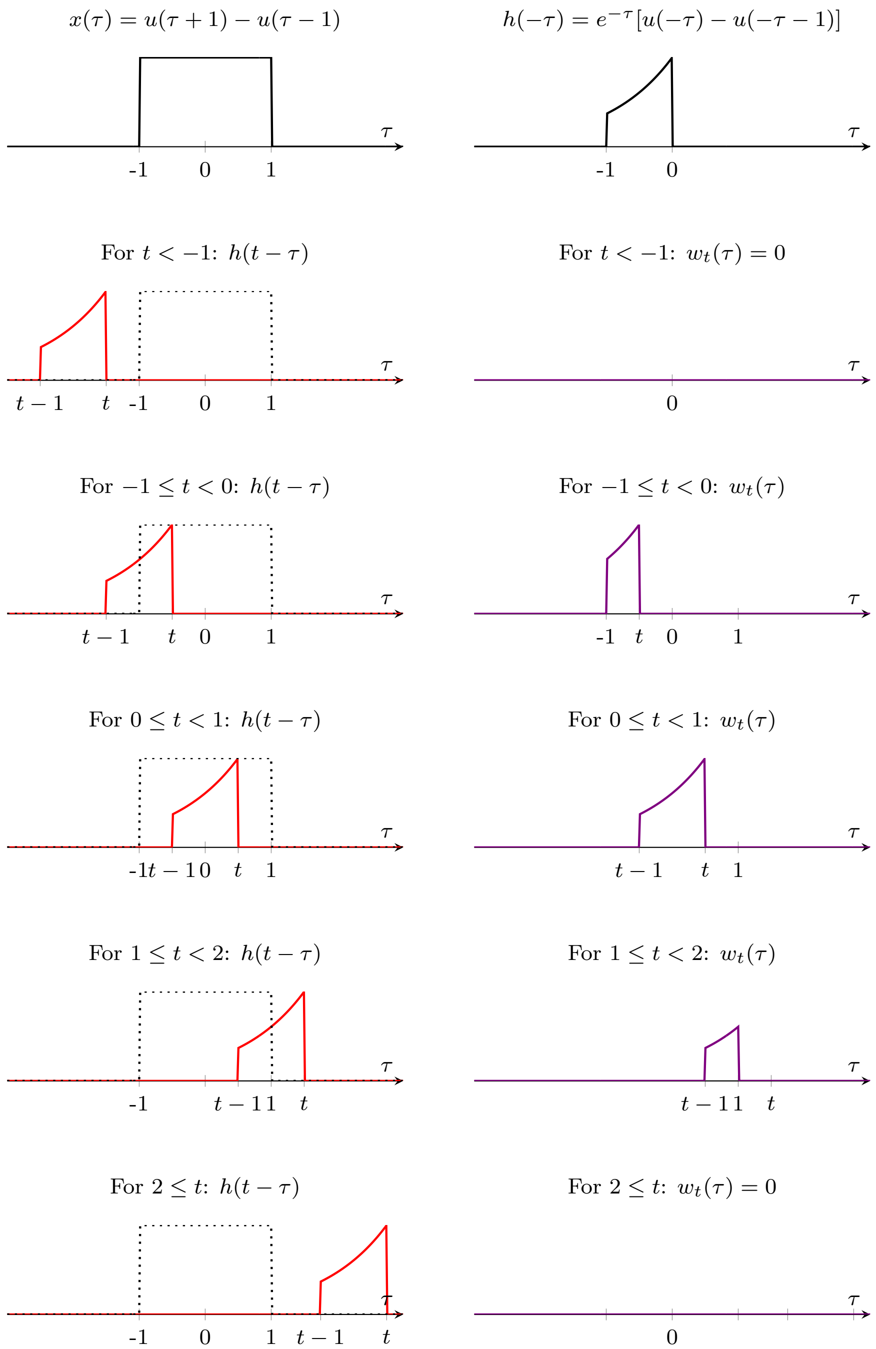


Figure 4.9: The result of convolving $x(t) = e^{-t}u(t)$ and $h(t) = \text{rect}\big(t - \frac{1}{2}\big)$.

Example 4.11.3. *$x(t) = u(t+1) - u(t-1)$ and $h(t) = e^{-t}[u(t) - u(t-1)]$. Compute $y(t) = x(t) \star h(t)$ and roughly sketch it.*



- *For $t < -1$, $w_t(\tau) = 0 \Rightarrow y(t) = 0$*
- *For $-1 \leq t < 0$, $y(t) = \int_{-1}^{t} e^{\tau - t} d\tau = e^{\tau - t}\Big|_{-1}^{t} = 1 - e^{-1-t}$*
- *For $0 \leq t < 1$, $y(t) = \int_{t-1}^{t} e^{\tau - t} d\tau = e^{\tau - t}\Big|_{t-1}^{t} = 1 - e^{-1}$*
- *For $1 \leq t < 2$, $y(t) = \int_{t-1}^{1} e^{\tau - t} d\tau = e^{\tau - t}\Big|_{t-1}^{1} = e^{1-t} - e^{-1}$*
- *For $2 \leq t$, $w_t(\tau) = 0 \Rightarrow y(t) = 0$.*

So,

$$y(t) = \begin{cases} 0 & if\, t < -1 \\ 1 - e^{-1-t} & if\, -1 \leq t < 0 \\ 1 - e^{-1} & if\, 0 \leq t < 1 \\ e^{1-t} - e^{-1} & if\, 1 \leq t < 2 \\ 0 & if\, t \geq 2 \end{cases}$$

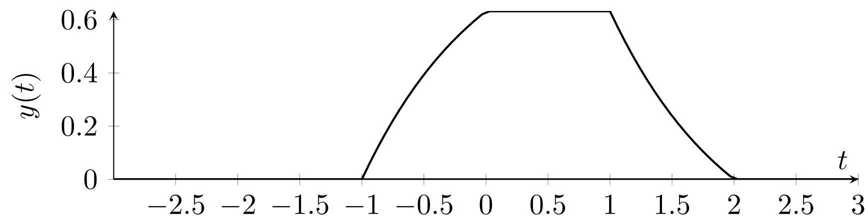


Figure 4.10: Plot of $y(t)$

4.12 Applications of convolution in Probability - Python notebook

In this chapter, we motivated convolution as the operation that is naturally performed by LTI systems. Convolution appears in probability (ECEN 303) also. Suppose X and Y are two independent random variables with density functions $f_X(x)$ and $f_Y(y)$, you may recall from ECEN 303 that the density function of $Z = X + Y$ is given by

$$f_Z(z) = f_X(x) \star f_Y(y) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

With a slight change in notation (think of $f_Z(z)$ as $y(t)$, $f_X(x)$ as $x(t)$, $f_Y(y)$ as $h(t)$), we see that the above integral is exactly the convolution integral in (4.8). Hence, this can be computed using the techniques we have discussed in this chapter.

We now consider an example.

Example 4.12.1. *Let X be a random variable that is exponentially distributed and let Y be a random variable, independent of X, that is uniformly distributed in $[0, 1)$, i.e., their densities are given by*

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \textit{otherwise.} \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \leq y < 1 \\ 0, & \textit{otherwise.} \end{cases} \tag{4.11}$$

What is the probability density function of $Z = X + Y$?

The density of Z can be computed in exactly the same way as we computed the convolution integral in Example 4.11.2. The density of Z would then be given as in Fig. 4.9.

In Section 7.3.6, we explore this connection further. In ECEN 303, you may have seen that the characteristic function of Z is the product of the characteristic functions of X and Y. This is no different from the fact that the Fourier transform of $x(t) \star h(t)$ is the product of the Fourier transform of $x(t)$ and the Fourier transform of $h(t)$.

4.13 Properties of Convolution

Convolution is a operator that acts on two signals and produces a third signal. In this sense, it is similar to addition, subtraction, multiplication and division of two signals which also operate on two signals and produce a signal as the result. In this section, we will study some properties of the convolution operator. The proofs of some of these properties are omitted for now.

4.13.1 Identity

What is the signal $h[n]$ such that $x[n] \star h[n] = x[n]$ for all $x[n]$? Such a signal would be called the identify signal for the convolution operator. The identity signal for DT convolution and CT convolution are $\delta[n]$ and $\delta(t)$, respectively. That is,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] &= x[n] \\ \int\limits_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &= x(t) \end{aligned}$$

4.13.2 Commutativity

Convolution is commutative, i.e.

$$\begin{aligned} x[n] \star h[n] &= h[n] \star x[n] \\ x(t) \star h(t) &= h(t) \star x(t) \end{aligned}$$

or, equivalently,

$$\begin{aligned} \sum_k x[k]h[n-k] &= \sum x[n-k]h[k] \\ \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\ d\tau &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\ d\tau \end{aligned}$$

A practically useful implication of this result is that when you are asked to convolve two signals, you can choose to flip and shift each of the two. Often, it will be useful to flip and shift one of the two since it will have a simpler description.

4.13.3 Homogenity

$$\begin{aligned} ax[n] \star h[n] &= a\big(x[n] \star h[n]\big) \\ ax(t) \star h(t) &= a\big(x(t) \star h(t)\big) \end{aligned}$$

This is simply a restatement of the fact that for an LTI system, scaling the input signal by a simply scales the output by a.

4.13.4 Distributivity (Linearity)

$$
\begin{aligned}
x[n] \star (h_1[n] + h_2[n]) &= x[n] \star h_1[n] + x(t) \star h_2[n] \\
x(t) \star (h_1(t) + h_2(t)) &= x(t) \star h_1(t) + x(t) \star h_2(t)
\end{aligned}
$$

Homogenity and distributivity together implies linearity.

This implies that if an input $x[n]$ is passed through two LTI systems in parallel with impulse responses $h_1[n]$ and $h_2[n]$ and the outputs are summed, it is equivalent to have passed the input $x[n]$ through an equivalent LTI system with impulse response $h_{eq}[n] = h_1[n] + h_2[n]$.

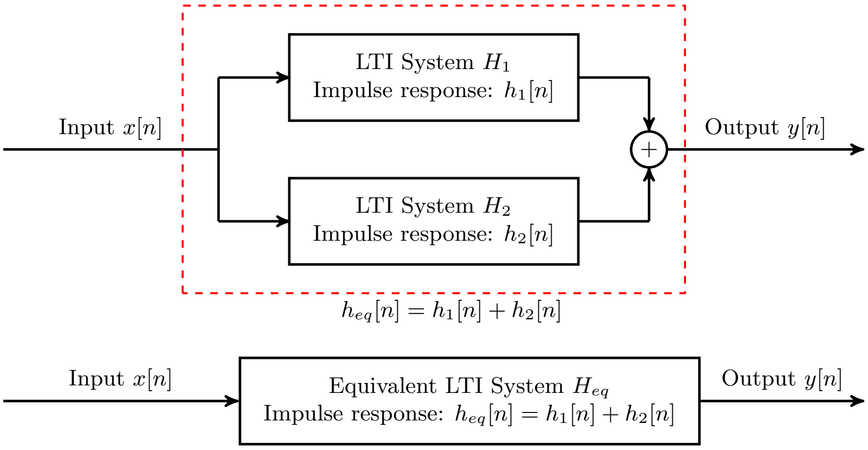


Figure 4.11: Equivalent impulse of two LTI systems in parallel.

4.13.5 Associativity

$$
\begin{aligned}
(x[n] \star h_1[n]) \star h_2[n] &= x[n] \star (h_1[n] \star h_2[n]) \\
(x(t) \star h_1(t)) \star h_2(t) &= x(t) \star (h_1(t) \star h_2(t))
\end{aligned}
$$

This implies that if an input $x(t)$ is passed through two LTI systems in series with impulse

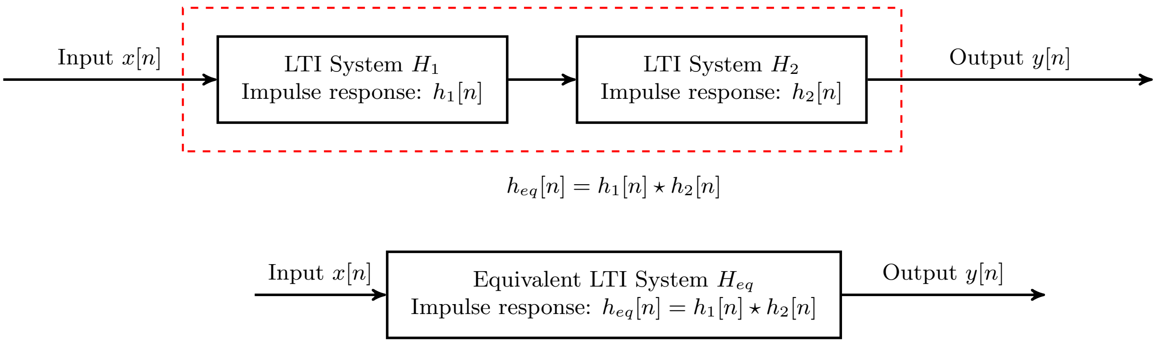


Figure 4.12: Equivalent impulse of two LTI systems in series.

responses $h_1(t)$ and $h_2(t)$, it is equivalent to having passed the input $x(t)$ through an equivalent LTI system with impulse response $h_{eq}(t) = h_1(t) \star h_2(t)$.

4.13.6 Time Invariance

Suppose $x(t) \star h(t) = y(t)$. Then,

$$\begin{aligned} x(t-t_0) \star h(t) &= y(t-t_0) \\ x(t) \star h(t-t_0) &= y(t-t_0) \\ x(t-t_1) \star h(t-t_2) &= y(t-(t_1+t_2)) \end{aligned}$$

A special case of the above result when $h(t) = \delta(t)$ gives the following result which will be used a few times in this course.

$$\begin{aligned} x[n] \star \delta[n-n_0] &= x[n-n_0] \\ x(t) \star \delta(t-t_0) &= x(t-t_0) \end{aligned}$$

Example 4.13.1. *What is $x(t) \star (\delta(t-1) + 2\delta(t) + \delta(t-2)$?*

Using the distributive property and the time invariance property, we can see that this is $x(t-1) + 2x(t) + x(t-2)$.

Example 4.13.2. *For a given T, and $x(t)$, what is $y(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT)$? How is it related to $x(t)$?*

$$y(t) = x(t) \star \left(\sum_{n=-\infty}^{\infty} \delta(t-nT) \right) = \sum_{n=-\infty}^{\infty} x(t-nT)$$

Thus, $y(t)$ is a periodic version of $x(t)$ obtained by repeating $x(t)$ every T seconds.

Example 4.13.3. *Let $x(t) = (1-|2t|)(u(t+1/2) - u(t-1/2))$ and $h(t) = \sum\limits_{n=-\infty}^{\infty} \delta(t-n))$. The result of convolving $x(t)$ and $h(t)$ is shown below*

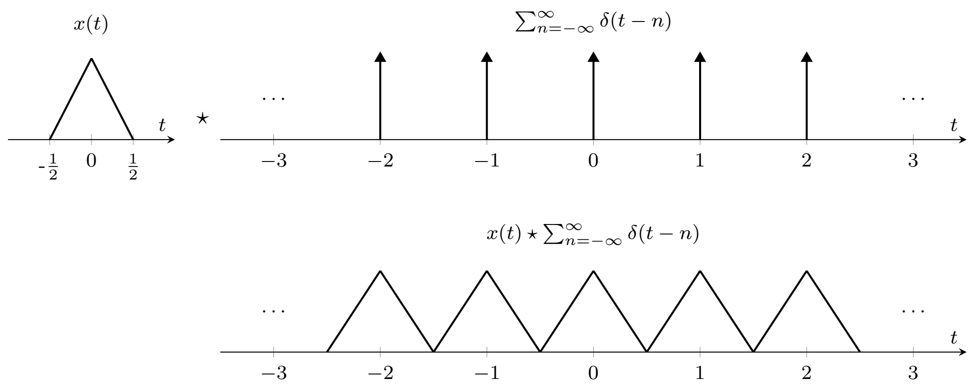


Figure 4.13: Example showing convolution with an impulse train

4.13.7 Support of $y[n]$

Suppose $x[n]$ is a signal that is zero for $n < 0$ and $n > N_1$, and $h[n]$ is a signal that is zero for $n < 0$ and $n > N_2$. Then, it can be seen that $y[n]$ is zero for $n < 0$ and $n > N_1 + N_2$. Thus, the support of $y[n]$ is $N_1 + N_2 - 1$. Similarly, if $x(t)$ and $h(t)$ are signals with contiguous support of length T_1 and T_2, $y(t)$ has a support that is of length at most $T_1 + T_2$.

4.14 Inferring Properties of LTI Systems from the Impulse Response

In the previous section, we have shown that the impulse response of an LTI system completely characterizes the system. Therefore, given the impulse response $h[n]$ of a DT LTI system or $h(t)$ of a CT LTI System, it should be possible to determine if the system is memoryless, causal, stable, and invertible.

4.14.1 Memoryless property

The output of an LTI system, $y[n]$, is related to the input through

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{4.12}$$

A system is memoryless iff $y[n]$ does not depend on $x[n-k]$ for any $k \neq 0$. From the above equation, we can see that the memoryless condition implies that $h[k] = 0$ for $k \neq 0$. In other words, $h[k]$ must be of the form $c\delta[k]$. Using the same argument, we can show that a CT system is memoryless iff $h(t)$ is of the form $c\delta(t)$.

Example 4.14.1. *Consider an LTI system with $h[n]$ as shown. Is this system memoryless?*

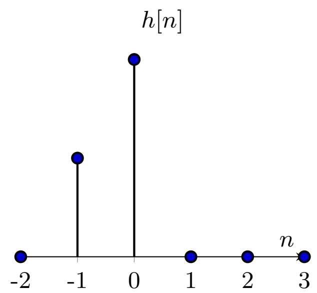


Figure 4.14: Example of an impulse response of an LTI system with memory

Since $h[n]$ is not of the form $c\delta[n]$, it is not memoryless. This can be seen by explicitly writing the output as a function of the input. In this case, we get $y[n] = 2x[n] + x[n-1]$. Since $y[n]$ depends on $x[n-1]$, this system is not memoryless.

4.14.2 Causality

A system is causal iff $y[n]$ does not depend on $x[n+k]$ for any $k > 0$. From (4.12), this implies that $h[k]$ must be zero for all $k > 0$.

Example 4.14.2. *Consider the impulse response shown in Fig. 4.15. This corresponds to a non-causal system.*

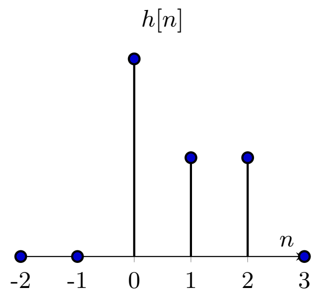


Figure 4.15: Example of an impulse response of an LTI system that is causal

4.14.3 Stability

We will show that if $\sum_{n=-\infty}^{\infty} |h[n]$ is finite (or bounded) then the LTI system is stable. It can be shown that this condition is also necessary. Recall that a system is stable if every bounded input produces a bounded output. Consider the output of an LTI system and its relationship to the input as given below.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The absolute value of $y[n]$ for any n can be bounded according to

$$|y[n]| = |\sum h[k]x[n-k]| \leq \sum |h[k]x[n-k]|$$

For any bounded input with bound M_x, by definition, $|x[n-k]| < M_x$. Therefore,

$$|y[n]| \leq M_x\Big(\sum_{k=-\infty}^{\infty} |h[k]|\Big) \tag{4.13}$$

If $\sum_{n=-\infty}^{\infty} |h[n]| < M_h$ (is bounded), then $|y[n] \leq M_x M_h$ and hence, is bounded. Therefore, the system will be stable. Using a similar argument, it can be shown that a CT LTI system is stable if $\int_{-\infty}^{\infty} |h(t)|dt$ is bounded.

Example 4.14.3. *Consider an LTI system with $h(t) = u(t)$. Is the system stable?*

Since $\int_{-\infty}^{\infty} |u(t)|dt$ is not bounded, the system is not stable. To give a concrete example of a bounded input signal that will produce an unbounded output signal, consider the input $x(t) = u(-t)$ and compute the output $y(t)$ at $t = 0$. We can see that it is unbounded.

Example 4.14.4. *Consider an LTI system with $h[n]\left(\frac{1}{2}\right)^n u[n]$. Is this system stable?*

$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$. *This implies that the system is stable.*

4.14.4 Invertibility

An LTI system is invertible if the output of the system can be passed through another system whose output is the same as the original signal $x[n]$. An LTI system may not have an inverse. We will state without proof that if the inverse of an LTI system exists, it must also be linear and time invariant. Thus, the question of whether an LTI system H is invertible or not is the same as asking if there is another LTI system H_{inv} such that the cascade of H and H_{inv} acts like an identity system (i.e., whose impulse response is $\delta[n]$).

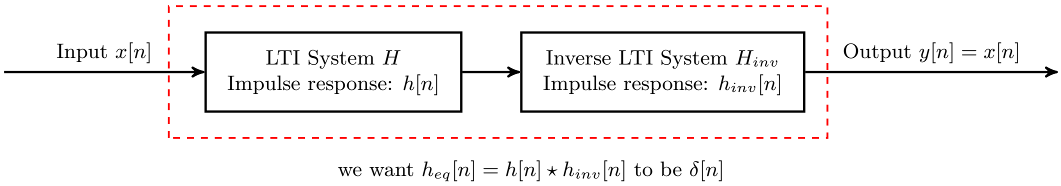


Figure 4.16: An LTI system is invertible if $h[n] \star h_{inv}[n] = \delta[n]$.

Therefore, LTI System with impulse response $h[n]$ or $h(t)$ is invertible if $\exists$ an $h_1[n]$ or $h_1(t)$ such that:

$$\begin{aligned} h[n] * h_1[n] &= \delta[n] \\ h(t) * h_1(t) &= \delta(t) \end{aligned}$$

Example 4.14.5. *Consider the system with impulse response $h(t) = \delta(t - t_0)$. Is this system invertible and if so, what is it's inverse?*

This system simply shifts the input by t_0 to the right. Hence, a system that shifts the output by t_0 to the left would be it's inverse. Hence $h_{inv}(t) = \delta(t + t_0)$. Indeed, it can be seen that

$$\delta(t - t_0) \star \delta(t + t_0) = \delta(t)$$

Example 4.14.6. *Consider a DT system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$. To find if the system is invertible, we ask if there is a system with impulse response $h_{inv}[n]$ such that $h[n] * h_{inv}[n] = \delta[n]$? By guess and check, we see that $h_{inv}[n] = \delta[n] - \frac{1}{2}\delta[n-1]$.*

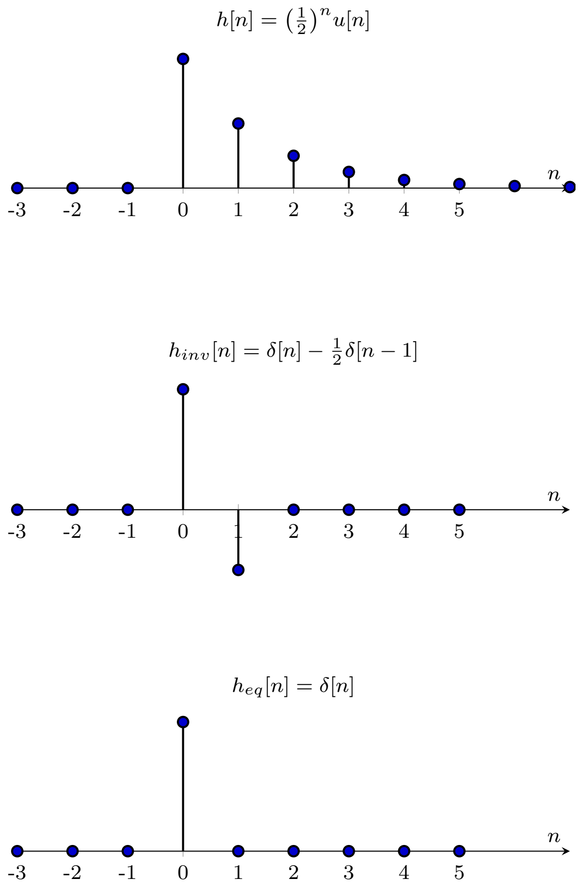


Figure 4.17: The inverse a system with $h[n] = \left(\frac{1}{2}\right)^n u[n]$ is a system with impulse response $h_{inv}[n] = \delta[n] - \frac{1}{2}\delta[n-1]$.

It can be seen that $h_{eq}[n] = \delta[n]$. *For example,* $h_{eq}[2] = 1 \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} = 0$.

4.15 Step response of an LTI System

The output of an LTI system when the input is $x(t) = u(t)$ is called the step response of the system (response of the system to a unit-step input). It is given by

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\ d\tau = \int_{-\infty}^{t} h(\tau)\ d\tau \tag{4.14}$$

A visual representation of this is shown in Fig. 4.18.

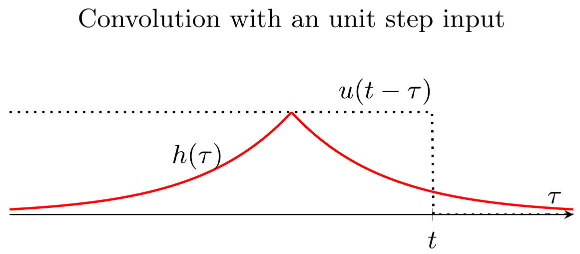


Figure 4.18: Figure showing convolution of $h(t)$ and $u(t)$ to compute the step response of an LTI sytem.

Example 4.15.1. *Let $h(t) = e^{-|t|}$. What is the step response of the system?*

The step response is given by $y(t) = \int_{-\infty}^{t} e^{-|\tau|}\ d\tau$ which is

$$y(t) = \begin{cases} \int_{-\infty}^{t} e^{\tau}\ d\tau = e^{t}, & t < 0 \\ \int_{-\infty}^{0} e^{\tau}\ d\tau + \int_{0}^{t} e^{-\tau}\ d\tau = 2 - e^{-t}, & t \geq 0 \end{cases} \tag{4.15}$$

4.16 Response of an LTI system to complex exponentials

The response of an LTI system to a complex exponential input deserves special study. Let $s = \sigma + j\omega$ be any complex number and consider the signal $x(t) = e^{st}$, which is sometimes referred to as the everlasting complex exponential. Suppose this signal is input to an LTI system with impulse response $h(t)$. The output of the LTI system is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau && (4.16)\\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\, d\tau = \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}\, d\tau && (4.17)\\ &= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau}_{H(s)} = H(s)e^{st} && (4.18) \end{aligned}$$

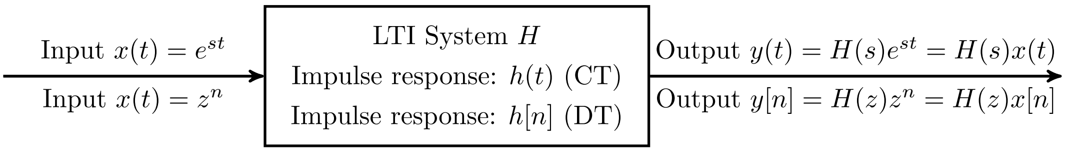


Figure 4.19: When the input to an LTI system is e^{st}, the output is simply a scaled version of the input.

Complex exponentials are eigen functions of LTI systems

Thus, when the input is of the form e^{st}, the LTI system simply scales the input by a (possibly complex) constant and does not alter the signal otherwise. The complex constant can be thought of the gain/attenuation of the system. This depends on s and the impulse response $h(t)$ and can be thought of as a transfer function of the system at frequency s. Notice the similarity between this and eigenvectors of a matrix whose direction remains unchanged when multiplied by the matrix. Hence, we call complex exponentials as eigen functions of an LTI system.

For the special case of $s = j\omega$, the output $y(t) = H(j\omega)e^{j\omega t}$. Here $H(j\omega)$ is the gain or transfer function associated with the LTI system at a frequency of ω rad/s.

Example 4.16.1. *Consider an LTI system with $h(t) = \delta(t) + \delta(t-1)$. What is the response of the system when the input is - a) e^{j5t} and b) e^{j3t} ?*

When $x(t) = e^{j5t}$,

$$\begin{aligned} y(t) = (\delta(t) + \delta(t-1)) \star e^{j5t} &= e^{j5t} + e^{j5(t-1)}\\ &= e^{j5t} + e^{j5t-j5} = \underbrace{(1 + e^{-j5})}_{H(s)}\underbrace{e^{j5t}}_{x(t)} \end{aligned}$$

When $x(t) = e^{j3t}$,

$$\begin{aligned} y(t) = (\delta(t) + \delta(t-1)) \star e^{j3t} &= e^{j3t} + e^{j3(t-1)} \\ &= e^{j3t} + e^{j3t-j3} = \underbrace{(1 + e^{-j3})}_{H(s)} \underbrace{e^{j3t}}_{x(t)} \end{aligned}$$

Notice that in both cases the output is simply a scaled version of the input, however the scaling is different.

For DT systems, the input $x[n] = z^n$ is an eigen function and the output will be $y[n] = H(z)x[n]$.

$$y[n] = \sum_{-\infty}^{\infty} h[k]x[n-k] \tag{4.19}$$

$$= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \sum_{k=-\infty}^{\infty} h[k]z^n z^{-k} \tag{4.20}$$

$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]z^{-k}}_{H(z)} = H(z)z^n \tag{4.21}$$

One-sided exponentials are not eigen functions

The signal e^{st} is different from $e^{st}u(t)$ and similarly, z^n is different from $z^n u[n]$. While e^{st} and z^n are eigen functions, $e^{st}u(t)$ and $z^n u[n]$ are not eigen functions.

Example 4.16.2. *Show that for any LTI system, if the input is* $x(t) = \cos(\omega_0 t)$*, the output can be expressed as* $a\cos(\omega_0 t) + b\sin(\omega_0 t)$*. Thus, any LTI system cannot create new frequencies at the output that are not present in the input.*

Chapter 5

Continuous-time Fourier Series

5.1 Prelude

1. What is the frequency of the signal $x(t) = \cos(2\pi t)$ if we were to measure the frequency in Hz?

2. What is the angular frequency of the signal above if we were measure the angular frequency in radians/s ?

3. Consider the signal $x(t) = e^{j2\pi t}$. Is this a periodic signal? If so, what is its fundamental time period? What is its fundamental frequency?

Complex Exponentials

We will refer to the signal $e^{j\omega t}$ as a complex exponential with angular frequency ω radians/s regardless of whether ω is positive or negative.

Let us now write $x(t) = \cos(2\pi t)$ using Euler's formula as

$$\cos(2\pi t) = \frac{1}{2}e^{-j2\pi t} + \frac{1}{2}e^{j2\pi t}$$

We will think of $x(t)$ as the sum of complex exponentials, one with frequency -2π rad/s and one with frequency 2π rad/s.

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5.2 Introduction (video1, video2, Python notebook)

Let $x(t)$ be a real periodic signal with a fundamental time period of T seconds. We will define the fundamental frequency as $\omega_0 = \frac{2\pi}{T}$ rad/s and we will refer to $k\omega_0$ as the kth-harmonic frequency.

Trigonometric Fourier series

Fourier showed that a large class of periodic signals with a time period of T seconds can be written as linear combinations of sinusoids and cosinusoids whose frequencies are $k\omega_0, k \in \mathbb{Z}^+$, i.e.,

$$x(t) = B_0 + \sum_{k=1}^{\infty} B_k \cos(k\omega_0 t) + A_k \sin(k\omega_0 t) \tag{5.1}$$

This is called the trigonometric Fourier series representation.

Since $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$ can be written as linear combinations of $e^{jk\omega_0 t}$ and $e^{-jk\omega_0 t}$ by Euler's formula according to

$$\begin{aligned} \cos(k\omega_0 t) &= \frac{1}{2}e^{jk\omega_0 t} + \frac{1}{2}e^{-jk\omega_0 t} \\ \sin(k\omega_0 t) &= \frac{1}{2j}e^{jk\omega_0 t} - \frac{1}{2j}e^{-jk\omega_0 t} \end{aligned}$$

Exponential Fourier series

we can write (5.1) as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \tag{5.2}$$

i.e., we can write $x(t)$ as a linear combination of the signals $e^{jk\omega_0 t}$ and $e^{-jk\omega_0 t}$ for $k = 0, 1, 2, \ldots$. Such a representation is called the exponential Fourier series representation. In some books, the coefficients are denoted by a_k, but we will use $X[k]$ to represent the coefficients.

We will also refer to this as simply the Fourier series representation or the synthesis equation of the Fourier series. We will also use $X[k]$ instead of a_k and these will be called the Fourier series coefficients of $x(t)$. By equating the coefficients in (5.1) and (5.2), we can see that

$$X[0] = B_0 \tag{5.3}$$

$$X[k] = \frac{B_k}{2} - j\frac{A_k}{2}, \quad k > 0 \tag{5.4}$$

$$X[-k] = \frac{B_k}{2} + j\frac{A_k}{2}, \quad k > 0 \tag{5.5}$$

and similarly,

$$B_0 = X[0] \tag{5.6}$$
$$B_k = X[k] + X[-k], \ k > 0 \tag{5.7}$$
$$A_k = j(X[k] - X[-k]), \ k > 0 \tag{5.8}$$

In general, we can construct a *complex* periodic signal $x(t) = x_R(t) + jx_I(t)$ where $x_R(t)$ and $x_I(t)$ themselves have an exponential F.S. representation $X_R[k]$ and $X_I[k]$, respectively. Then $X[k] = X_R[k] + jX_I[k]$. Thus, the exponential F.S. representation can be used directly with complex periodic signals $x(t)$.

5.3 Parseval's theorem

Parseval's theorem states that the power in a periodic CT signal $x(t)$ can be computed using the F.S. coefficients.

$$P_x = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2 \tag{5.9}$$

This tells us that the total power in a CT periodic signal is distributed in the different harmonics. The amount of power contained in the k-th harmonic is $|X[k]|^2$.

5.3.1 Linear Algebra interpretation

Consider the space of all CT periodic signals with fundamental time period T. Consider the set of signals

$$\Phi = \{\ldots, e^{-j2\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, e^{j2\omega_0 t}, \ldots\}$$

Since every $x(t)$ in the space can be written as a linear combination of the elements of the set Φ, we say that Φ is a basis for the signal space.

5.3.2 What do the F.S. coefficients signify?

Loosely speaking, $X[k]$ denotes the amount of a certain harmonic or the amount of a certain frequency that is present in the signal $x(t)$. I am intentionally using loosely defined words like 'the amount of a certain harmonic or frequency' that is present in the signal. Such a loose language provides some insight into the fact that periodic signals are composed of a complex exponentials (sinusoids and cosinsusoids) of frequencies $0, \omega_0, 2\omega_0, \ldots,$. The more precise way to interpret the F.S. coefficients is through the synthesis equation, i.e., $X[k]$ is the coefficient of the term $e^{jk\omega_0 t}$ in representing $x(t)$ as a linear combination of complex exponentials. $|X[k]|^2$ denotes the amount of power that is contained in the kth harmonic. The term with $k = 0$ corresponds to a complex exponential with zero frequency which is also referred to as a the D.C. (direct current) term.

5.3.3 Fourier Series coefficients as a discrete-frequency signal

$X[k]$ can be thought of as a discrete-frequency signal where k represents the kth harmonic. We can, therefore, express $X[k]$ using $\delta[k]$'s to make the presentation succint.

Example 5.3.1. *Consider the signal $x(t) = \cos(\omega_0 t)$. This can be written as $\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$ and hence,*

$$X[k] = \begin{cases} 1/2, & \textit{if } k = -1 \\ 1/2, & \textit{if } k = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

$X[k]$ can be succintly written as $X[k] = \frac{1}{2}\delta[k+1] + \frac{1}{2}\delta[k-1]$.

5.4 Synthesis and Analysis Equations

Let $x(t)$ be a periodic signal with a time period T and fundamental frequency $w_0 = \frac{2\pi}{T}$.Then, Fourier's result tells us that

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jkw_0 t} \tag{5.10}$$

This is called the synthesis equation since it tells us that we can synthesize or form $x(t)$ by taking linear combinations of complex exponentials at different harmonics.

Naturally, we are interested in the relationship between $x(t)$ and $X[k]$, i.e., for what set of $X[k]$s, can we write $x(t)$ as in (5.10)? We will now show that if (5.10) is to be satisfied, then

$$X[k] = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt, \text{ where } \omega_0 = \frac{2\pi}{T}. \tag{5.11}$$

This equation is called the analysis equation.

To prove this, let us first show an important result concerning complex exponentials. Let $\omega_0 = \frac{2\pi}{T} \Rightarrow \omega_0 T = 2\pi$. Consider the set of complex exponentials

$$\{\phi_k(t)\} = \{...., e^{-j2w_0 t}, e^{-jw_0 t}, 1, e^{jw_0 t}, e^{j2w_0 t}, ...\},$$

where $\phi_k(t) = e^{jk\omega_0 t}$. Say we pick two complex exponentials from this set $\phi_m(t) = e^{jm\omega_0 t}$ and $\phi_l(t) = e^{jl\omega_0 t}$ and we compute $\int_{-\infty}^{\infty} \phi_k(m)\phi_l^*(t)dt = \int_0^T e^{jm\omega_0 t}e^{-jl\omega_0 t}dt$.

$$\int_0^T e^{jm\omega_0 t}e^{-jl\omega_0 t}dt = \int_0^T e^{j(m-l)\omega_0 t}dt \quad = \left[\frac{e^{j(m-l)\omega_0 t}}{j(m-l)\omega_0}\right]_0^T = \frac{e^{j(m-l)\omega_0 t - 1}}{j(m-l)\omega_0}$$

Since m and l are integers and $\omega_0 T = 2\pi$,

$$e^{j(m-l)\omega_0 T} = e^{j(m-l)2\pi} = 1.$$

Therefore,

$$\int_0^T e^{jm\omega_0 t}e^{-jl\omega_0 t}dt = \begin{cases} 0 & \text{if } m \neq l \\ T & \text{if } m = l \end{cases} \tag{5.12}$$

Hence,

$$\begin{aligned}\int_0^T x(t)e^{-jl\omega_0 t}dt &= \int_0^T \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}e^{-jl\omega_0 t}dt \\ &= \sum_{k=-\infty}^{\infty} \int_0^T X[k]e^{j(k-l)\omega_0 t}dt \\ &= \sum_{k=-\infty}^{\infty} X[k]\int_0^T e^{j(k-l)\omega_0 t}dt\end{aligned}$$

Substituting the value from equation 3.2,

$$\int_0^T x(t)e^{-jl\omega_0 t}dt = X[l]T$$

So,the analysis equation is given by:-

$$\boxed{X[k] = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt,\ \omega_o = \frac{2\pi}{T}} \tag{5.13}$$

Notice that when $k = 0$, $X[k]$ is given by

$$\boxed{X[0] = \frac{1}{T}\int_T x(t)dt}$$

Sometimes when computing $X[k]$, you may run into divide by zero when $k = 0$ and in those cases, you may have to evaluate $X[0]$ using the above equation directly. $X[0]$ represent the average value of $x(t)$ over a period and is called the DC component of $x(t)$.

$X[k]$ can be complex even when $x(t)$ is a real signal. It is not unusual or unreasonable for this to happen. Take for example, $\sin(k\omega_0 t)$ which can be written as $\sin(k\omega_0 t) = \frac{1}{2j}e^{jk\omega_0 t} - \frac{1}{2j}e^{-jk\omega_0 t}$. Notice that the coefficients in the linear combination are $\frac{1}{2j}$ and $-\frac{1}{2j}$ which are both imaginary. But $\sin(k\omega_0 t)$ is strictly real. Thus, the complex numbers multiplying complex exponentials add up in such a way that the final result of the linear combination is real when $x(t)$ is real.

5.5 Computing Fourier Series Coefficients

In this section, we consider the question - if we are given an $x(t)$, how can we compute $X[k]$? There are basically two ways to do this.

1. Direct Method - directly plug $x(t)$ into (5.11) and evaluate the integral

2. Method of Inspection - sometimes we don't have to evaluate the integral and we can 'see' that $x(t)$ is a linear combination of complex exponentials. This typically happens when the signal itself is a complex exponential or is a sum of sinusoids and cosinusoids.

5.5.1 Direct Method - (video)

Example 5.5.1. *Let $x(t)$ be a continuous-time periodic signal with fundamental time period T given by*

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

We can compute $X[k]$ using (5.13) (detailed calculation is shown below) and we get

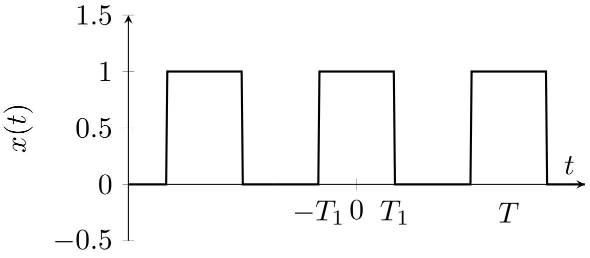
$$X[k] = \frac{2\sin(k\omega_0 T_1)}{Tk\omega_0}, k \neq 0 \tag{5.14}$$

$$X[0] = \frac{2T_1}{T} \tag{5.15}$$

This can be written using the sinc function as

$$X[k] = \frac{2T_1}{T} sinc\left(\frac{2kT_1}{T}\right)$$

In this case, the CTFS coefficients turn out to be strictly real (imaginary part is zero).



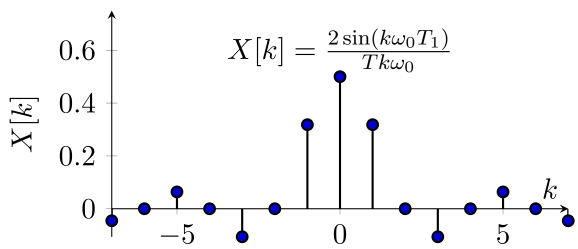


Figure 5.1: The CTFS coefficients for a rectangular signal with a fundamental time period T

A detailed calculation of the Fourier series coefficients $X[k]$ is shown below

$$X[k] = \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-jk\omega_0 t}\,dt$$

Substituting $x(t) = 1$ for $-T_1 \leq t \leq T_1$:

$$X[k] = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt$$

Evaluating the integral:

$$X[k] = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}$$

$$X[k] = \frac{1}{T}\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{-jk\omega_0}$$

Using Euler's formula:

$$X[k] = \frac{1}{T}\frac{-2j\sin(k\omega_0 T_1)}{-jk\omega_0} = \frac{1}{T}\frac{2j\sin(k\omega_0 T_1)}{jk\omega_0}$$

$$X[k] = \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$

For $k = 0$:

$$X[0] = \frac{1}{T}\int_{-T_1}^{T_1} 1\, dt = \frac{2T_1}{T}$$

Combining the results:

$$\boxed{X[k] = \frac{2T_1}{T} sinc\left(\frac{k\omega_0 T_1}{\pi}\right)}$$

where

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}.$$

Example 5.5.2. *Let $x(t)$ be a periodic signal with fundamental time period $T = 2$ one period given by*

$$x(t) = \begin{cases} e^{-t}, & \text{if } 0 \leq t < 1 \\ 0, & 1 \leq 2. \end{cases}$$

Compute the CTFS coefficients and the fundamental frequency.

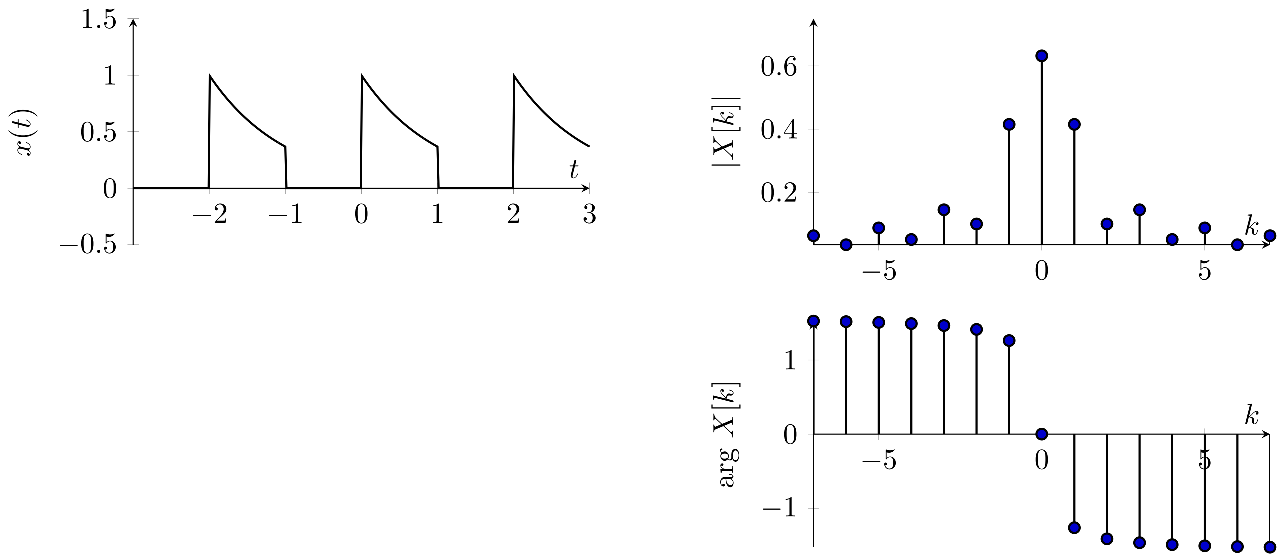


Figure 5.2: The CTFS coefficients for the exponential signal in Example 5.5.2 with fundamental time period $T = 2$.

$$T = 2, \qquad \omega_0 = \frac{2\pi}{2} = \pi \; rad/s$$

$$X[k] = \frac{1}{2}\int_0^1 e^{-t} \cdot e^{-jk\pi t}\, dt \tag{5.16}$$

$$= \frac{1}{2}\int_0^1 e^{-(1+jk\pi)t}\, dt \tag{5.17}$$

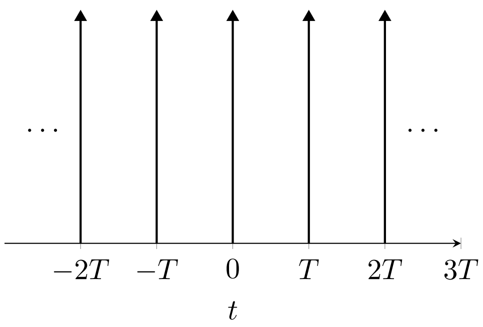
$$= \frac{1}{2}\left[\frac{e^{-(1+jk\pi)t}}{-(1+jk\pi)}\right]_0^1 \tag{5.18}$$

$$= \frac{1}{2}\left[\frac{1 - e^{-(1+jk\pi)}}{1+jk\pi}\right] \tag{5.19}$$

Example 5.5.3. *Let $x(t)$ be an impulse train with fundamental time period T one period given by*

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Compute the CTFS coefficients and the fundamental frequency.



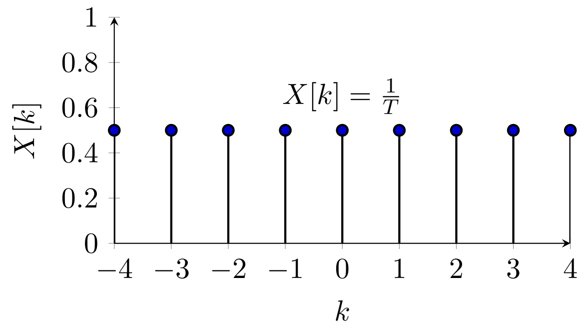


Figure 5.3: The CTFS coefficients for the impulse train in Example 5.5.3 with fundamental time period T.

The fundamental time period is T and hence, the fundamental frequency is $\omega_0 = \frac{2\pi}{T}$.

$$X[k] = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_o t} \, dt \tag{5.20}$$

$$= \frac{1}{T} \quad \forall k \tag{5.21}$$

(5.21) *follows from* (5.20) *due to the sifting property.*

5.5.2 Method of Inspection- (video)

When you are asked to compute the F.S. coefficients of signals involving trigonometric functions or complex exponentials, we can use Euler's formula or sometimes directly read off the F.S. coefficients by inspection. The following examples will elucidate this idea.

Example 5.5.4. *Compute the fundamental frequency and the FS coefficients of* $x(t) = \sin(3\pi t)$. *The fundamental frequency of* $x(t)$ *is* $\omega_0 = 3\pi$.

$$\sin(3\pi t) = \frac{1}{2j}e^{j3\pi t} - \frac{1}{2j}e^{-3\pi t}$$
$$\sin(3\pi t) = \frac{1}{2j}e^{j1.\omega_0 t} - \frac{1}{2j}e^{-1\omega_0 t}$$
$$X[k] = \begin{cases} \frac{1}{2j}, & \textit{if } k = 1 \\ -\frac{1}{2j}, & \textit{if } k = -1 \\ 0, & \textit{otherwise.} \end{cases}$$
$$\Rightarrow X[k] = \frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1]$$

Example 5.5.5. *Let* $x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right)$. *Compute the CTFS representation for* $x(t)$.

We first note that time period $T = 4$ *and hence,* $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$x(t) = 3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

$$3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) = ... + x[-2]e^{-j2\omega_0 t} + x[-1]e^{-j\omega_0 t} + x[0] + x[1]e^{j\omega_0 t} + x[2]e^{j2\omega_0 t} + ...$$

Using Euler's formula, we can write

$$3\cos\left(\frac{\pi t}{2} + \frac{\pi}{4}\right) = \frac{3}{2}\left[e^{j(\frac{\pi t}{2}+\frac{\pi}{4})} + e^{-j(\frac{\pi t}{2}+\frac{\pi}{4})}\right]$$

which can be expanded as

$$\frac{3}{2}\left[e^{j(\frac{\pi t}{2}+\frac{\pi}{4})}+e^{-j(\frac{\pi t}{2}+\frac{\pi}{4})}\right] = \frac{3}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{\pi t}{2}}+\frac{3}{2}e^{j\frac{\pi}{4}}e^{j\frac{\pi t}{2}} = ...+x[-1]e^{-j\frac{\pi t}{2}}+x[0]+x[1]e^{j\frac{\pi t}{2}}+x[2]e^{j\pi t}+.......$$

So, the Fourier Series coefficients are given by:

$$X[k] = \begin{cases} \frac{3}{2}e^{-j\frac{\pi}{4}} & \textit{if } k = -1 \\ \frac{3}{2}e^{j\frac{\pi}{4}} & \textit{if } k = 1 \\ 0 & \textit{if } k \neq -1, 1 \end{cases}$$

Therefore,

$$\boxed{X[k] = \frac{3}{2}e^{-j\frac{\pi}{4}}\delta[k+1] + \frac{3}{2}e^{j\frac{\pi}{4}}\delta[k-1]} \tag{5.22}$$

Example 5.5.6. *Compute the CTFS of* $x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$.

$$x(t) = 1 + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + \frac{2}{2}e^{j\omega_0 t} + \frac{2}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\frac{\pi}{4}}e^{j2\omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2\omega_0 t}.$$

The Fourier coefficients $X[k]$ *are determined as follows:*

$$\begin{aligned} X[0] &= 1, \\ X[1] &= \frac{1}{2j} + 1 = 1 - \frac{1}{2}j, \\ X[-1] &= \frac{-1}{2j} + 1 = 1 + \frac{1}{2}j, \\ X[2] &= \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j), \\ X[-2] &= \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j), \\ X[k] &= 0, \quad |k| > 2. \end{aligned}$$

The compact representation of $X[k]$ *is:*

$$X[k] = \begin{cases} 1 & k = 0, \\ 1 - \frac{1}{2}j & k = 1, \\ 1 + \frac{1}{2}j & k = -1, \\ \frac{\sqrt{2}}{4}(1+j) & k = 2, \\ \frac{\sqrt{2}}{4}(1-j) & k = -2, \\ 0 & |k| > 2. \end{cases}$$

5.6 Inverse CTFS

Sometimes we may be given the Fourier series coefficients $X[k]$ and the fundamental frequency ω_0 and asked to recover the time domain signal $x(t)$. This can be done by simply plugging the coefficients into the synthesis equation (5.2) and simplifying the expressions. Here are a couple of examples.

Example 5.6.1. *Let* $X[k] = \delta[k+3] - j\delta[k+1] + j\delta[k-1] + \delta[k-3]$ *and* $\omega_0 = 4\pi$.

The time-domain signal $x(t)$ can be computed using the inverse Fourier series formula, also called the synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}.$$

Substituting the given $X[k]$:

$$x(t) = X[-3]e^{-j3\omega_0 t} + X[-1]e^{-j\omega_0 t} + X[1]e^{j\omega_0 t} + X[3]e^{j3\omega_0 t}.$$

From the given coefficients:

$$X[-3] = 1, \quad X[-1] = -j, \quad X[1] = j, \quad X[3] = 1.$$

Thus,

$$x(t) = e^{-j3\cdot 4\pi t} - je^{-j\cdot 4\pi t} + je^{j\cdot 4\pi t} + e^{j3\cdot 4\pi t}.$$

Simplifying:

$$x(t) = e^{-j12\pi t} - je^{-j4\pi t} + je^{j4\pi t} + e^{j12\pi t}.$$

We now use Euler's formula to simplify this into the final answer given by

$$x(t) = 2\cos(12\pi t) - 2\sin(4\pi t)$$

Example 5.6.2. *What is the signal $x(t)$ whose Fourier series coefficients are given by*

$$X[k] = \left(-\frac{1}{3}\right)^{|k|}$$

and $\omega_0 = 1$.

The Fourier series representation of a signal $x(t)$ is given by the synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}.$$

Substituting the given $X[k]$ and $\omega_0 = 1$, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} \left(-\frac{1}{3}\right)^{|k|} e^{jkt}.$$

We now separate the summation into two parts: $k \geq 0$ and $k < 0$:

$$x(t) = \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^{k} e^{jkt} + \sum_{k=-\infty}^{-1} \left(-\frac{1}{3}\right)^{|k|} e^{jkt}.$$

For $k < 0$, note that $|k| = -k$. Therefore, the second summation becomes:

$$\sum_{k=-\infty}^{-1} \left(-\frac{1}{3}\right)^{|k|} e^{jkt} = \sum_{k=-\infty}^{-1} \left(-\frac{1}{3}\right)^{-k} e^{jkt}.$$

Changing the variable $k' = -k$, where k' runs from 1 to ∞, the summation becomes:

$$\sum_{k=-\infty}^{-1} \left(-\frac{1}{3}\right)^{-k} e^{jkt} = \sum_{k'=1}^{\infty} \left(-\frac{1}{3}\right)^{k'} e^{-jk't}.$$

Substituting back, we get:

$$x(t) = \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^{k} e^{jkt} + \sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^{k} e^{-jkt}. = \sum_{k=0}^{\infty} \left(-\frac{1}{3}e^{jt}\right)^{k} + \sum_{k=1}^{\infty} \left(-\frac{1}{3}e^{-jt}\right)^{k}$$

$$\begin{aligned}
x(t) &= \sum_{k=1}^{\infty} \left(\frac{1}{3}e^{-jt}\right)^{k} + \sum_{k=0}^{\infty} \left(\frac{1}{3}e^{jt}\right)^{k} \\
&= \frac{\frac{1}{3}e^{-jt}}{1 - \frac{1}{3}e^{-jt}} + \frac{1}{1 - \frac{1}{3}e^{jt}} \\
&= \frac{\frac{1}{3}e^{-jt} - \frac{1}{9} + 1 - \frac{1}{3}e^{-jt}}{1 - \frac{2}{3}\cos t + \frac{1}{9}} \\
&= \frac{\frac{8}{9}}{\frac{10}{9} - \frac{2}{3}\cos t}
\end{aligned}$$

5.7 Convergence of the CTFS - (Python notebook)

What exactly do we mean when we say $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$? Notice that on both sides of this equation we have signals. What does it mean to say that one signal is equal to another signal? What does it mean for an infinite series to converge to a signal? To understand the sublteties, we define the truncated signal, error signal and the truncation error as follows,

$$\begin{aligned} x_N(t) &:= \sum_{k=-N}^{N} X[k]e^{jk\omega_0 t} && (5.23) \\ e_N(t) &:= x(t) - x_N(t) && (5.24) \\ E_N &:= \int_T |x(t) - x_N(t)|^2 \, dt && (5.25) \end{aligned}$$

There are two notions for how the infinite series converges to $x(t)$.

Convergence in squared error: Under this notion, we say that $x_N(t) \to x(t)$ as $N \to \infty$ if $E_N \to 0$ as $N \to \infty$. It turns out that a sufficient condition for this to happen is that $x(t)$ has finite energy over one time period.

Point-wise convergence - Dirichlet conditions (See Section 3.4 from OWN) Under this notion of convergence, $x_N(t) \to x(t), \forall t \in [0, T_0]$. This is also called convergence everywhere (for all values of t). It turns out that if the following set of conditions, called Dirichlet conditions, are satisfied $x(t)$ matches $x_N(t)$ in the limit as $N \to \infty$ everywhere except at the discontinuities.

Dirichlet conditions

1. Over any period, $x(t)$ must be absolutely integrable; $\int_T |x(t)|dt < \infty$
2. In any finite interval of time, $x(t)$ is of bounded variation, i.e., it has a finite number of maxima and minima. An example of $x(t)$ that does not satisfy this condition is

$$x(t) = \sin\left(\frac{2\pi}{t}\right)$$

3. In any finite interval of time, there are finite number of discontinuities

Most signals of interest in engineering satisfy the conditions required for convergence in the mean-squared error and/or point-wise convergence. This is why Fourier analysis is a powerful tool in engineering.

Chapter 6

Discrete-time Fourier Series

In the previous chapter, we saw that a periodic continuous signal can be written as a linear combination of complex exponentials. In this chapter, we discuss a similar result for discrete-time periodic signals. Just like how we were able to write a CT signal as a linear combination of complex exponentials, we can write a periodic DT signal also as a linear combination of discrete-time complex exponential signals.

Summary of DTFS result Suppose $x[n]$ is a DT periodic signal with fundamental time period N. Define the fundamental frequency $\boxed{\Omega_0 := \frac{2\pi}{N}}$. Then, the DFTS representation for $x[n]$ says that

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}. \tag{6.1}$$

From a signal space perspective, let $\phi_k[n] = e^{jk\Omega_0 n}$, and consider the set $\Phi = \{\phi_0[n], \phi_1[n], \ldots, \phi_{N-1}[n]\}$. The DTFS results says that $x[n]$ can be written as a linear combination of the signals in Φ and the Fourier series coefficients are the coefficients of the linear combination. $X[k]$ can be computed using

$$X[k] = \frac{1}{N} \sum_{\langle N \rangle} x[n] e^{-jk\Omega_0 n}, \quad k = 0, 1, 2, \ldots, N-1. \tag{6.2}$$

It is instructive to compare the results on the continuous-time Fourier series and its discrete-time counterpart and so we quickly review the CTFS result here.

Review of CTFS result If $x(t)$ is a periodic CT signal with fundamental time period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, then

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}. \tag{6.3}$$

From the signal space perspective, let $\phi_k(t) = e^{jk\omega_0 t}$ and consider the set $\Phi = \{\ldots, \phi_{-1}(t), \phi_0(t), \phi_1(t), \ldots\}$. $x(t)$ can be written as a linear combination of the signals in Φ and the Fourier series coefficients are the coefficients of the linear combination. $X[k]$ can be computed using

$$X[k] = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \mathrm{d}t, \quad k = \ldots, -1, 0, 1, \ldots. \tag{6.4}$$

Comparison between CTFS and DTFS

Comparing (6.3) with (6.1), we see several similarities but one major difference. In (6.3), the summation on the RHS has infinitely many terms whereas in (6.1), there are only N terms on the RHS. That is, while we may need to sum infinitely many complex exponentials (harmonics) to synthesize a CT signal, we need to sum only N complex exponentials to synthesize a DT signal with fundamental time period N.

This difference is the result of the fact that there are only N different discrete-time complex exponentials with frequencies that are integer multiples of Ω_0. We discuss this below

6.0.1 Limited Range of Frequencies of DT Complex Exponentials $e^{j\Omega_0 kn}$

Let $\phi_k[n] := e^{jk\Omega_0 n}$ where $\Omega_0 = 2\pi/N$ denote the DT complex exponential with frequency Ω_0.

$$\text{For any } k, \quad \phi_{k+N}[n] = \phi_k[n] \tag{6.5}$$

$$i.e., \; e^{j(k+N)\Omega_0 n} = e^{jk\Omega_0 n} \cdot \underbrace{e^{jN\Omega_0 n}}_{=1} = e^{jk\Omega_0 n} \tag{6.6}$$

To see a concrete example, consider $k = 0$. $\phi_0[n] = 1\,\forall n$ (Note that n takes only integer values). We can see that $\phi_N[n] = e^{jN\Omega_0 n} = e^{j2\pi n} = 1\forall n$. (6.5) implies that there are only N distinct frequencies of the form $\frac{2\pi}{N}k$ corresponding to $k = 0, \ldots, N-1$. Therefore, we need only N terms to represent $x[n]$ i.e.,

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}.$$

6.0.2 Linear Algebra Perspective

From a linear algebra perspective, we can write one period of $x[n]$ from $n = 0, \ldots, N-1$ as

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{j\Omega_0} & e^{j2\Omega_0} & \ldots & e^{j(N-1)\Omega_0} \\ 1 & e^{j2\Omega_0} & e^{j4\Omega_0} & \ldots & e^{j2(N-1)\Omega_0} \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & e^{j(N-1)\Omega_0} & e^{j2(N-1)\Omega_0} & \ldots & e^{j(N-1)^2\Omega_0} \end{bmatrix}}_{A} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

The matrix A is invertible and the columns of A form a basis for the space spanned by DT signals of length N. To get a matrix representation for the periodic signal $x[n]$ notice that $x[n]$ as well as each column of A is a periodic signal in n. Thus the columns repeat every N.

$$
\begin{bmatrix} \vdots \\ x[-N]=x[0] \\ \vdots \\ x[-2]=x[N-2] \\ x[-1]=x[N-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \\ x[N]=x[0] \\ x[N+1]=x[1] \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
\vdots & \vdots & \vdots & \dots & \vdots \\
1 & 1 & 1 & \dots & 1 \\
1 & e^{j\Omega_0} & e^{j2\Omega_0} & \dots & e^{j(N-1)\Omega_0} \\
1 & e^{j2\Omega_0} & e^{j4\Omega_0} & \dots & e^{j2(N-1)\Omega_0} \\
\vdots & \vdots & \vdots & \vdots & \\
1 & e^{j(N-1)\Omega_0} & e^{j2(N-1)\Omega_0} & \dots & e^{j(N-1)^2\Omega_0} \\
1 & 1 & 1 & \dots & 1 \\
1 & e^{j\Omega_0} & e^{j2\Omega_0} & \dots & e^{j(N-1)\Omega_0} \\
1 & e^{j2\Omega_0} & e^{j4\Omega_0} & \dots & e^{j2(N-1)\Omega_0} \\
\vdots & \vdots & \vdots & \vdots & \\
1 & e^{j(N-1)\Omega_0} & e^{j2(N-1)\Omega_0} & \dots & e^{j(N-1)^2\Omega_0} \\
1 & 1 & 1 & \dots & 1 \\
1 & e^{j\Omega_0} & e^{j2\Omega_0} & \dots & e^{j(N-1)\Omega_0}
\end{bmatrix}
\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}
=
\begin{bmatrix} \vdots \\ A \\ A \\ A \\ \vdots \end{bmatrix}
\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}
$$

From this, we can see that the dimension of $\{x[n]\}$ is only N and hence only N coefficients are required to represent $x[n]$ on the left.

6.0.3 Parseval's Relation

Similar to Parseval's relationship for CT signals, for DT signals, the power can be computed in the time domain or in the frequency domain according to

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |X[k]|^2. \tag{6.7}$$

6.0.4 Computing the DTFS coefficients

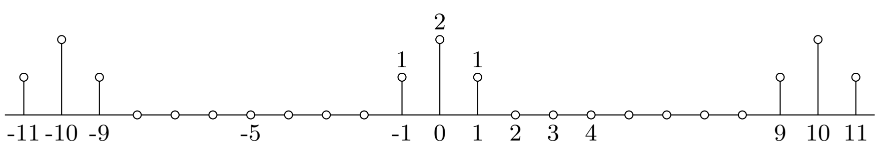
There are two ways to compute $X[k]$:

1. Direct method: evaluate $X[k]$ using $X[k] = \frac{1}{N} \sum_{N=\langle N \rangle} x[n] e^{-jk\Omega_0 n}$

2. Method of inspection: Use Euler's formula and read off the coefficients

Example Consider the periodic signal $x[n] = \sum_{l=-\infty}^{\infty} \delta[n+1-10l] + 2\delta[n-10l] + \delta[n-1-10l]$

$x[n]$ is sketched below

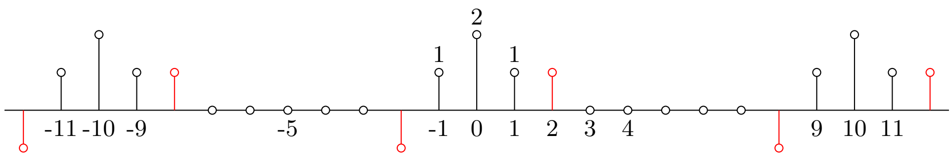


For this signal, $N = 10$, $\Omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$. $X[k]$ can be computed using the direct method as follows

$$X[k] = \frac{1}{10} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} = \frac{1}{10} \left(1 \cdot e^{+jk\Omega_0} + 2 + 1 \cdot e^{-jk\Omega_0} \right)$$

$$= X[k] = \frac{1}{10} \left(2 + 2\cos\Omega_0 k \right) = \frac{1}{10} \left(2 + 2\cos\frac{\pi}{5}k \right)$$

Example



$N = 10$

$\Omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$

$$X[k] = \frac{1}{10}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n} = \frac{1}{10}\left(1\cdot e^{+jk\Omega_0} + 2 + 1\cdot e^{-jk\Omega_0}\right)$$
$$= X[k] = \frac{1}{10}\left(2 + 2\cos\Omega_0 k\right) = \frac{1}{10}\left(2 + 2\cos\frac{\pi}{5}k\right)$$

When the two new values are added, the new F.S. coefficients are

$$X[k] = \frac{1}{10}\left(2 + 2\cos\frac{\pi}{5}k - 2j\sin\frac{2\pi}{5}k\right)$$

Example

$$x[n] = 1 + \sin\left(\frac{2\pi n}{N}\right) + 3\cos\left(\frac{4\pi n}{N} + \frac{\pi}{2}\right)$$

Find $X[k]$

$$\Omega_0 = \frac{2\pi}{N}$$

$$x[n] = 1 + \sin\left(\Omega_0 n\right) + 3\cos\left(2\Omega_0 n + \frac{\pi}{2}\right)$$
$$= X[k] = 1 + \frac{1}{2j}e^{j\Omega_0 n} - \frac{1}{2j}e^{-j\Omega_0 n} + \frac{3}{2}e^{j\left(2\Omega_0 n + \frac{\pi}{2}\right)} + \frac{3}{2}e^{-j\left(2\Omega_0 n + \frac{\pi}{2}\right)}$$
$$= \underbrace{\frac{3}{2}e^{-j\frac{\pi}{2}}}_{X[-2]} \cdot e^{-j2\Omega_0 n} - \underbrace{\frac{1}{2j}}_{X[-1]}e^{-j\Omega_0 n} + \underbrace{1}_{X[0]} + \underbrace{\frac{1}{2j}}_{X[1]}e^{j\Omega_0 n} + \underbrace{\frac{3}{2}e^{j\frac{\pi}{2}}}_{X[2]} \cdot e^{j2\Omega_0 n}$$
$$= \sum X[k]e^{jk\Omega_0 n}$$

Chapter 7

Continous-time Fourier Transform

In the previous chapter, we saw the Fourier series representation of a periodic signal $x\tilde{(t)}$ with time period T_f . In this chapter, we consider the case when the signal is not necessarily periodic and we ask if there is a way to express the signal as a linear combination of complex exponentials and if there is, what would be linear combination look like?

We begin by noting that the Fourier series representation of a CT periodic signal with fundamental time period T_f is given by:

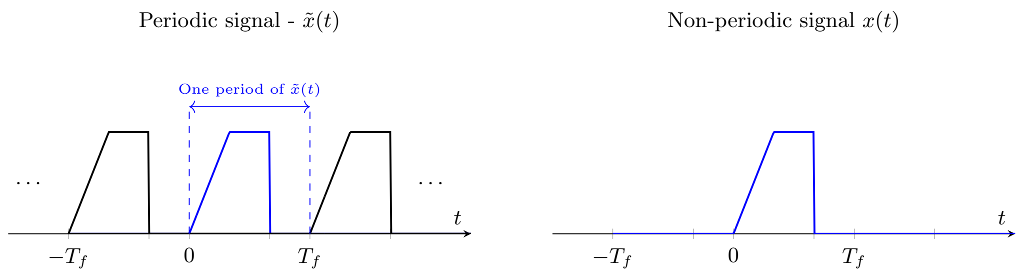


Figure 7.1: A non-periodic signal $x(t)$ can be viewed as the $\lim_{T_f\to\infty}$ of a periodic version of the signal $\tilde{x}(t)$.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \tilde{X}[k]e^{jk\omega_0 t} \tag{7.1}$$

$$\tilde{X}[k] = \frac{1}{T}\int_0^T \tilde{x}(t)e^{-jk\omega_0 t}dt \tag{7.2}$$

Suppose, $x(t)$ is not periodic. Is there a representation for $x(t)$ as a linear combination of complex exponentials? The answer is yes and it is called the continuous-time Fourier transform. The main idea in obtaining such a represetnation is to think of $x(t)$ as the limit of $\tilde{x}(t)$ when $T \to \infty$ i.e $x(t) = \lim_{T\to\infty} \tilde{x}(t)$ and see what happens to the Fourier series in the limit. To explain this further, consider the following example.

Example 7.0.1. *Consider the periodic signal $\tilde{x}(t)$ which is a rectangular signal repeated every T seconds and let $x(t)$ be one period of the signal. The Fourier series coefficients of $\tilde{x}(t)$ are given by $X[k] = \frac{2T_1}{T} sinc\left(\frac{2T_1 k}{T}\right)$. Fig. 7.2, we plot the Fourier series coefficients as*

$T \to \infty$. Instead of plotting $X[k]$ versus k, let us plot $X[k]$ as a function of $\omega = k\omega_f$. Since $\omega_f = \frac{2\pi}{T_f}$ as T_f increases ω_f decreases. Since the spacing between the harmonics is ω_f, when $T_f \to \infty$, $\omega_f \to 0$ and $X[k\omega_f]$ becomes a continuous function of ω. This is represented by $X(j\omega)$ and is called the continuous-time Fourier transform.

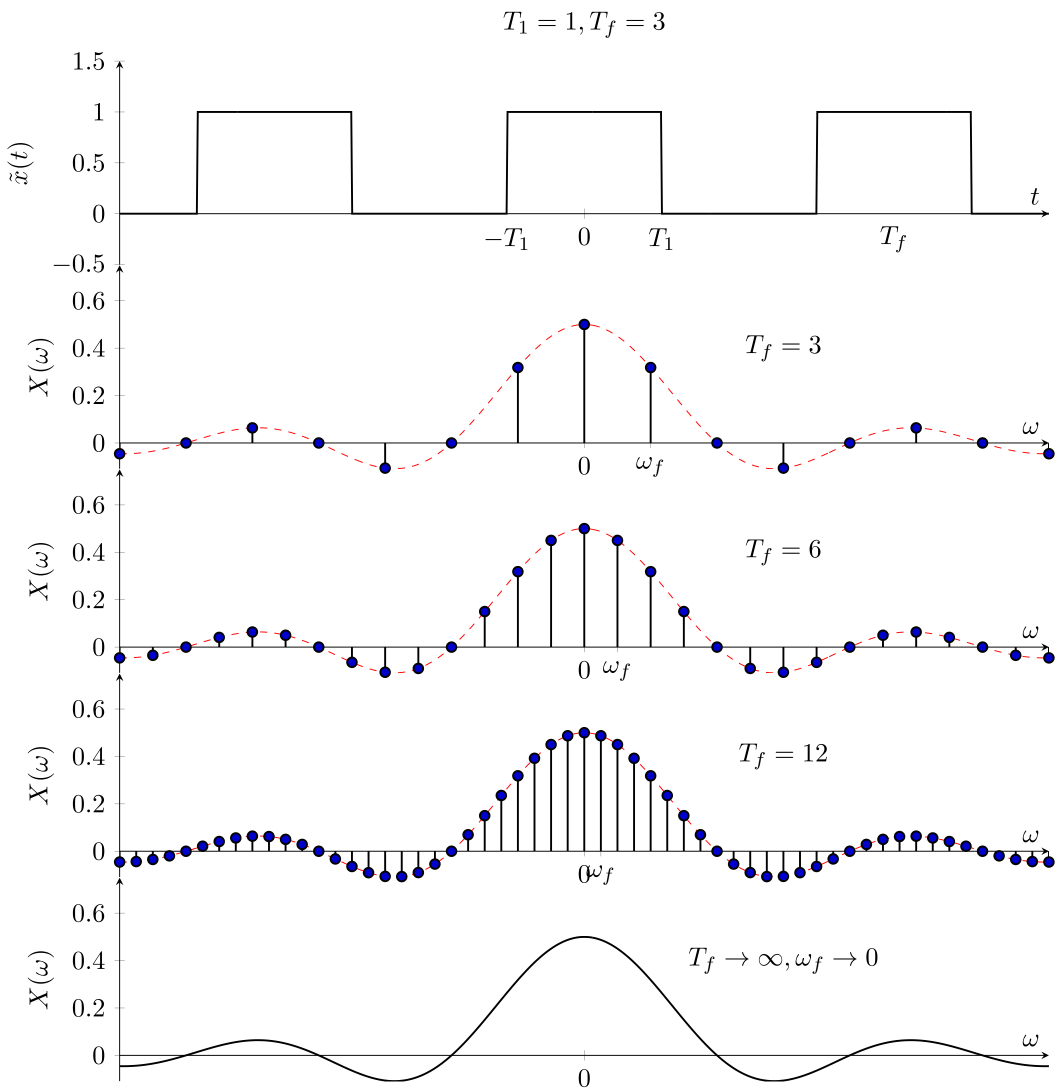


Figure 7.2: Figure shows how the discrete harmonics tend to a continuous time of ω

Thus,

- Fourier series representation applies to periodic signals - i.e a signal contains only frequencies which are integer multiples of a fundamental frequency.
- Fourier transform representation applies to Non-periodic (and periodic) signals and periodic signals i.e., the signal may contain a continuum of frequencies $X(j\omega)$ refers to as the Fourier transform,where ω is a continuously changing variable .

Analysis and synthesis equations for the FT

$$\begin{aligned} Analysis: X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt && (7.3) \\ Synthesis: x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega && (7.4) \end{aligned}$$

Magnitude and Phase spectra

The Fourier transform of a signal is, in general, a complex function $X(j\omega)$ of a real variable ω. Thus, in order to plot the FT, we have to separately plot the magnitude $|X(j\omega)|$ as a function of ω (called the magnitude spectrum) and the phase $\angle X(j\omega)$ as a function of ω (called the phase spectrum). In some cases, $X(j\omega)$ becomes a real function of ω or strictly imaginary function of ω and in these cases, we may just plot $X(j\omega)$ or $X(j\omega)/j$.

Relation to the Laplace transform

You may have seen the Laplace transform in earlier classes (circuits, differential equations, etc.) and you might recall that the Laplace transform of a signal $x(t)$ is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

Comparing (7.3) and (7), we can see that the Fourier transform is a special case of the Laplace transform where $s = j\omega$.

Another form of the FT

In some books, we choose to measure frequency in Hertz instead of rad/s and the FT may be expressed in terms of f. This can be obtained by replacing ω in (7.3) and (7.4) by $\omega = 2\pi f$ and noting that $d\omega = 2\pi df$. Then, (7.3) and (7.4) become

$$\begin{aligned} Analysis: X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt && (7.5) \\ Synthesis: x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df. && (7.6) \end{aligned}$$

7.1 Parseval's theorem

Similar to Parseval's theorem for periodic signals, Parseval's theorem for a periodic signals tells us that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega. \tag{7.7}$$

Interpreting $|X(j\omega)|^2$ as the energy density

This essentially tells us that the total energy in the signal $x(t)$ is distributed among different frequency components and that $|X(j\omega)|^2$ should be interpreted as the energy density of $x(t)$, i.e., the amount of energy that is contained in the frequency interval $[\omega_0, \omega_0 + d\omega)$ is given by $\frac{1}{2\pi}|X(j\omega_0)|^2 d\omega$.

7.2 Computing the Fourier Transform of some basic signals

Let us now compute the Fourier transform of some basic signals using the analysis and synthesis equations.

Example 7.2.1. *One-sided exponential: Compute the Fourier transform of* $x(t) = e^{-at}u(t), a > 0$

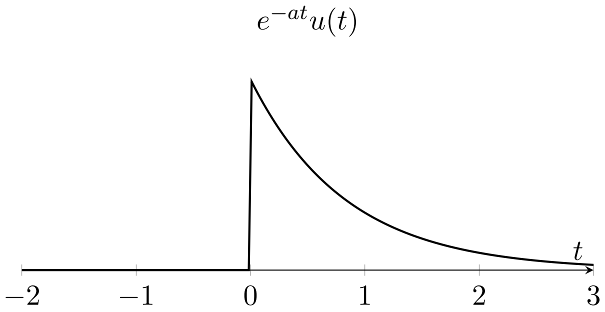


Figure 7.3: Plot of $x(t) = e^{-at}u(t)$ for $a > 0$.

$$\begin{aligned} X(jw) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt \\ &= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{0-1}{-(a+j\omega)} \\ &= \frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2} \\ |X(jw)| &= \left|\frac{1}{a+j\omega}\right| = \frac{1}{|a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}} \end{aligned}$$

$$\angle X(jw) = 0 - tan^{-1}(\frac{w}{a})$$

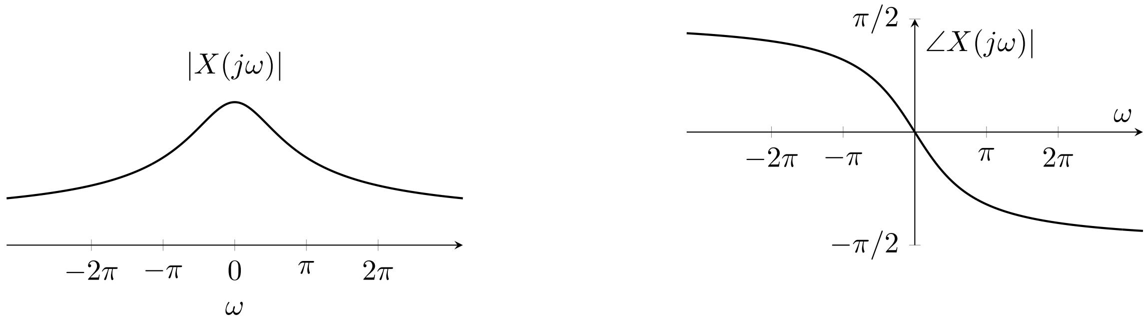


Figure 7.4: Magnitude spectrum and Phase spectrum of $X(j\omega)$ corresponding to $x(t) = e^{-at}u(t)$.

Example 7.2.2. *Two-sided exponential: Compute the Fourier transform of*

$$x(t) = e^{-a|t|},\ a > 0$$

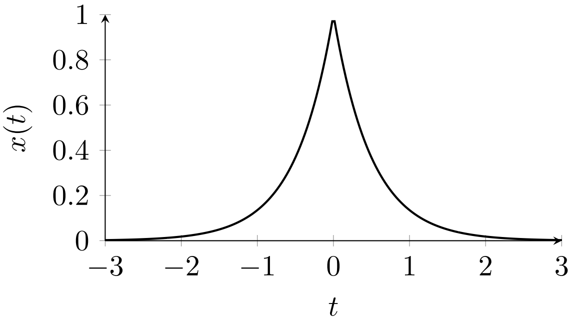


Figure 7.5: Two sided exponential signal

$$x(t) = \begin{cases} e^{-at} & \textit{if } t \geq 0 \\ e^{at} & \textit{if } t < 0 \end{cases}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega t)}dt \\ &= \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty} \\ &= \frac{1-0}{a-j\omega} + \frac{0-1}{-(a+j\omega)} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

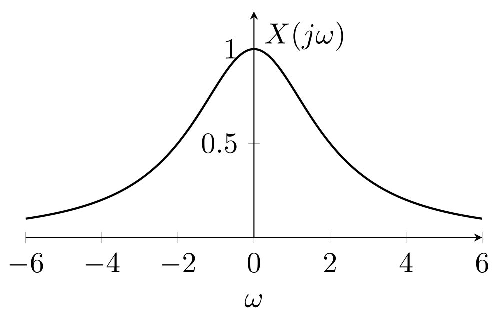


Figure 7.6: CTFT of the two-sided exponential signal

Example 7.2.3. *Compute the Fourier transform of $x(t) = \delta(t)$*

$$X(jw) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega 0} = 1$$

Think of $e^{-j\omega t}$ as some signal $g(t)$ and recall, $\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$.
Therefore,

$$X(j\omega) = 1 \tag{7.8}$$

Therefore,

$$x(t) = \delta(t) \longleftrightarrow X(j\omega) = 1$$

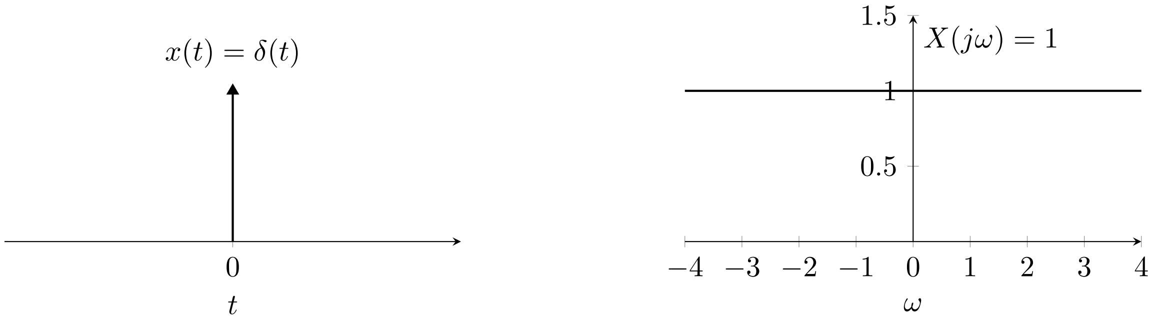


Figure 7.7: CTFT of a continuous-time delta signal.

Example 7.2.4. *Compute the inverse Fourier transform of $X(j\omega) = 2\pi\delta(w)$*

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} \, d\omega = 1$$

Thus,

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

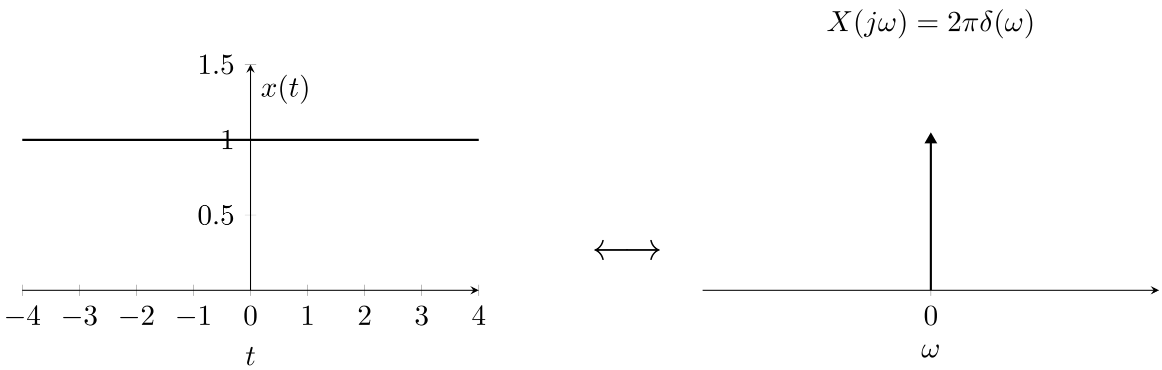


Figure 7.8: CTFT of $x(t) = 1$.

Example 7.2.5. *Compute the inverse Fourier transform of* $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

Example 7.2.6. *Consider the rectangular signal*

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

Then

$$X(j\omega) = 2\frac{\sin \omega T_1}{\omega} = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

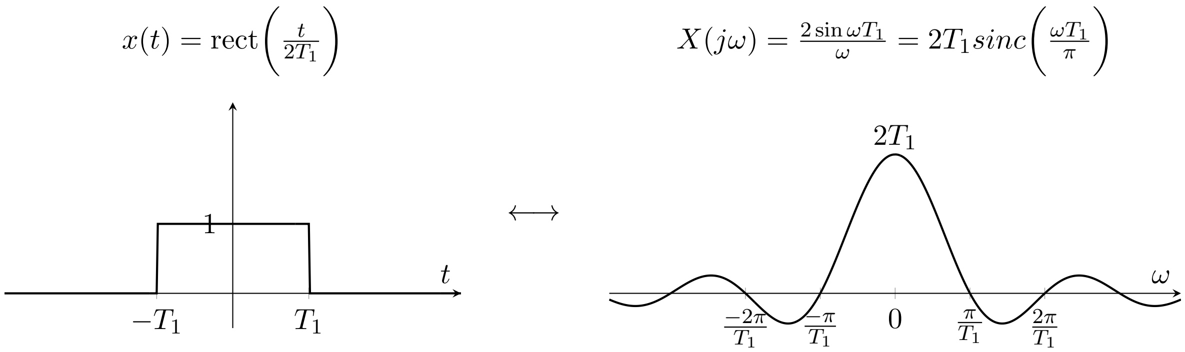


Figure 7.9: CTFT of a rectangular signal.

The CTFT of the rectangular signal can be derived starting from (7.3) as follows

$$\begin{aligned} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt &= \int_{-T_1}^{T_1} 1.e^{-j\omega t}dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1} \\ &= \frac{e^{-j\omega T_1} - e^{j\omega T_1}}{-j\omega} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} \\ &= \frac{2\sin(\omega T_1)}{\omega} \\ &= 2T_1 sinc\left(\frac{\omega T_1}{\pi}\right). \end{aligned}$$

The last step follows from our earlier discussion on the sinc signal (2.9.11).

Example 7.2.7. *Consider a scaled version of the rectangular signal in the previous example*

$$x(t) = \begin{cases} \frac{1}{\sqrt{2T_1}}, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

Notice that $x(t)$ has unit energy for any chosen value of T_1. Therefore, this scaled version lets us understand the effect of changing T_1 while maintaining the same energy. Then

$$X(j\omega) = \sqrt{2T_1}\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$x(t) = \frac{1}{\sqrt{2T_1}}\text{rect}\left(\frac{t}{2T_1}\right) \qquad X(j\omega) = \sqrt{2T_1}\frac{\sin\omega T_1}{\omega} = \sqrt{2T_1}sinc\left(\frac{\omega T_1}{\pi}\right)$$

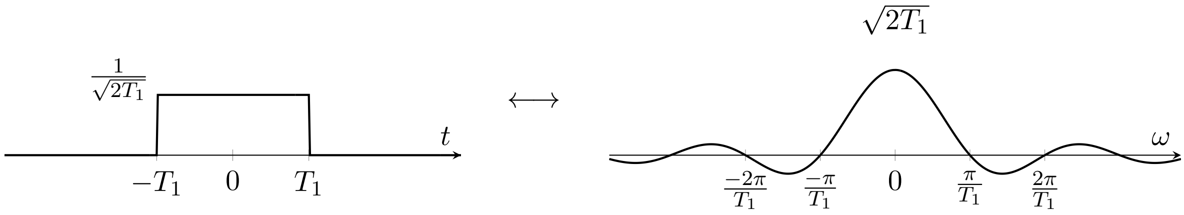


Figure 7.10: CTFT of a rectangular signal with unit energy.

As can be seen from the figure, increasing T_1 makes the time domain signal wider whereas the Fourier transform or frequency domain signal becomes narrower and taller.

Example 7.2.8. *Find the signal $x(t)$ whose Fourier transform $X(j\omega)$ is the rectangular signal given below*

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

The ICTFT of the rectangular signal can be derived starting from (7.4) *as follows*

$$\begin{aligned} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega &= \frac{1}{2\pi}\int_{-W}^{W} 1.e^{j\omega t}d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W} \\ &= \frac{1}{2\pi}\frac{e^{jWt} - e^{-jWt}}{jt} \\ &= \frac{1}{2\pi}\frac{2j\sin(Wt)}{jt} \\ &= \frac{\sin Wt}{\pi t} = \frac{W}{\pi}\frac{\sin(Wt)}{Wt} = \frac{W}{\pi}sinc\left(\frac{Wt}{\pi}\right). \end{aligned}$$

The last step follows from our earlier discussion on the sinc signal (2.9.11).

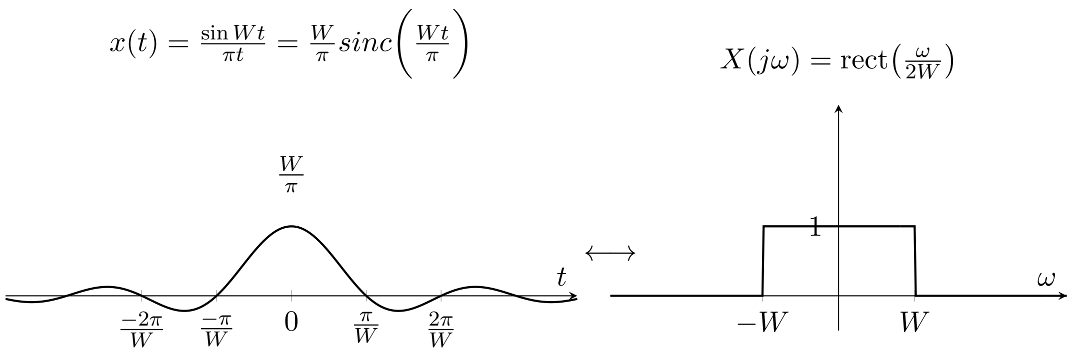


Figure 7.11: Inverse CTFT of a rectangular signal.

When can we evaluate the Fourier transform?

We have defined the Fourier transform of $x(t)$ as $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. For what $x(t)$ can we actually evaluate the integral (without the integral diverging)? If we computed $X(j\omega)$ and $\hat{x}(t)$ as per the equations below, in what sense would $\hat{x}(t)$ be equal to $x(t)$?

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{7.9}$$

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{7.10}$$

The answer to these questions is similar to what happened with the Fourier series. If $x(t)$ has finite energy, then $\int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt$ will be zero. If $x(t)$ satisfies Dirichlet conditions, then $\hat{x}(t)$ will be equal to $x(t)$ pointwise except at the discontinuities.

Sometimes, we are interested in computing the Fourier transform of signals that do not have finite energy. For example, $x(t) = \delta(t)$ does not have finite energy. Periodic signals such as $\cos(t)$ or $\sin(t)$ do not have finite energy either. However, we can compute the Fourier transform of some of these signals if we allow for generalized functions like the δ function in the Fourier transforms. This is covered in the next section.

However, there are signals for which we cannot compute the Fourier transform since the integral will diverge. An example of such a signal is $x(t) = e^t u(t)$.

7.2.1 Fourier transform of periodic signals

We can compute the Fourier transform of periodic signals even though they do not have finite energy. In this case, the Fourier transform has impulses and hence, it is a generalized function. Recall that a generalized function is a function that is undefined for some values of t. The Fourier transform in this case is not defined at the points where the delta functions occur, but we can understand that there is non-zero energy in those frequencies.

Let $x(t)$ be a periodic signal with fundamental time period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Then $x(t)$ has a Fourier series representation. We can take the Fourier transform of both sides to get $X(j\omega)$, i.e.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \tag{7.11}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} X[k]2\pi\delta(\omega - k\omega_0) \tag{7.12}$$

$$= \sum_{k=-\infty}^{\infty} (2\pi X[k])\delta(\omega - k\omega_0) \tag{7.13}$$

To compute the Fourier transform a periodic signal, first compute the fundamental frequency ω_0 and the Fourier series coefficients, $X[k]$. Then, the Fourier transform is given by

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi X[k]\delta(\omega - k\omega_0)$$

$X(j\omega)$ being a sum of delta functions means that the signal contains only discrete frequency components

Example 7.2.9. *Compute the Fourier transform of* $x(t) = \cos\omega_0 t$ *The fundamental frequency of* $x(t) = \cos\omega_0 t$ *is* ω_0 *and the Fourier series coefficients are* $X[k] = \frac{1}{2}\delta[k+1] + \frac{1}{2}\delta[k-1]$. *Hence,*

$$X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

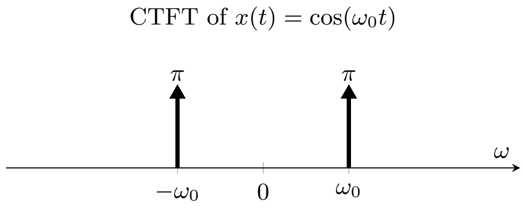


Figure 7.12: Fourier transform of $x(t) = \cos\omega_0 t$

Example 7.2.10. *Compute the Fourier transform of* $x(t) = \sin\omega_0 t$ *The fundamental frequency of* $x(t) = \cos\omega_0 t$ *is* ω_0 *and the Fourier series coefficients are* $X[k] = \frac{1}{2j}\delta[k+1] - \frac{1}{2j}\delta[k-1]$. *Hence,*

$$X(j\omega) = \frac{\pi}{j}\delta(\omega + \omega_0) - \frac{\pi}{j}\delta(\omega - \omega_0)$$

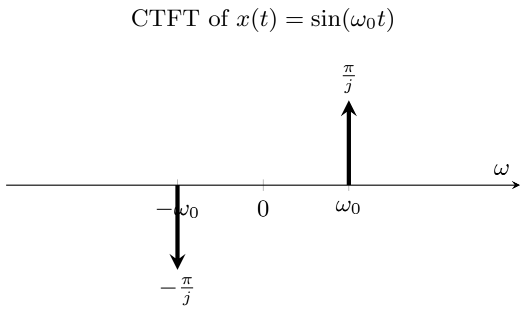


Figure 7.13: Fourier transform of $x(t) = \sin\omega_0 t$

Example 7.2.11. *Compute the Fourier transform of* $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

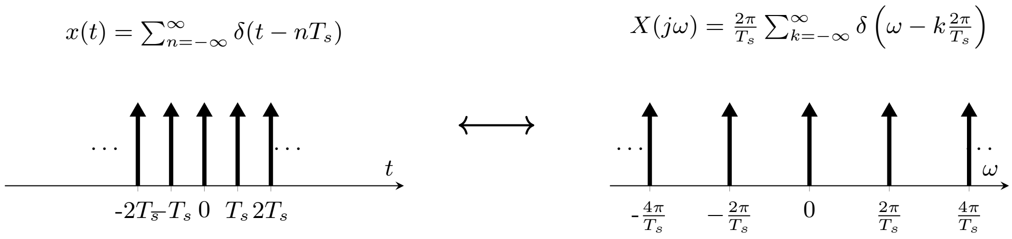


Figure 7.14:

Since the fundamental time period is T_s*, the fundamental frequency is* $\omega_0 = \frac{2\pi}{T_s}$*. Now, we can compute* $X[k]$ *according to*

$$X[k] = \frac{1}{T_s} \int_{\frac{-T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-j\omega t}\, dt = \frac{1}{T_s}$$

Therefore,

$$X(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

Fourier transform tables and properties

So far, we have computed several basic Fourier transform pairs. In the following section, we discuss several important properties of the continuous-time Fourier transform. Our main approach to computing the CTFT of more complicated signals will be to use the Fourier transform of some basic signals in conjunction with these properties. On the exam, you will be allowed to use these tables. Therefore, it is very important that you look at the tables of Fourier transform pairs and all the properties listed in the attached tables and be familiar with the entries in the table.

PROPERTIES OF THE CONTINUOUS TIME FOURIER TRANSFORM

Section	**Property**	**Aperiodic Signal**	**Fourier Transform**
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.4	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.6	Duality	$X(jt)$	$2\pi x(-\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \Re\{X(j\omega)\}=\Re\{X(-j\omega)\} \\ \Im\{X(j\omega)\}=-\Im\{X(-j\omega)\} \\ \|X(j\omega)\|=\|X(-j\omega)\| \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re\{X(j\omega)\}$ $j\Im\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

BASIC FOURIER TRANSFORM PAIRS

Signal	**Fourier Transform**	**Fourier Series Coefficients (if periodic)**
$\sum_{k=-\infty}^{+\infty} a_k e^{j\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{jw_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0, \quad$ otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0, \quad$ otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0, \quad$ otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, a_k = 0, k \neq 0$ (this is the Fourier series representation for any choice $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$	——
$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right)$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	——
$\delta(t)$	1	——
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	——
$\delta(t - t_0)$	$e^{-j\omega t_0}$	——
$e^{-at}u(t), \Re\{a\} > 0$	$\frac{1}{a + j\omega}$	——
$te^{-at}u(t), \Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	——
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	——
$e^{-a\lvert t\rvert}, a > 0$	$\frac{2a}{(a^2 + \omega^2)}$	——

7.3 Properties of the Continuous-Time Fourier Transform

7.3.1 Linearity

Let $x(t) \longleftrightarrow X(j\omega)$ and $y(t) \longleftrightarrow Y(j\omega)$ then,

$$z(t) = ax(t) + by(t) \longleftrightarrow aX(j\omega) + bY(j\omega) \tag{7.14}$$

Proof:

$$\begin{aligned} Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} (ax(t) + by(t))e^{-j\omega t} &= a\int_{-\infty}^{\infty} x(t)e^{-j\omega t} + b\int_{-\infty}^{\infty} y(t)e^{-j\omega t} \\ &= aX(j\omega) + bY(j\omega) \end{aligned}$$

Example 7.3.1. *Let* $x(t) = \mathrm{rect}(t) + \mathrm{rect}\left(\frac{t}{3}\right)$. *The CTFT of* $x(t)$ *is given by*

$$x(t) \longleftrightarrow X(j\omega) = \frac{2\sin\omega/2}{\omega} + \frac{2\sin 3\omega/2}{\omega}$$

Example 7.3.2. *Compute the Fourier transform of $x(t) = \cos(\omega_0 t)$.*

From Example 7.2.5, we see that the Fourier transform of $x(t) = e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$. Since

$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{-j\omega_0 t} + e^{j\omega_0 t}\right] \leftrightarrow \frac{1}{2}\left[2\pi\delta(\omega+\omega_0) + 2\pi\delta(\omega-\omega_0)\right] = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

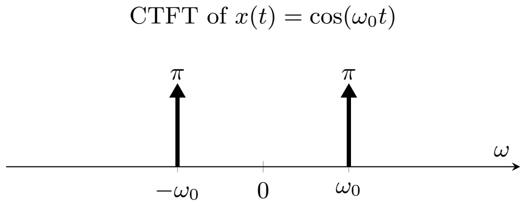


Figure 7.15: CTFT of a cosine signal $x(t) = \cos(\omega_0 t)$

Example 7.3.3. *Compute the Fourier transform of $x(t) = \sin(\omega_0 t)$.*

From Example 7.2.5, we see that the Fourier transform of $x(t) = e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$. Since

$$\sin(\omega_0 t) = \frac{1}{2j}\left[-e^{-j\omega_0 t} + e^{j\omega_0 t}\right] \leftrightarrow \frac{1}{2j}\left[-2\pi\delta(\omega+\omega_0) + 2\pi\delta(\omega-\omega_0)\right] = \frac{\pi}{j}\left[-\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

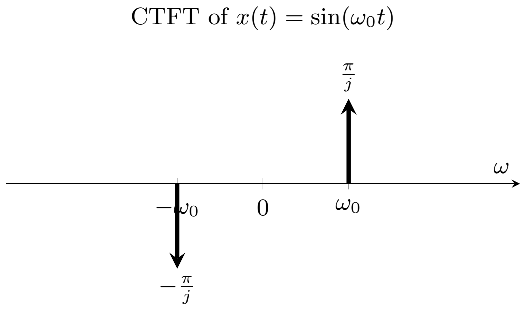


Figure 7.16: CTFT of a sine signal $x(t) = \sin(\omega_0 t)$

7.3.2 Time shifting/Shift in time

Let $x(t) \longleftrightarrow X(j\omega)$ then,

$$z(t) = x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega) \tag{7.15}$$

Proof:

$$Z(j\omega) = \int x(t - t_0) e^{-j\omega t_0} dt$$

Substituting, $t' = t - t_0$,we get

$$Z(j\omega) = \int x(t') e^{-j\omega(t' + t_0)} dt = e^{-j\omega t_0} X(j\omega).$$

Notice that

$$\begin{aligned} |Z(j\omega)| &= |X(j\omega)| \\ \angle Z(j\omega) &= \angle X(j\omega) - \omega t_0 \end{aligned}$$

This implies that shifting $x(t)$ in time does not change the magnitude of the Fourier transform, but subtracts/adds a phase that changes linearly with frequency.

Example 7.3.4. *Compute the Fourier transform of the signal $x(t)$ shown in Figure 7.17.*

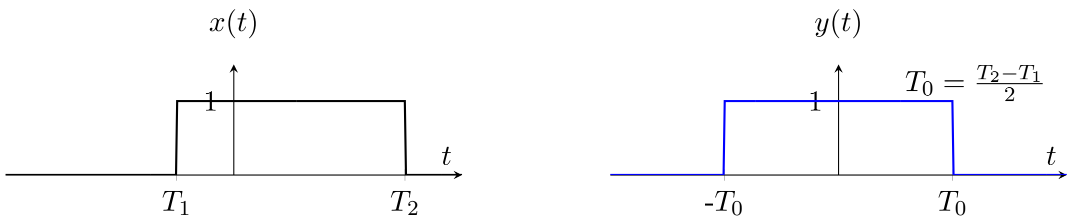


Figure 7.17: $x(t)$ is a shifted rectangular signal.

Start with the signal $y(t)$ show in the figure above and notice that $x(t) = y(t - \tau)$ where $\tau = \frac{T_1 + T_2}{2}$. Now, $y(t)$ is a signal whose Fourier transform is easy to compute (we have computed this already). This is also available in the table of Fourier transform pairs.

$$y(t) \longleftrightarrow 2T_0 sinc\left(\frac{\omega T_0}{\pi}\right) = \frac{2}{\omega} \sin\left(\omega T_0\right)$$

So,

$$x(j\omega) = e^{-j\omega\tau} 2T_0 sinc\left(\frac{\omega T_0}{\pi}\right) \tag{7.16}$$

We can substitute $T_0 = \frac{T_2 - T_1}{2}$ and $\tau = \frac{T_1 + T_2}{2}$ to get the final result.

Example 7.3.5. *Let* $x(t) = \sum_{n=0}^{5} sinc\left(\frac{W(t-nT_s)}{\pi}\right)$ *where* T_s *is any real number. Is it true that* $X(j\omega) = 0$ *for* $|\omega| > W$ *?*

The answer is yes. The Fourier transform of $sinc\left(\frac{Wt}{\pi}\right)$ *is 0 for* $|\omega| > W$*. By the time shift property, the Fourier transform of* $sinc\left(\frac{W(t-nT_s)}{\pi}\right)$ *is also 0 for* $|\omega| > W$*. By linearity, the Fourier transform of* $x(t)$ *is the sum of the Fourier transforms of the* $sinc\left(\frac{W(t-nT_s)}{\pi}\right)$ *for each* n *and hence,* $X(j\omega) = 0$ *for* $|\omega| > W$*.*

7.3.3 Frequency Shifting (Modulation Property)

Let $x(t) \longleftrightarrow X(j\omega)$ then,

$$e^{j\gamma t}x(t) \longleftrightarrow X(j(\omega-\gamma)) \tag{7.17}$$

Proof: The inverse Fourier transform of $Z(j\omega) = X(j(\omega-\gamma))$ is given by

$$\begin{aligned} z(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega-\gamma))e^{j\omega t}\,d\omega && (7.18)\\ &\overset{(a)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\nu)e^{j(\gamma+\nu)t}\,d\nu && (7.19)\\ &= e^{j\gamma t}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\nu)e^{j\nu t}\,d\nu}_{x(t)} = e^{j\gamma t}x(t) && (7.20)\end{aligned}$$

where we substituted $\nu = \omega - \gamma$ and hence, $\omega = \nu + \gamma$ in (a).

Example 7.3.6. *Let $x(t) = \frac{\sin Wt}{\pi t}$ and the corresponding $X(j\omega) = rect\Big(\frac{\omega}{2W}\Big)$. Using the shift in frequency (or, modulation property) and linearity, we can see that*

$$\begin{aligned} x(t) = \frac{\sin Wt}{\pi t} &\longleftrightarrow X(j\omega) = rect\Big(\frac{\omega}{2W}\Big)\\ y(t) = e^{j\gamma t}\frac{\sin Wt}{\pi t} &\longleftrightarrow Y(j\omega) = X(j(\omega-\gamma))\\ z(t) = \cos(\gamma t)\frac{\sin Wt}{\pi t} &\longleftrightarrow Z(j\omega) = \frac{1}{2}X(j(\omega+\gamma)) + \frac{1}{2}X(j(\omega-\gamma))\end{aligned}$$

This is shown in Fig. 7.18.

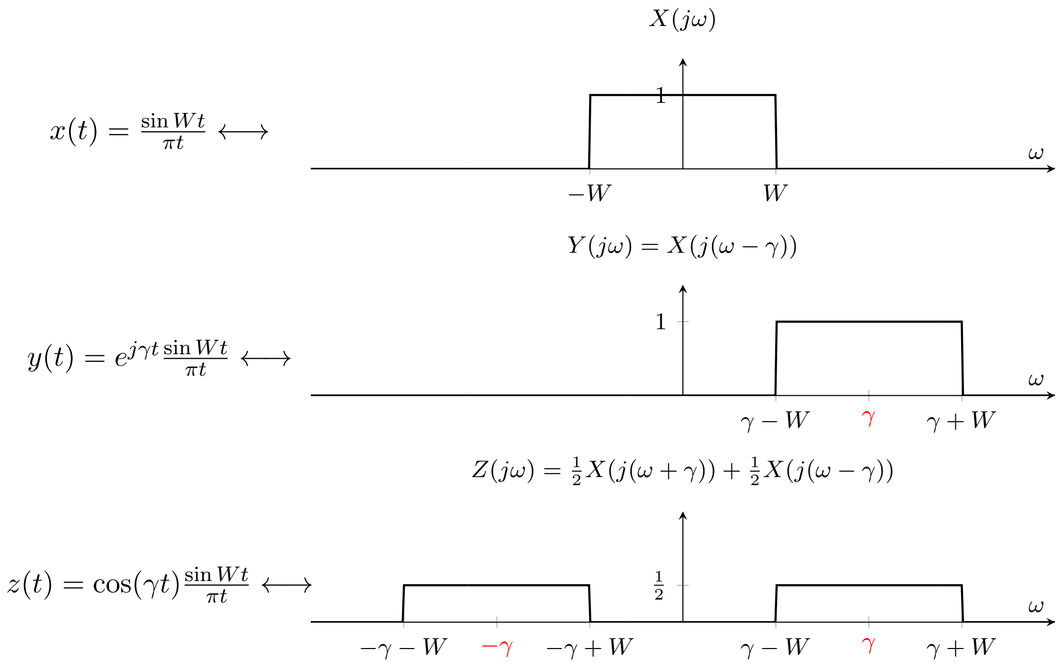


Figure 7.18: Figure showing shifting in frequency or the modulation property.

Example 7.3.7. *Describe how upconversion works in a communication system. Use Wi-Fi or 5G as an example to get exact carrier frequencies and bandwidths.*

7.3.4 Time and Frequency Scaling:

Let $x(t) \longleftrightarrow X(j\omega)$ then,

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) \quad a \neq 0 \text{ is a real constant} \tag{7.21}$$

A special case of the above result occurs when $a = -1$. In this case,

$$x(-t) \longleftrightarrow X(-j\omega)$$

Proof: Let $z(t) = x(at)$. Then, the CTFT of $z(t)$ is given by

$$Z(j\omega) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

Let us substitute $t' = at$ and hence, $t = t'/a$ and $dt = dt'/a$. Thus,

$$Z(j\omega) = \begin{cases} \int_{-\infty}^{\infty} \frac{1}{a} x(t')e^{-j(\omega/a)t'}dt', & a > 0; \\ \int_{\infty}^{-\infty} \frac{1}{a} x(t')e^{-j(\omega/a)t'}dt', & a < 0. \end{cases}$$

The second expression can also be written as $\int_{-\infty}^{\infty} \frac{1}{-a} x(t')e^{-j(\omega/a)t'}dt'$. Both these can be seen to simplified into one expression that is valid for both $a > 0$ and $a < 0$ given by

$$Z(j\omega) = \int_{-\infty}^{\infty} \frac{1}{|a|} x(t')e^{-j(\omega/a)t'}dt = \frac{1}{|a|} X\big(j\frac{\omega}{a}\big).$$

Example 7.3.8. *Consider the signal $x(at)$ where $x(t) = \frac{\sin Wt}{\pi t}$. The time domain and frequency domain signals are plotted for $a = 1, 2, 3$ in Fig. 7.19.*

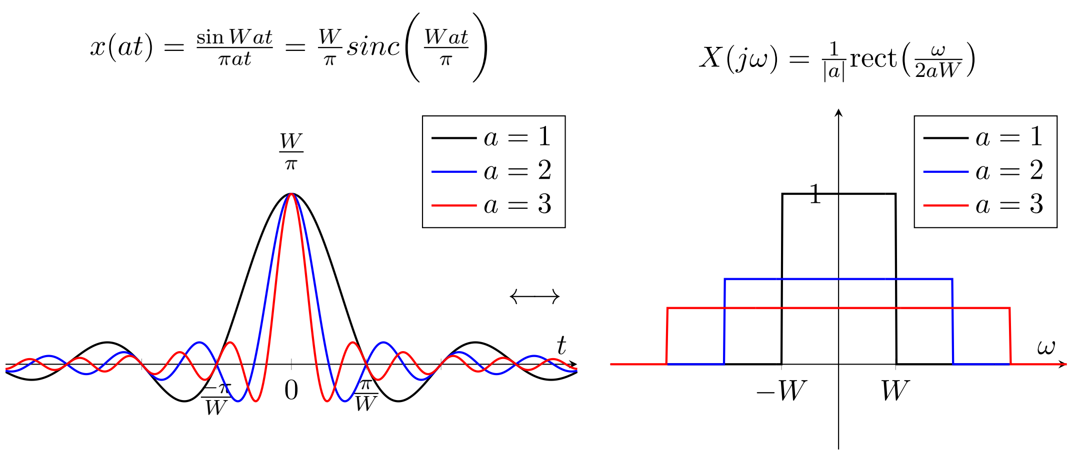


Figure 7.19: Figure showing the CTFT of time scaled sinc functions

Why does YouTube video played at 2x speed sound shriller?

You can observe that when $a = 2$, the time domain signal becomes narrower, the frequency axis is expanded and the Fourier transform is wider. This means that the signal $x(2t)$ has more energy in the high frequency components. High frequency components corresponds to shriller sounds and this is why YouTube videos played at 2x sounds shriller. This is easy to see with $x(t) = \cos(a\omega_0 t)$. As a increases, the frequency of the cosinusoid increases.

7.3.5 Conjugation and Symmetry:

Let $x(t) \longleftrightarrow X(j\omega)$ then,

$$x^*(t) \longleftrightarrow X^*(-j\omega) \tag{7.22}$$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$
$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

Substitute, $\gamma = -\omega$ and $d\gamma = -d\omega$

$$\begin{aligned} x^*(t) &= \frac{-1}{2\pi}\int_{\infty}^{-\infty} X^*(-j\gamma)e^{j\gamma t}d\gamma \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-j\gamma)e^{j\gamma t}d\gamma \end{aligned}$$

If $x(t)$ is real, i.e., $x(t) = x^*(t)$, then we can observe the following symmetry in $X(j\omega)$

$$X(j\omega) = X^*(-j\omega) \tag{7.23}$$
$$X^*(j\omega) = X(-j\omega) \tag{7.24}$$
$$|X(j\omega)| = |X(-j\omega)| \tag{7.25}$$
$$\angle X(j\omega) = -\angle X(-j\omega) \tag{7.26}$$

This means that if $x(t)$ is strictly real, $|X(j\omega)|$ is an even function and $\angle X(j\omega)$ is an odd function.

Example 7.3.9. $x(t) = e^{-at}u(t) \longleftrightarrow X(j\omega) = \frac{1}{a+j\omega}$. *Then* $X^*(j\omega) = X(-j\omega) = \frac{1}{a-j\omega}$. *The magnitude* $|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$ *is indeed an even function and the phase* $\angle X(j\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$ *is indeed an odd function.*

If $x(t)$ is strictly imaginary, i.e $x(t) = -x^*(t)$, then we can observe the following symmetry

$$X(j\omega) = -X^*(-j\omega) \tag{7.27}$$
$$-X^*(j\omega) = X(-j\omega) \tag{7.28}$$
$$|X(j\omega)| = |X(-j\omega)| \tag{7.29}$$
$$\angle X(j\omega) = \pi - \angle X(-j\omega) \tag{7.30}$$

A more general result is true. Recall that any signal $x(t)$ or $X(j\omega)$ can be written either a sum of a real part and j times the imaginary part and also as the sum of a conjugate symmetric part and anti-symmetric part as shown below.

$$\begin{array}{cc} x(t) = x_R(t) + jx_I(t) & x(t) = x_{sym}(t) + x_{anti-sym}(t) \\ X(j\omega) = X_R(j\omega) + jX_I(j\omega) & X(j\omega) = X_{sym}(j\omega) + X_{anti-sym}(j\omega) \end{array}$$

It can be seen that

$$x_R(t) = \frac{1}{2}[x(t) + x^*(t)] \leftrightarrow \frac{1}{2}[X(j\omega) + X^*(-j\omega)] = X_{sym}(j\omega) \quad (7.31)$$

$$x_I(t) = \frac{1}{2j}[x(t) - x^*(t)] \leftrightarrow \frac{1}{2j}[X(j\omega) - X^*(-j\omega)] = -jX_{anti-sym}(j\omega) \quad (7.32)$$

$$jx_I(t) = \frac{1}{2}[x(t) - x^*(t)] \leftrightarrow \frac{1}{2}[X(j\omega) - X^*(-j\omega)] = X_{anti-sym}(j\omega) \quad (7.33)$$

Thus the Fourier transform of the real part of $x(t)$ is the conjugate-symmetric part of $X(j\omega)$ and the Fourier transform of the imaginary part of $x(t)$ is $-j$ times the conjugate anti-symmetric part of $X(j\omega)$. Equivalently, the Fourier transform of j times the imaginary part is the conjugate anti-symmetric part of $X(j\omega)$.

Similarly, the conjugate symmetric part of $x(t)$ and the conjugate anti-symmetric part of $x(t)$ have Fourier transforms that corresponds to the real part and j times the imaginary part of $X(j\omega)$, i.e.,

$$\frac{1}{2}[x(t) + x^*(-t)] \leftrightarrow X_R(j\omega) \quad (7.34)$$

$$\frac{1}{2}[x(t) - x^*(-t)] \leftrightarrow jX_I(j\omega) \quad (7.35)$$

It can be seen from these properties that

$x(t)$ is strictly real	$X(j\omega)$ is conjugate symmetric
$x(t)$ is strictly imaginary	$X(j\omega)$ is conjugate anti-symmetric
$x(t)$ is conjugate symmetric	$X(j\omega)$ is strictly real
$x(t)$ is conjugate anti-symmetric	$X(j\omega)$ is strictly imaginary
$x(t)$ is real and even	$X(j\omega)$ is even and strictly real
$x(t)$ is real and odd	$X(j\omega)$ is even and strictly imaginary

Example 7.3.10. *Let* $x(t) = e^{-2t}u(t)$*.* $X(j\omega) = \frac{1}{2+j\omega}$*. The even part and odd part of* $x(t)$ *and their F.T.s are given by*

$$\begin{aligned} x_e(t) = \frac{1}{2}e^{-2|t|} &\quad\longleftrightarrow\quad \frac{2}{4+\omega^2} \\ x_o(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{t}u(-t) &\quad\longleftrightarrow\quad j\frac{-2\omega}{4+\omega^2} \end{aligned}$$

7.3.6 Convolution-Multiplication Property

Let $x(t) \longleftrightarrow X(j\omega)$ and $h(t) \longleftrightarrow H(j\omega)$. The convolution-multiplication property tells that

$$y(t) = x(t) \star h(t) \quad = \quad Y(j\omega) = X(j\omega)H(j\omega) \tag{7.36}$$

$$x(t)h(t) \quad = \quad \frac{1}{2\pi} X(j\omega) \star H(j\omega) \tag{7.37}$$

The convolution-multiplication property also provides insight into how LTI systems operate on signals. It tells us that when $x(t)$ is input to an LTI system with impulse response $h(t)$, we can equivalently think of *multiplying* $X(j\omega)$ by $H(j\omega)$. The function $H(j\omega)$ can be interpreted as the gain provided by the system at a frequency of ω. Thus, every LTI system can be thought of as something that provides different gains at different frequencies and this frequency dependent gain function is what differentiates one LTI system from another. The black and blue paths in Fig. 7.20 represent equivalent operations of the LTI system on the input signal $x(t)$ in the time domain and frequency domain, respectively.

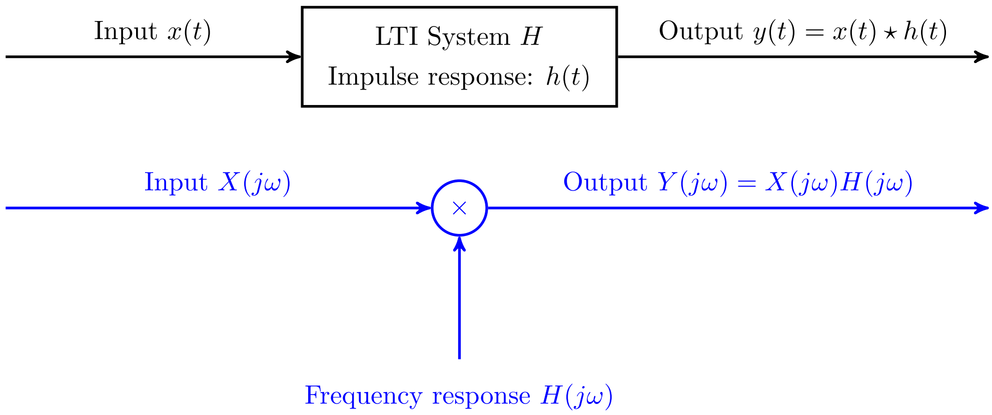


Figure 7.20: Time and frequency domain representations of an LTI system

Frequency response or transfer function of an LTI system

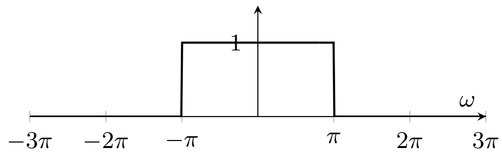
$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$ is called the frequency response or transfer function of an LTI system with impulse response $h(t)$.

When asked to compute $y(t)$ for a given $x(t)$ and $h(t)$, quite often, it will be easier to compute $X(j\omega)$ and $H(j\omega)$ and then compute $Y(j\omega) = X(j\omega)H(j\omega)$ and then take the inverse Fourier transform of $Y(j\omega)$ to compute $y(t)$. The following example will illustrate this.

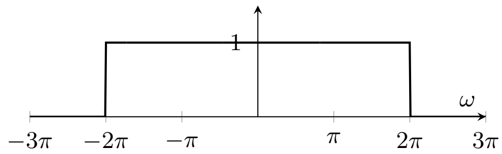
Example 7.3.11. *Given* $x(t) = \frac{1}{\pi t}\sin \pi t$ *and* $h(t) = \frac{1}{\pi t}\sin 2\pi t$. *Find* $y(t) = x(t) \star h(t)$.

We will find $Y(j\omega) = X(j\omega)H(j\omega)$ *and then find* $y(t)$.

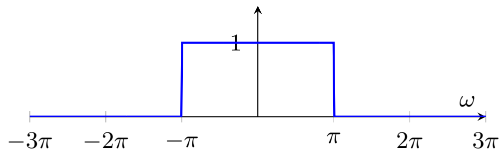
$X(j\omega)$



$H(j\omega)$



$Y(j\omega) = X(j\omega)H(j\omega)$

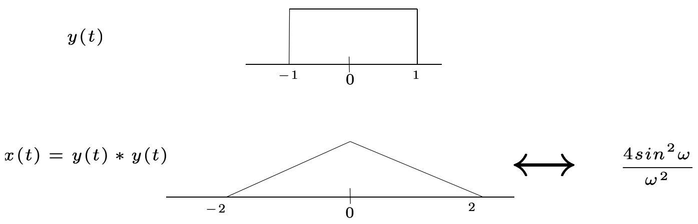


Notice that $Y(j\omega) = X(j\omega)H(j\omega) = X(j\omega)$. *Thus,* $y(t) = x(t)$.

Example 7.3.12. *Find $x(t)$ where $X(j\omega) = \frac{4}{\omega^2} sin^2\omega$.*

$$X(j\omega) = \left(\frac{2}{\omega}\sin\omega\right)\left(\frac{2}{\omega}\sin\omega\right) = Y(j\omega)Y(j\omega)$$

Note that $Y(j\omega)$ can also be written as $2sinc\left(\frac{\omega}{\pi}\right)$ and $X(j\omega)$ as $X(j\omega) = 4sinc^2\left(\frac{\omega}{\pi}\right)$.



Example 7.3.13. *Multiplication in time (Windowing): Suppose that we have $x(t) \longleftrightarrow X(j\omega)$.*

We say that $x(t)$ is timelimited if $x(t) = 0$ for $|t| > T_0$ and we say that $x(t)$ is bandlimited if $X(j\omega) = 0$ for $|\omega| > \omega_0$.

Multiplying a signal $x(t)$ by $y(t) = rect(t/2T_0)$ will make $x(t)$ time-limited. We call this process windowing. Let $z(t) = x(t)y(t)$ denote the windowed signal.

In this example, we will get an intuitive understanding of the effect of windowing on the Fourier transform of $x(t)$.

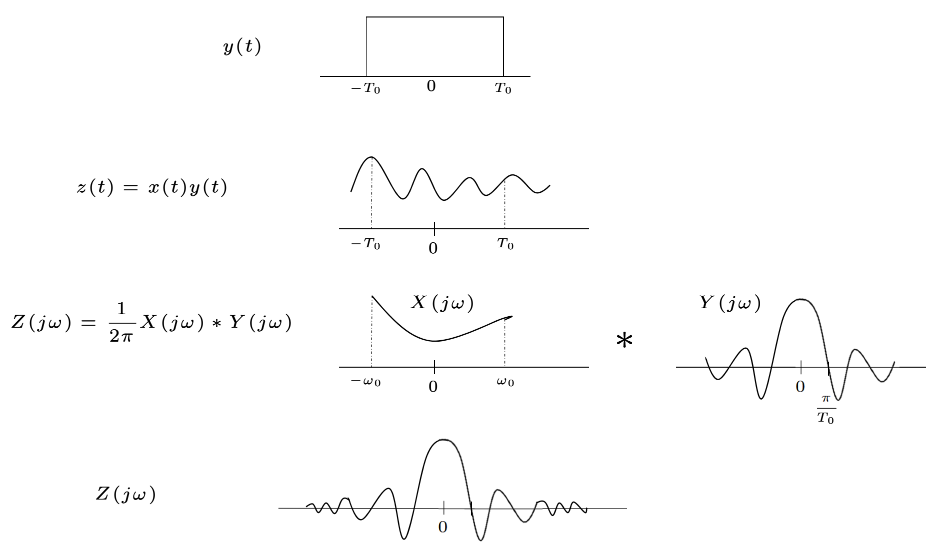


Figure 7.21: Figure showing the effect of windowing in the frequency domain.

$$Z(j\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

Conclusion: Any time-limited signal cannot be band-limited.

7.3.7 Total area under the curve

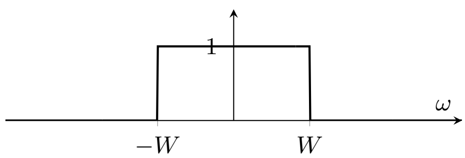
Sometimes, we are interested in computing $\int_{-\infty}^{\infty} x(t)dt$. If the Fourier transform of $x(t)$ is known, then we can compute the integral without explicitly performing the integration by noting that $\int_{-\infty}^{\infty} x(t)dt$ is simply the Fourier transform $X(j\omega)$ evaluated at $\omega = 0$, i.e.,

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ X(j0) &= \int_{-\infty}^{\infty} x(t)dt \end{aligned}$$

Similarly, to evaluate $\int_{-\infty}^{\infty} X(j\omega)\, d\omega$, we can evaluate $2\pi x(t)$ at $t = 0$ since

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ 2\pi x(0) &= \int_{-\infty}^{\infty} X(j\omega)d\omega \end{aligned}$$

Example 7.3.14. *What is* $I = \int_{-\infty}^{\infty} \frac{1}{\pi t} \sin Wtdt$ *? Notice that* $I = X(j0)$ *where* $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$.



Example 7.3.15. *Evaluate* $I = \int_{-\infty}^{\infty} \frac{2\sin \omega T}{\omega}\, d\omega$. *By using*

$$\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi x(0)$$

and setting $X(j\omega) = \frac{2\sin \omega T}{\omega}$ *in the above expression, we see that*

$$x(t) = \begin{cases} 1, & |t| < T; \\ 0, & \textit{otherwise.} \end{cases}$$

Hence $I = 2\pi.1 = 2\pi$

The results can be generalized by realizing that

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega_0 t}dt = X(j\omega_0) \quad (7.38)$$

$$\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t_0}d\omega = 2\pi x(t_0) \quad (7.39)$$

7.3.8 Differentiation Properties

If $x(t) \longleftrightarrow X(j\omega)$, then

$$\begin{aligned} \frac{d}{dt}x(t) &\longleftrightarrow j\omega X(j\omega) \\ -jtx(t) &\longleftrightarrow \frac{d}{d\omega}X(j\omega) \end{aligned} \tag{7.40}$$

Proof:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ \frac{d}{dt}x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega \\ \frac{d}{dt}x(t) &\longleftrightarrow j\omega X(j\omega) \end{aligned}$$

Example 7.3.16. *Find the CTFT of the given signal $y(t)$ using the F.T of $x(t)$.*

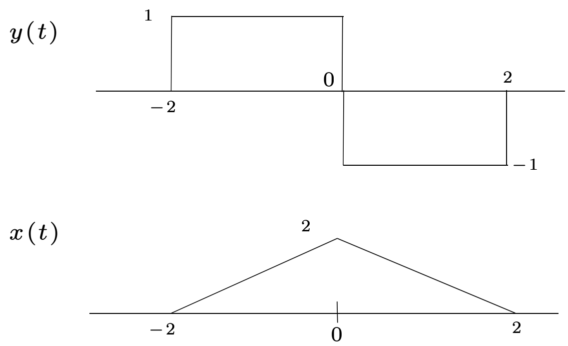


Figure 7.22:

Solution: The F.T of $x(t)$ is

$$X(j\omega) = \frac{4sin^2\omega}{\omega^2}$$

Notice that

$$y(t) = \frac{d}{dt}x(t) \Rightarrow Y(j\omega) = \frac{j\omega 4sin^2\omega}{\omega^2}$$

The Fourier transform of $y(t)$ can also be computed using the time shifting and linearity properties. Let $z(t)$ be the signal given by

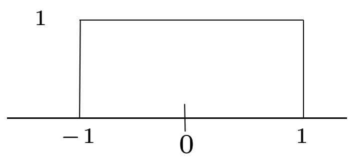


Figure 7.23:

Notice that $y(t) = z(t+1) - z(t-1)$. Hence,$Y(j\omega) = e^{j\omega}Z(j\omega) - e^{-j\omega}Z(j\omega)$ and hence,

$$X(j\omega) = (e^{j\omega} - e^{-j\omega})\frac{2sin\omega}{\omega} = \frac{j\omega 4sin^2\omega}{\omega^2}$$

In the next subsection, we will discuss yet another way to compute this F.T using the integration property. Before that, we consider another example using the differentiation in frequency property.

Example 7.3.17. *Compute the Fourier transform of $x(t) = t^{n-1}e^{-at}u(t)$.*

Let us begin with $n = 1$. We know that $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$. The CTFT of $te^{-at}u(t)$ can be computed by taking the derivative of $\frac{1}{a+j\omega}$ w.r.t ω as follows

$$te^{-at}u(t) \longleftrightarrow j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) = j\frac{-1\times j}{(a+j\omega)^2} = \frac{1}{(a+j\omega)^2}$$

To compute the CTFT of $t^n e^{-at}u(t)$, we can continue to use the differentiation property and see that

$$\begin{aligned}
e^{-at}u(t) &\longleftrightarrow \frac{1}{a+j\omega} \\
te^{-at}u(t) &\longleftrightarrow \frac{1}{(a+j\omega)^2} \\
t^2e^{-at}u(t) &\longleftrightarrow j\frac{-2j}{(a+j\omega)^3} = \frac{2}{(a+j\omega)^3} \\
\vdots &\longleftrightarrow \vdots \\
t^{n-1}e^{-at}u(t) &\longleftrightarrow \frac{(n-1)!}{(a+j\omega)^n} \textit{ equivalently, } \frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^n}
\end{aligned}$$

7.3.9 Integration property

The integration property states that if $x(t) \longleftrightarrow X(j\omega)$, then

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow Y(j\omega) = \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega) \tag{7.41}$$

Here $X(j0)$ is the CTFT of $x(t)$, namely $X(j\omega)$ evaluated at 0.

The first term on the right hand side seems intuitive as it is the inverse of what the differentiation property indicates. Where did the second term come from? The second term comes from the fact that if $\int_{-\infty}^{\infty} x(t)dt = a \neq 0$, then $y(t)$ becomes a constant a as $t \to \infty$. This must lead to a $\delta(\omega)$ term and that is where the second term comes from. Since $X(j0) = a$, we see that the second term depends on $X(j0)$. The reason we get a $\pi X(j0)\delta(\omega)$ instead of $2\pi X(j0)\delta(\omega)$ is that $y(t) = a$ only as $t \to \infty$ and not as $t \to -\infty$. This is certainly not a mathematically precise argument. The idea is to provide intuition for where the term comes from rather than just the proof. To do - add the proof.

Example 7.3.18. *Compute the CTFT of $u(t)$.*

Let us start with $x(t) = \delta(t) \longleftrightarrow X(j\omega) = 1$. Notice that $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ and that $X(j0) = 1$. Therefore,

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \longleftrightarrow \frac{1}{j\omega} + \pi.1.\delta(\omega)$$

7.3.10 Parseval's theorem revisited

Practically speaking, when we want to compute the energy of a signal, we have two options to consider. Sometimes, it will be easier to compute $\int_{-\infty}^{\infty} |x(t)|^2 dt$. Sometimes, it will be easier to compute $\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$. Parseval's theorem tells us that we can choose either of these, depending on whichever is easier, to compute the energy of a signal. The following example will clarify this.

Example 7.3.19. *Let $x(t) = \frac{\sin Wt}{\pi t}$. What is the energy of $x(t)$?*

In this example, it is quite difficult to directly compute $\int_{-\infty}^{\infty} |x(t)|^2 dt$. However, we know that $X(j\omega)$ is a simple rectangular function. Thus, computing the energy in the frequency domain using $X(j\omega)$ is easier. The energy is then given by

$$E_x = \frac{1}{2\pi} \int_{-W}^{W} 1^2 d\omega = \frac{W}{\pi}.$$

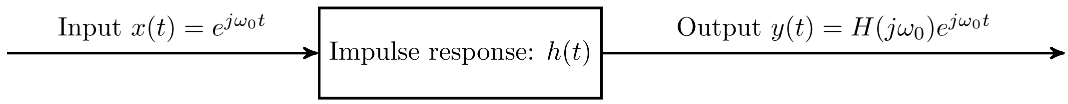


Figure 7.24: Complex exponentials are eigen functions of LTI systems

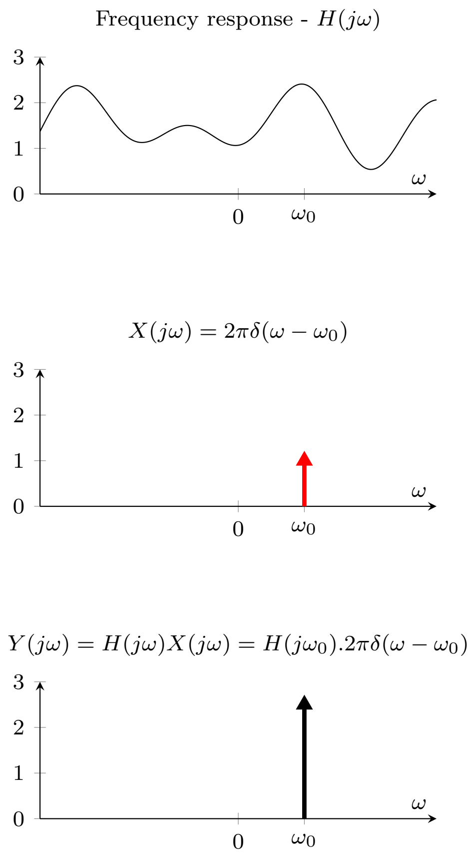


Figure 7.25: Frequency domain perspective for why $y(t) = H(j\omega_0)x(t)$ when $x(t) = e^{j\omega_0 t}$.

7.4 Passing complex exponentials and sinusoids through LTI systems

In the chapter on time domain analysis of LTI systems, we showed that complex exponentials are eigen functions of LTI systems. Here, we revisit this from the frequency domain perspective and see why this true (at least when $s = j\omega$).

To get some insight into this, we use the convolution-multiplication property which states that $Y(j\omega) = X(j\omega)H(j\omega)$. Consider the case when $x(t) = e^{j\omega_0 t}$ and therefore, $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. As shown in Fig. 7.25, we can see that

$$Y(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

Hence, $y(t) = H(j\omega_0)e^{j\omega_0 t}$.

Example 7.4.1. *Consider an LTI system with impulse response* $h(t) = \frac{\sin 4(t-1)}{\pi(t-1)}$. *Compute the output when the input is - a)* $x(t) = e^{j2t}$, *b)* $x(t) = e^{j5t}$, *c)* $x(t) = \cos(2t)$.

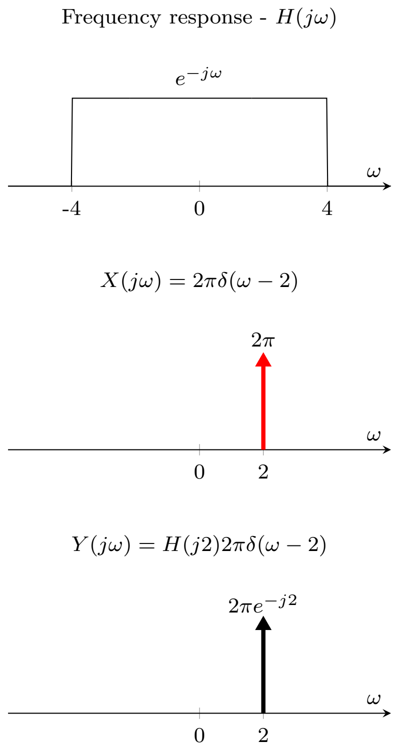


Figure 7.26: Computing the output of H when $x(t) = e^{j2t}$.

Solution: Part a) We begin by computing $H(j\omega)$ which, in this case is

$$
\begin{aligned}
H(j\omega) &= \begin{cases} e^{-j\omega}, & -4 \leq \omega < 4 \\ 0, & \textit{otherwise.} \end{cases} \\
X(j\omega) &= 2\pi\delta(\omega - 2) \\
Y(j\omega) &= H(j\omega)X(j\omega) = H(j2)2\pi\delta(\omega - 2) = e^{-j2}2\pi\delta(\omega - 2) \\
y(t) &= e^{-j2}e^{j2t} = e^{j2(t-1)}
\end{aligned}
$$

Part b) The answer is zero

Part c)

$$
\begin{aligned}
H(j\omega) &= \begin{cases} e^{-j\omega}, & -4 \leq \omega < 4 \\ 0, & \textit{otherwise.} \end{cases} \\
X(j\omega) &= \pi\delta(\omega + 2) + \pi\delta(\omega - 2) \\
Y(j\omega) &= H(j\omega)X(j\omega) = H(-j2)\pi\delta(\omega + 2) + H(j2)\pi\delta(\omega - 2) = e^{j2}\pi\delta(\omega + \omega_0) + e^{-j2}\pi\delta(\omega - 2) \\
y(t) &= \frac{1}{2}e^{j2}e^{-j2t} + \frac{1}{2}e^{-j2}e^{j2t} = \cos(2(t-1))
\end{aligned}
$$

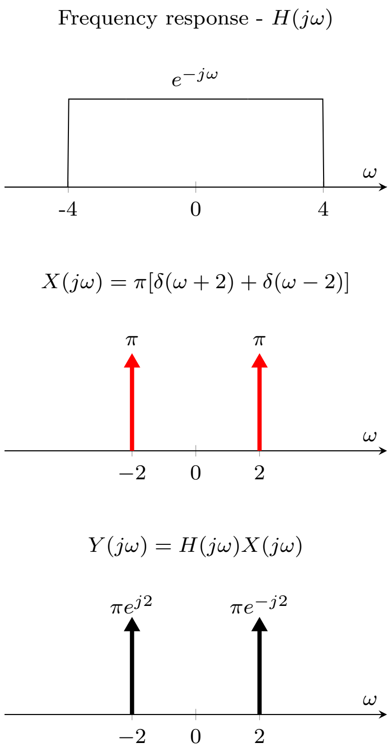


Figure 7.27: Computing the output of H when $x(t) = \cos(2t)$.

7.5 Inverse Fourier Transform of Rational Functions

Given a $X(j\omega)$, how can we compute its inverse Fourier transform?

1. Use the definition:
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

2. Use tables and properties.

3. Use partial fractions to compute inverse Fourier transforms of rational functions.

We have already seen examples of using methods 1 and 2. Here we consider method 3 in more detail. We begin with two examples for why we should care about rational transfer functions.

A Fourier transform $X(j\omega)$ is a rational function if it is a ratio of two polynomials

$$X(j\omega) = \frac{\gamma_m(j\omega)^m + \gamma_{m-1}(j\omega)^{m-1} + \ldots + \gamma_1 j\omega + \gamma_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1 j\omega + a_0}$$

A rational function $H(j\omega)$ is

Proper: if deg(numerator) $<$ deg(denominator).

Improper: if deg(numerator) $\geq$ deg(denominator).

Example 7.5.1. *Consider an LTI system whose input $x(t)$ and output $y(t)$ are related according to the differential equation*

$$\frac{d^2y(t)}{dt} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \tag{1}$$

Find the impulse response of the system.

Solution: *Our approach is to find*

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

and then find the inverse Fourier transform of $H(j\omega)$ to find $h(t)$.

We can find $\frac{Y(j\omega)}{X(j\omega)}$ by noting that (using the differentiation property)

$$(j\omega)^2Y(j\omega) + 4(j\omega)Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega),$$

$$Y(j\omega)\left[(j\omega)^2 + 4j\omega + 3\right] = X(j\omega)(j\omega + 2),$$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}}$$

Impulse response is the inverse FT of $H(j\omega)$. Before computing the inverse FT, recall that

$$e^{-at}u(t) \;\longleftrightarrow\; \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \;\longleftrightarrow\; \frac{1}{(a + j\omega)^2}$$

$$\frac{t^2}{2}e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^3}$$

i.e., we know the inverse FT of $\frac{1}{(a+j\omega)^n}$.

How do we compute the inverse FT of $H(j\omega) = \frac{j\omega+2}{(j\omega)^2+4j\omega+3}$*?*

The main idea is to use the method of partial fractions and write $H(j\omega)$ *as*

$$H(j\omega) = \frac{j\omega+2}{(j\omega)^2+4j\omega+3} = \frac{j\omega+2}{(j\omega+1)(j\omega+3)} = \frac{A_1}{j\omega+1} + \frac{A_2}{j\omega+3}. \tag{2}$$

Then $h(t)$ *is given by*

$$h(t) = A_1 e^{-t}u(t) + A_2 e^{-3t}u(t).$$

Substituting $s = j\omega$ *in (2) we get*

$$\frac{s+2}{(s+1)(s+3)} = \frac{A_1}{s+1} + \frac{A_2}{s+3}.$$

Multiplying both sides by $(v+1)$ *we get*

$$\frac{s+2}{s+3} = A_1 + \frac{A_2}{s+3}(s+1).$$

Substitute $v = -1$*:*

$$\frac{-1+2}{-1+3} = A_1 \Longrightarrow A_1 = \frac{1}{2}.$$

To find A_2*:*

$$\frac{s+2}{s+1} = A_1\frac{s+3}{s+1} + A_2.$$

Substitute $v = -3$*:*

$$\frac{-3+2}{-3+1} = A_2 \Longrightarrow A_2 = \frac{1}{2}.$$

Therefore,

$$\boxed{h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).}$$

Example 7.5.2. *The input to an LTI system with impulse response $h(t) = e^{-t}u(t)$ is $x(t) = e^{-2t}u(t)$. What will be the output of the system?*

Solution:

$$\begin{aligned}
Y(j\omega) = X(j\omega)H(j\omega) &= \frac{1}{2+j\omega}\cdot\frac{1}{1+j\omega} \\
&= \frac{1}{(2+j\omega)(1+j\omega)} = \frac{A_1}{2+j\omega} + \frac{A_2}{1+j\omega} \\
&= \frac{A_1(1+j\omega)+A_2(2+j\omega)}{(2+j\omega)(1+j\omega)} = \frac{(A_1+2A_2)+j\omega(A_1+A_2)}{(2+j\omega)(1+j\omega)}.
\end{aligned}$$

Hence

$$\begin{aligned}
A_1 + 2A_2 &= 1, \\
A_1 + A_2 &= 0,
\end{aligned}$$

and we find $A_1 = -1$, $A_2 = 1$.

$$\boxed{\begin{aligned}
Y(j\omega) &= \frac{-1}{2+j\omega} + \frac{1}{1+j\omega} \\
y(t) &= -e^{-2t}u(t) + e^{-t}u(t).
\end{aligned}}$$

Example 7.5.3. *Compute the inverse Fourier transform of*

$$H(j\omega) = \frac{j\omega+2}{(j\omega+3)(j\omega+1)(j\omega+4)}.$$

Solution: *To keep the notation simple let us set $s = j\omega$.*

$$H(s) = \frac{s+2}{(s+3)(s+1)(s+4)} = \frac{A_1}{s+3} + \frac{A_2}{s+1} + \frac{A_3}{s+4}.$$

$$H(s)(s+3) = \frac{s+2}{(s+1)(s+4)} = A_1 + \frac{A_2(s+3)}{s+1} + \frac{A_3(s+3)}{s+4}.$$

$$\left.\frac{s+2}{(s+1)(s+4)}\right|_{s=-3} = A_1 = \frac{1}{2}.$$

$$\left.\frac{s+2}{(s+3)(s+4)}\right|_{s=-1} = A_2 = \frac{1}{2\cdot 3} = \frac{1}{6}.$$

$$\left.\frac{s+2}{(s+3)(s+1)}\right|_{s=-4} = A_3 = \frac{-2}{(-1)(-3)} = -\frac{2}{3}.$$

Thus

$$\begin{aligned}
H(j\omega) &= \frac{1}{2}\frac{1}{2+j\omega} + \frac{1}{6}\frac{1}{1+j\omega} - \frac{2}{3}\frac{1}{4+j\omega}. \\
h(t) &= \frac{1}{2}e^{-2t}u(t) + \frac{1}{6}e^{-t}u(t) - \frac{2}{3}e^{-4t}u(t).
\end{aligned}$$

Example 7.5.4. *Consider an example of $H(j\omega)$ with repeated roots in the denominator.*

$$H(j\omega) = \frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}.$$

Solution: *Let's set $s = j\omega$ and write a partial fraction expansion for $H(s)$ as below.*

$$H(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{s+3}.$$

Then

$$\boxed{h(t) = A_{11}e^{-t}u(t) + A_{12}te^{-t}u(t) + A_{21}e^{-3t}u(t).}$$

The coefficients A_{11}, A_{12} and A_{21} can be computed as follows:

$$A_{21} = H(s)(s+3)\Big|_{s=-3} = \frac{s+2}{(s+1)^2}\Big|_{s=-3} = \frac{-1}{4}.$$

$$\frac{s+2}{s+3}\Big|_{s=-1} = \left[A_{11}(s+1) + A_{12} + A_{21}\frac{(s+1)^2}{s+3}\right]\Big|_{s=-1},$$

$$\frac{1}{2} = A_{12}.$$

$$\frac{s+2}{s+3} = A_{11}(s+1) + A_{12} + A_{21}\frac{(s+1)^2}{s+3},$$

$$\frac{d}{ds}\left[\frac{s+2}{s+3}\right] = A_{11} + A_{21}\frac{2(s+1)(s+3)-(s+1)^2}{(s+3)^2},$$

$$\frac{d}{ds}\left[\frac{s+2}{s+3}\right]\Big|_{s=-1} = A_{11},$$

$$\frac{s+2}{s+3} = \frac{s+3-1}{s+3} = 1 - \frac{1}{s+3},$$

$$\frac{d}{ds}\left[\frac{s+2}{s+3}\right] = \frac{1}{(s+3)^2},$$

$$\frac{1}{(s+3)^2}\Big|_{s=-1} = \frac{1}{4},$$

$$A_{11} = \frac{1}{4}.$$

Example 7.5.5. *Finally we will solve an example involving improper rational functions.*

$$H(j\omega) = \frac{(j\omega)^2 + 2j\omega + 2}{(j\omega+3)^2(j\omega+4)}.$$

Solution: *Let's set $s = j\omega$. Then*

$$H(s) = \frac{s^2+2s+2}{(s+3)(s+4)}.$$

We use long division to write $H(s)$ as

$$H(s) = \frac{s^2+2s+2}{s^2+7s+12} = 1 + \frac{-5s-11}{s^2+7s+12}.$$

Notice that $\frac{-5s-11}{s^2+7s+12}$ is a proper rational function and we can use partial fraction method to write $H(s)$ as

$$H(s) = 1 + \frac{A_1}{s+3} + \frac{A_2}{s+4}.$$

$$H(j\omega) = 1 + \frac{A_1}{j\omega+3} + \frac{A_2}{j\omega+4}.$$

$$\Longrightarrow h(t) = \delta(t) + A_1e^{-3t}u(t) + A_2e^{-4t}u(t).$$

A_1 and A_2 can be computed using the usual partial fractions expansion procedure.

7.6 Filtering

A filter is a system that selectively passes some frequency components and removes other frequency components of the input signal.

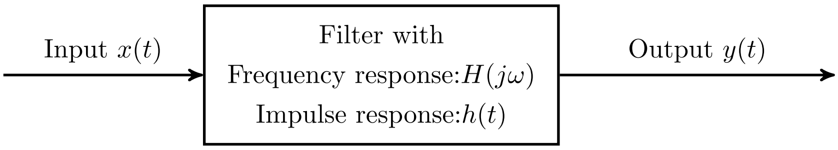


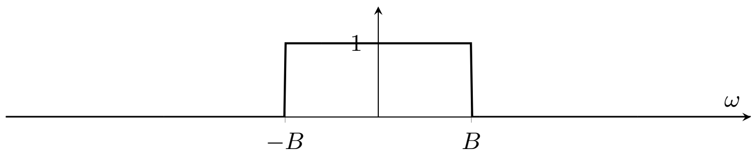
Figure 7.28: Block diagram of a filter

When you use a filter on instagram, you are performing a modification to the photo to achieve a desired effect. The origin for the term filter indeed comes from signal processing as we use in this course. Some filters that you use in instagram may require non-linear operations, regardless, in principle you perform some image processing operations similar in spirit what is considered here.

The impulse response and frequency responses of ideal filters are shown in the following figures.

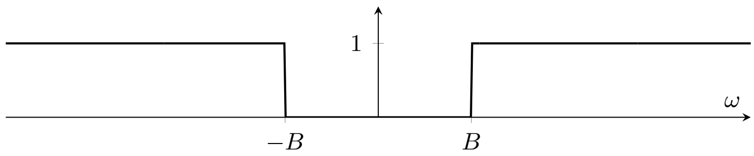
$H(j\omega)$ for an ideal low pass filter with cut-off freqeuency B

$$h(t) = \frac{\sin Bt}{\pi t} \longleftrightarrow$$



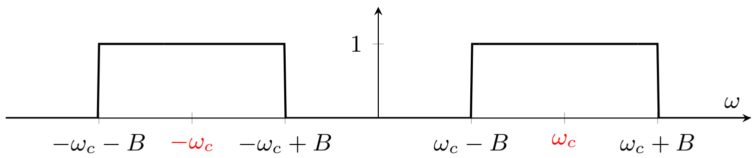
$H(j\omega)$ for an ideal high pass filter with cut-off freqeuency B

$$h(t) = \delta(t) - \frac{\sin Bt}{\pi t} \longleftrightarrow$$



$H(j\omega)$ for an ideal band pass filter with center frequency ω_c and bandwidth $2B$

$$h(t) = 2\cos(\omega_c t)\frac{\sin Bt}{\pi t} \longleftrightarrow$$



$H(j\omega)$ for an ideal band reject filter with center frequency ω_c and bandwidth $2B$

$$h(t) = \delta(t) - 2\cos(\omega_c t)\frac{\sin Bt}{\pi t} \longleftrightarrow$$

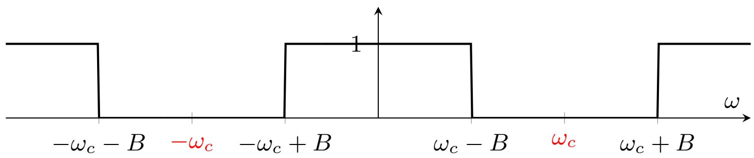


Figure 7.29: Frequency responses and impulse responses of ideal filters

Chapter 8

Discrete-time Fourier Transform

8.1 Definition of the DTFT

Let $x[n]$ be a discrete-time signal (not necessarily periodic). The discrete-time Fourier transform (DTFT) of $x[n]$, denoted by $X(e^{j\Omega})$ is defined as

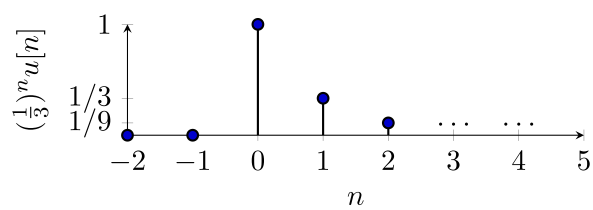
$$DTFT : X(e^{j\Omega}) := \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \tag{8.1}$$

The inverse DTFT provides a way to obtain $x[n]$ from $X(e^{j\Omega}$ and is given by

$$InverseDTFT : x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}\, d\Omega, \tag{8.2}$$

where the integral is over *any* contiguous frequency interval of length 2π. It is typically chosen to be $[-\pi, \pi)$ or $[0, 2\pi)$.

Example 8.1.1. *Consider the signal* $x[n] = \left(\frac{1}{3}\right)^n u[n]$ *shown in the figure below*



The DFTF of $x[n]$ *is given by*

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty} (\frac{1}{3})^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}e^{-j\Omega}\right)^n = \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$

The magnitude and phase of $X(e^{j\Omega})$ *is plotted in Fig. 8.1.*

It can be seen from Fig. 8.1 that both the magnitude and the phase (and, hence, $X(e^{j\Omega})$) are periodic in Ω with a fundamental period of 2π. This is not merely an isolated example where $X(e^{j\Omega})$ is periodic with a period of 2π. We will now show that this is true for any DT signal.

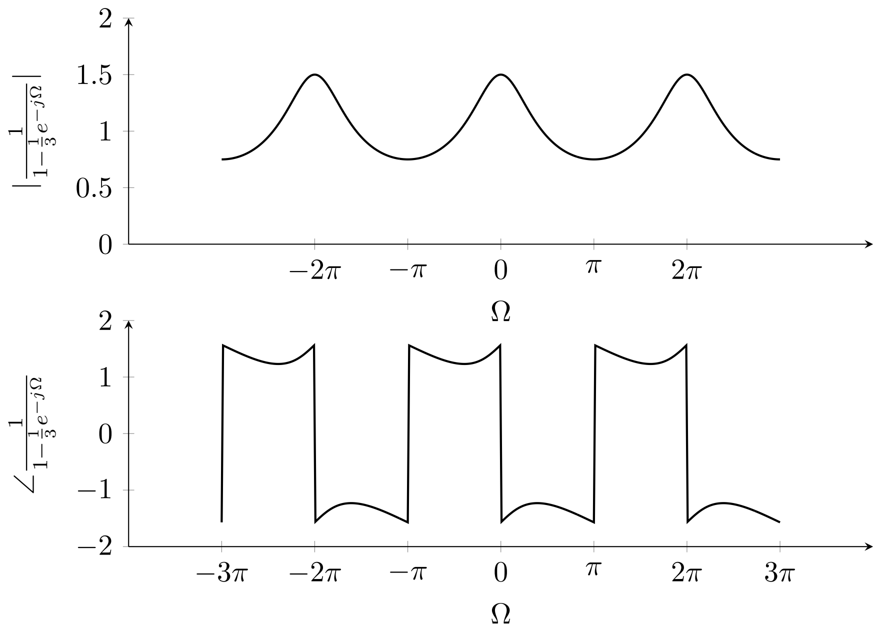


Figure 8.1: Plot of magnitude and phase of $X(e^{j\Omega})$

The DTFT of any DT signal is periodic with a period of 2π

To see this, let us consider $X(e^{j(\Omega+2\pi)})$.

$$X(e^{j(\Omega+2\pi)}) = \sum_{n=0}^{\infty} x[n]e^{-j(\Omega+2\pi)n} = \sum_{n=0}^{\infty} x[n]e^{-j\Omega n}.e^{-j2\pi n} = \sum_{n=0}^{\infty} x[n]e^{-j\Omega n} = X(e^{j\Omega})$$

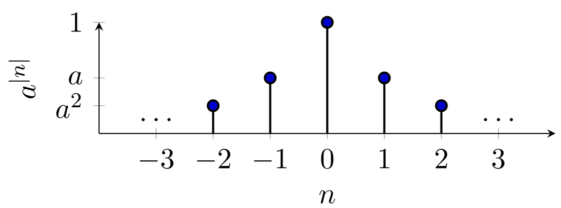
Here, we use the fact that $e^{-j2\pi n} = 1$ for any integer n.

Since $X(e^{j\Omega})$ is periodic with a period of 2π, it is common to consider only one period of $X(e^{j\Omega})$ when evaluating or plotting $X(e^{j\Omega})$. All the entries in the DTFT tables also reflect the fact that $X(e^{j\Omega})$ is periodic with a period of 2π.

Example 8.1.2. *Compute the DTFT of $x[n] = a^n u[n]$ where $|a| < 1$. The DTFT of $x[n]$ is given by*

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n = \frac{1}{1 - ae^{-j\Omega}}$$

Example 8.1.3. *Compute the DTFT of $x[n] = a^{|n|}$. The signal $x[n]$ is shown in the figure below*

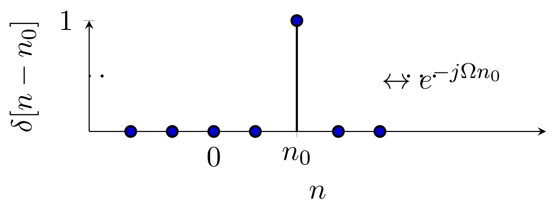


$X(e^{j\Omega})$ can be computed as follows

$$
\begin{aligned}
X(e^{j\Omega}) &= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\
&= \sum_{m=1}^{\infty} a^m e^{j\Omega m} + \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\
&= \frac{ae^{j\Omega}}{1 - ae^{j\Omega}} + \frac{1}{1 - ae^{-j\Omega}} \qquad (8.3) \\
&= \frac{1 - a^2}{1 + a^2 - 2a\cos\Omega} \qquad (8.4)
\end{aligned}
$$

Example 8.1.4. *Let $x[n] = \delta[n - n_0]$. The DTFT of $x[n]$ is given by*

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\Omega n} = e^{-j\Omega n_0}$$



when $n_0 = 0$, we get

$$\delta[n] \leftrightarrow 1$$

Example 8.1.5. *Let $X(e^{j\Omega}) = 2\pi\delta(\Omega - \Omega_0)$. What is the corresponding $x[n]$? Since*

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega - \Omega_0) e^{j\Omega n}\, d\Omega = e^{j\Omega_0 n}$$

Thus, we see that

$$e^{j\Omega_0 n} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

Since $X(e^{j\Omega})$ is periodic with a period of 2π, in the tables in the compendium, the DTFT is given as

$$e^{j\Omega_0 n} \leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi l)$$

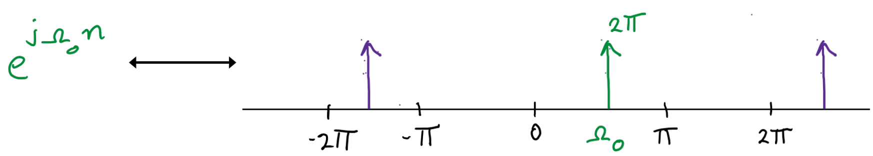


Figure 8.2: DTFT of a DT complex exponential

Example 8.1.6. *Compute the DTFT of $\cos(\Omega_0 n)$ and $\sin(\Omega_0 n)$ Using the result in the previous example, we see that*

$$\cos(\Omega_0 n) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n} \quad \leftrightarrow \quad \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$
$$\sin(\Omega_0 n) = \frac{1}{2j}e^{j\Omega_0 n} - \frac{1}{2j}e^{-j\Omega_0 n} \quad \leftrightarrow \quad \frac{\pi}{j}[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$

Again to emphasize that $X(e^{j\Omega})$ is periodic, it may be written as

$$\cos(\Omega_0 n) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n} \quad \leftrightarrow \quad \pi \sum_{l=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi l) + \delta(\Omega + \Omega_0 - 2\pi l)]$$
$$\sin(\Omega_0 n) = \frac{1}{2j}e^{j\Omega_0 n} - \frac{1}{2j}e^{-j\Omega_0 n} \quad \leftrightarrow \quad \frac{\pi}{j} \sum_{l=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi l) + \delta(\Omega + \Omega_0 - 2\pi l)]$$

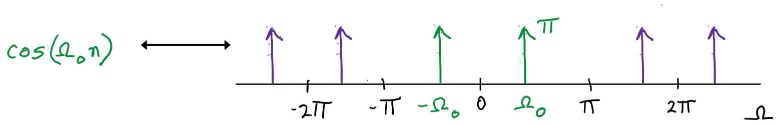


Figure 8.3: DTFT of a DT Cosinusoid

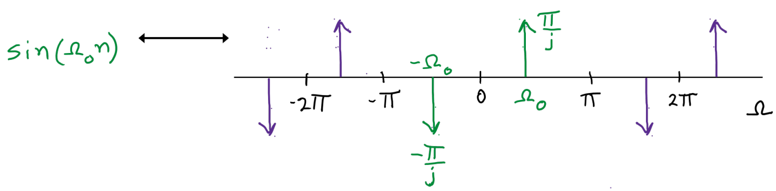


Figure 8.4: DTFT of a DT Sinusoid

Example 8.1.7. *Let*

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \Omega_c \le |\Omega| < \pi \end{cases}$$

as shown in the figure. What is $x[n]$?

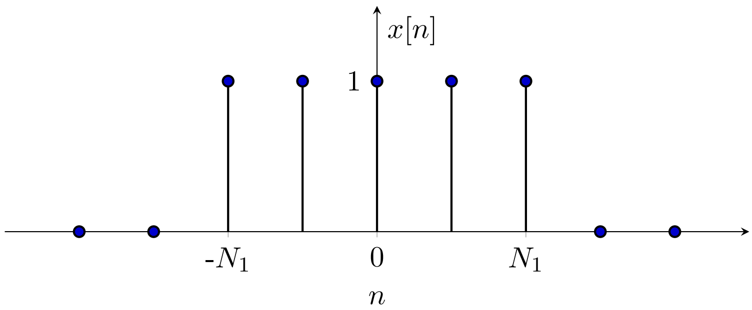
Using the IDTFT formula (8.2), we see that

$$x[n] = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1.e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}[\frac{e^{j\Omega n}}{jn}]_{-\Omega_c}^{\Omega_c} = \frac{\sin \Omega_c n}{\pi n}$$

Example 8.1.8. *Compute the DTFT of*

$$x[n] = \begin{cases} 1, & -N_1 \leq n \leq N_1 \\ 0, & \textit{otherwise.} \end{cases}$$

It is always a good idea to first visualize the signal by plotting it. The signal $x[n]$ looks like



The DTFT can be computed using the DTFT formula and it can be simplified as follows:

$$\begin{aligned} X(e^{j\Omega} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-N_1}^{N_1} 1.e^{-j\Omega n} \\ &= \frac{e^{j\Omega N_1} - e^{-j\Omega(N_1+1)}}{1 - e^{-j\Omega}} \\ &= \frac{e^{-j\Omega\frac{1}{2}}\left(e^{j\Omega(N_1+\frac{1}{2})} - e^{-j\Omega(N_1+\frac{1}{2})}\right)}{e^{-j\Omega\frac{1}{2}}\big(e^{j\Omega/2} - e^{-j\Omega/2}\big)} = \frac{\sin\Omega\big(N_1 + 1/2\big)}{\sin\Omega/2} \end{aligned}$$

Several properties of the DTFT can be proved similar to how we proved them in the case of the CTFT. Using the properties and the basic DTFT pairs, the DTFT of several commonly used signals can be computed and are given in the following tables. The reader should familiarize themselves with these properties and be aware of which DTFT pairs are listed in the tables. While many properties bear resemblance to those of CTFT's, there are a few differences. Unlike continuous-time signals, we cannot differentiate/integrate discrete-time signals. The equivalent operations are given by the difference operation and summation operation. Note that Ω is a continuous variable and hence, taking derivative with respect to Ω is a valid operation.

PROPERTIES OF THE DISCRETE TIME FOURIER TRANSFORM

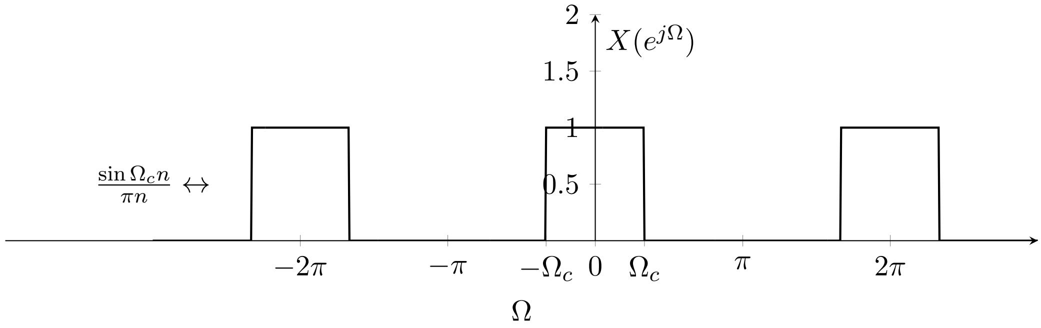
Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega}) - 2\pi\text{periodic}$
		$y[n]$	$Y(e^{j\omega}) - 2\pi\text{periodic}$
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.5	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.5	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k] & n \mod k = 0 \\ 0 & \text{otherwise} \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.4	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.4	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.6	Differentiation in Frequency	$nx[n]$	$j\dfrac{d}{d\omega}X(e^{j\omega})$
5.3.3	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re\{X(e^{j\omega})\} = \Re\{X(e^{-j\omega})\} \\ \Im\{X(e^{j\omega})\} = -\Im\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.3	Symmetry for Real and Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
5.3.3	Symmetry for Real and Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.3	Even-Odd Decomposition for Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\Re\{X(e^{j\omega})\}$ $j\Im\{X(e^{j\omega})\}$

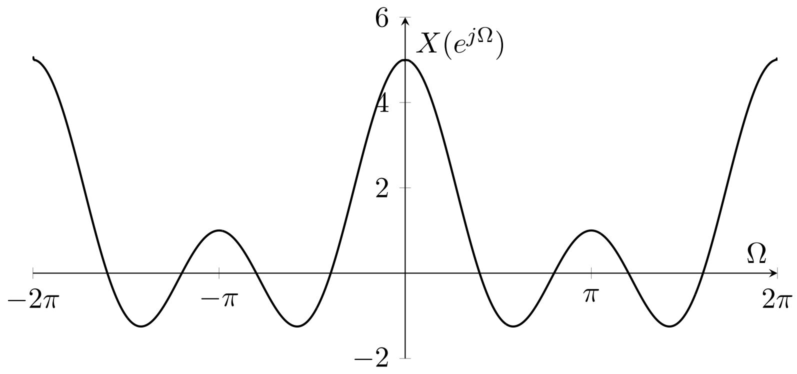
5.3.7 Parseval's Relation for Aperiodic Signals

$$\sum_{-\infty}^{+\infty} |x[n]|^2 dt = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - \frac{2\pi k}{N})$	a_k
$e^{jw_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} 1 & k = m, m \pm N, \ldots \\ 0 & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\rightarrow$ The signal is aperiodic
$\cos w_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$, $a_k = \begin{cases} \frac{1}{2} & k = m, m \pm N, \ldots \\ 0 & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\rightarrow$ The signal is aperiodic
$\sin w_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$, $a_k = \begin{cases} \frac{1}{2j} & k = r, r \pm N, \ldots \\ -\frac{1}{2j} & k = -r, -r \pm N, \ldots \\ 0 & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1 & k = 0, \pm N, \ldots \\ 0 & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \leq N_1 \\ 0, & N_1 < \|n\| \leq \frac{N}{2} \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - \frac{2\pi k}{N})$	$a_k = \frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N \sin[2\pi k/2N]}$ $k \neq 0, \pm N, \ldots$ $a_k = \frac{2N_1+1}{N}$ $k = 0, \pm N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \ \|a\| < 1$	$\frac{1}{1-ae^{-j\omega}}$	——
$x[n] = \begin{cases} 1, & \|n\| \leq N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin \omega(N_1 + \frac{1}{2})}{\sin(\omega/2)}$	——
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X\left(e^{j\omega}\right) = \begin{cases} 1, & 0 < \|\omega\| \leq W \\ 0, & W < \|\omega\| \leq \pi \end{cases}$ $X\left(e^{j\omega}\right)$ 2π periodic	——
$\delta[n]$	1	——
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	——
$\delta[n - n_0]$	$e^{-j\omega n_0}$	——
$(n+1)a^n u[n], \ \|a\| < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$	——
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \ \|a\| < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$	——





8.1.1 Frequency Response of Discrete-Time LTI Systems

Using the convolution-multiplication property of discrete-time LTI systems, we can obtain a frequency-domain representation of (or, insight into) the operation of an LTI system. If the input signal $x[n]$ is passed through an LTI system with impulse response $h[n]$ and the output is $y[n]$, then we know that $y[n] = x[n] \star h[n]$. In the frequency domain, the action of the LTI system can be viewed as multiplying $X(e^{j\Omega})$ by $H(e^{j\Omega})$ to obtain $Y(e^{j\Omega})$. This is shown in Fig 8.5

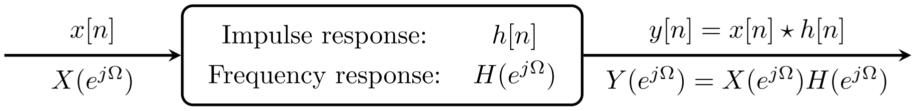


Figure 8.5: Time domain and frequency domain representations of an LTI system

> The frequency response of a Discrete-time LTI system is the DT Fourier transform of the impulse response given by $H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$.

8.1.2 Discrete-time Filters

Similar to LTI systems that perfom filtering of a continuous-time signal, we can define DT LTI system that perform filtering of a DT signal. An important difference between the frequency response of a CT filter and that of a DT filter is in what are considered high frequencies. Since the DTFT of every DT signal is periodic with a period of 2π, one period from $[-\pi, \pi)$ describes the entire frequency range. The highest frequency possible is π and therefore, frequencies close to π and $-\pi$ are the high frequency components.

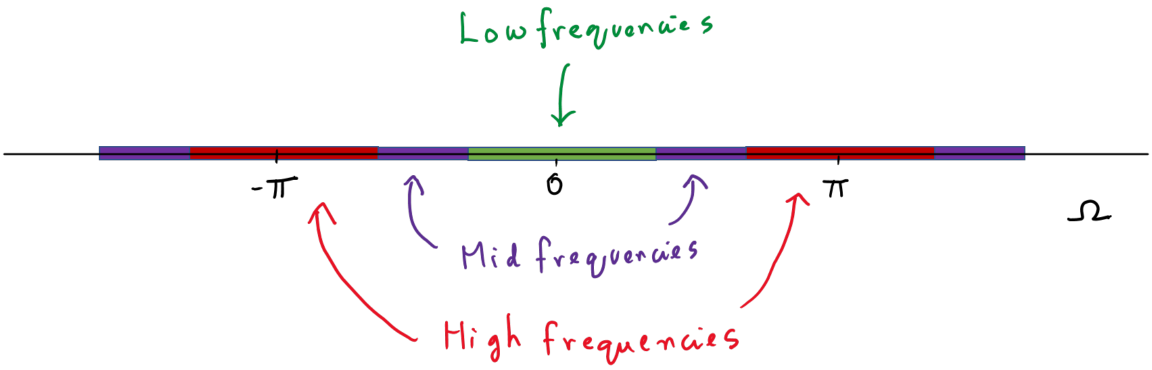


Figure 8.6: Figure showing low frequencies, mid frequencies and high frequencies in a DTFT

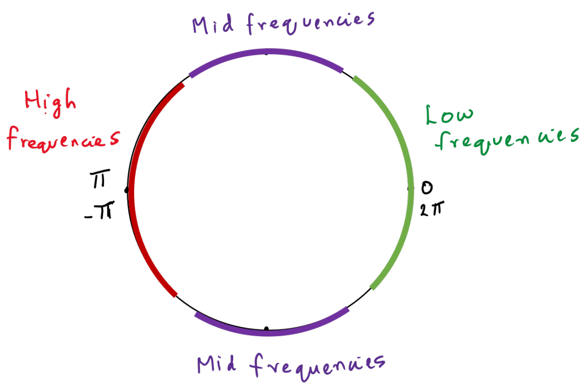


Figure 8.7: Figure showing low frequencies, mid frequencies and high frequencies when wrapping the frequency axis on a circle

This can be better visualized by wrapping the frequency axis on a circle as shown in Fig. 8.7. It can be seen from the figure that the periodicity of the DTFT is naturally accounted for when the different periods overlap in frequency and thus helps us visualize just the interval from $[-\pi, \pi)$.

We can now consider the frequency response and impulse response of ideal DT filters.

Ideal Low Pass Filter (LPF)

- The frequency response of an ideal LPF with cut-off frequency Ω_c is given by

$$H_{LPF}(e^{j\Omega}) = \begin{cases} 1, & 0 \leq |\Omega| < \Omega_c \\ 0, & \Omega_c \leq |\Omega| < \pi. \end{cases}$$

and is shown in Figure 8.8.

The impulse response of the filter is given by

$$h_{LPF}[n] = \frac{\sin \Omega_c n}{\pi n} = \frac{\Omega_c}{\pi} \text{sinc}\left(\frac{\Omega_c n}{\pi}\right)$$

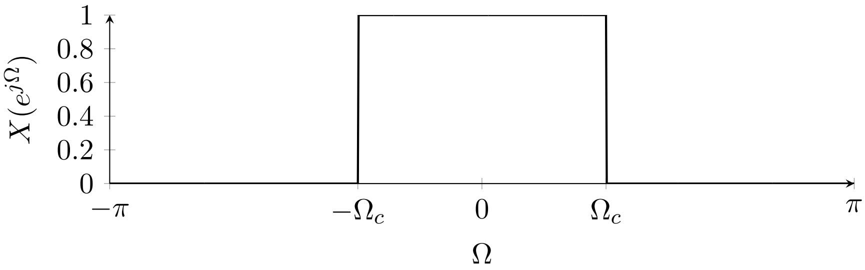


Figure 8.8: Frequency response of an ideal LPF

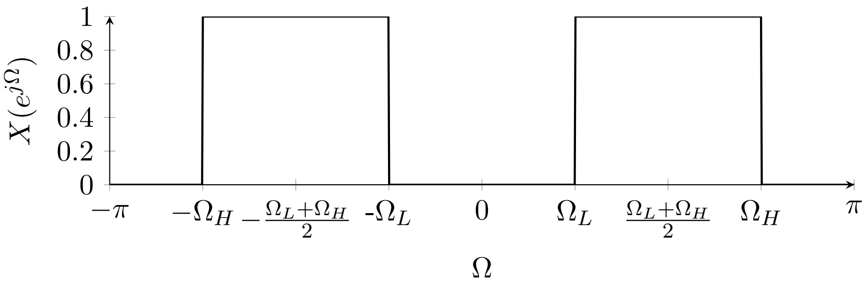


Figure 8.9: Frequency response of an ideal Bandpass filter (BPF)

Ideal Band Pass Filter (BPF)

- The frequency response of an ideal BPF that passes frequencies between Ω_L and Ω_H is given by

$$H_{LPF}(e^{j\Omega}) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_L \\ 1, & \Omega_L \leq |\Omega| < \Omega_H \\ 0, & \Omega_H \leq |\Omega| < \pi. \end{cases}$$

and is shown in Figure 8.9.

The impulse response of the filter is given by

$$h_{LPF}[n] = \cos\left(\frac{(\Omega_L + \Omega_H)n}{2}\right)\frac{\sin(\Omega_H - \Omega_L)n}{\pi n} = \cos\left(\frac{(\Omega_L + \Omega_H)n}{2}\right)\frac{\Omega_H - \Omega_L}{\pi}\text{sinc}\left(\frac{(\Omega_H - \Omega_L)n}{\pi}\right)$$

Ideal High Pass Filter (HPF)

- The frequency response of an ideal HPF with cut-off frequency Ω_c is given by

$$H_{HPF}(e^{j\Omega}) = \begin{cases} 0, & 0 \leq |\Omega| < \Omega_c \\ 1, & \Omega_c \leq |\Omega| < \pi. \end{cases}$$

and is shown in Figure 8.8.

The impulse response of the filter is given by

$$h_{LPF}[n] = \delta[n] - \frac{\sin \Omega_c n}{\pi n} = \delta[n] - \frac{\Omega_c}{\pi}\text{sinc}\left(\frac{\Omega_c n}{\pi}\right)$$

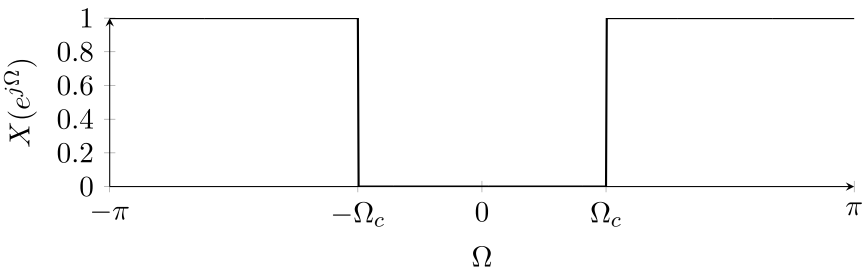


Figure 8.10: Frequency response of an ideal HPF

8.1.3 Problems - Homework 8

- **10 pt** – Problem 1: c
- **10 pt** – Problem 2: c
- **10 pt** – Problem 3: e
- **10 pt** – Problem 4: c
- **10 pt** – Problem 5: d
- **10 pt** – Problem 6: a, b
- **10 pt** – Problem 7: a, b
- **10 pt** – Problem 8: a, b

1. Compute the Fourier transform of each of the following, and sketch the magnitude spectrum and the phase spectrum of each of the following:

   (a) $x[n] = \delta[n-1] + \delta[n+2]$

   (b) $x[n] = (\frac{1}{2})^{n-2}u[n-2]$

   (c) $x[n] = (\frac{1}{2})^{|n-2|}$

   (d) $x[n] = \delta[n+1] - \delta[n-2]$

   (e) $x[n] = \sin(\pi n/4 + \pi/3)$

   (f) $x[n] = 2 + \cos(\pi n/5 + \pi/6)$

2. Compute the inverse Fourier transform of each of the following:

   (a) $X(e^{j\Omega}) = \begin{cases} 3j, & 0 < \Omega < \pi \\ -3j, & -\pi < \Omega < 0 \end{cases}$, with $\Omega_0 = 2\pi$

   (b) $X(e^{j\Omega}) = \begin{cases} 1, & 0 < |\Omega| < \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < |\Omega| < \pi \end{cases}$, with $\Omega_0 = 2\pi$

   (c) $X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} (2\pi\delta(\Omega - \pi k) + \pi\delta(\Omega - \pi/4 - \pi k) + \pi\delta(\Omega + \pi/4 - \pi k))$

   (d) $X(e^{j\Omega}) = (\frac{1}{1-e^{-j\Omega}})(\frac{\sin(3\Omega/2)}{\sin(\Omega/2)}) + 5\pi\delta(\Omega)$ for $-\pi < \Omega < \pi$, with $\Omega_0 = 2\pi$

3. Compute the Fourier transform of each of the following, and use the Parseval's relation to find the energy of each of the following:

   (a) $x[n] = (-\frac{1}{2})^n u[1-n]$

   (b) $x[n] = (n-1)\left(\frac{2}{3}\right)^{|n|}$

   (c) $x[n] = 3^n \sin(3\pi n/4) u[-n]$

   (d) $x[n] = \sin(4\pi n/3) + \cos(5\pi n/3)$

   (e) $x[n] = (\frac{\sin(\pi n/4)}{\pi n}) \cos(3\pi n/2)$

   (f) $x[n] = \begin{cases} n, & |n| \leq 3 \\ 0, & \text{otherwise} \end{cases}$

4. Compute the inverse Fourier transform of each of the following:

   (a) $X(e^{j\Omega}) = \begin{cases} 1, & \frac{\pi}{5} \leq |\Omega| \leq \frac{4\pi}{5} \\ 0, & \text{otherwise} \end{cases}$

   (b) $X(e^{j\Omega}) = e^{-j\Omega/2}$ for $|\Omega| \leq \pi$

   (c) $X(e^{j\Omega}) = \cos^2(2\Omega) + \sin^2(3\Omega)$

   (d) $X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(\Omega - \pi k/2)$

   (e) $X(e^{j\Omega}) = \frac{e^{-j\Omega} - \frac{1}{2}}{1 - \frac{1}{4}e^{-j\Omega}}$

   (f) $X(e^{j\Omega}) = \frac{1 - \frac{1}{2}e^{-j\Omega}}{1 - \frac{1}{2}e^{-j\Omega} - e^{-j2\Omega}}$

5. Consider an LTI system with the impulse response $h[n] = (\frac{1}{3})^n \cos(\pi n/2) u[n]$. Compute the Fourier transform of the output $y[n]$ for each of the following inputs:

   (a) $x[n] = (\frac{1}{3})^n u[n]$

   (b) $x[n] = n(\frac{1}{3})^n u[n]$

   (c) $x[n] = \cos(3\pi n/2)$

   (d) $x[n] = (n+2)(\frac{1}{2})^n u[n]$

6. Suppose that the input $x[n]$ and the output $y[n]$ of a system follow the difference equation $y[n] + \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$. Using the Fourier transform and the inverse Fourier transform, compute each of the following:

   (a) The frequency response of the system, $H(e^{j\Omega})$.

   (b) The impulse response of the system, $h[n]$.

   (c) The output $y[n]$ for the input $x[n] = 1$.

7. Consider a system S whose input $x[n]$ and output $y[n]$ are related through the equation $y[n] = nz[n] * (4^{-n}u[n])$, where $z[n] = x[n] * (\delta[n] + \frac{1}{2}\delta[n-1])$.

   (a) Find the Fourier transform $Z(e^{j\Omega})$ of the signal $z[n]$ for the input

   $$x[n] = n(-2)^{-n+1}u[n-1].$$

   (b) Find the output $y[n]$ of the system S for the input $x[n]$ defined in the part (a).

8. Let

   $$x[n] = \begin{cases} \dfrac{j^n}{\pi^2 n^2}, & n \neq 0 \\ 0, & n = 0 \end{cases}.$$

   Without explicitly evaluating $X(e^{j\Omega})$, find:

(a) $\displaystyle\int_0^{2\pi} X(e^{j2\Omega})\cos(4\Omega)\ d\Omega$

(b) $\displaystyle\int_0^{2\pi} \frac{d}{d\Omega}\left(X(e^{j(\Omega+\pi/2)})U(e^{j\Omega})\right) e^{-j(\Omega+\pi/2)}\ d\Omega$

Your final answers must be simple (real or complex) numbers (e.g., $\pi^3/2$ or $5+j\pi/3$). (Hint: The following formula can be useful: $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.)

9. Without explicitly evaluating $X(e^{j\Omega})$, compute (i) $X(e^{j0})$, (ii) $\angle X(e^{j\Omega})$, (iii) $\int_{-\pi}^{\pi} X(e^{j\Omega})d\Omega$, (iv) $X(e^{j\pi})$, (v) $\int_{-\pi}^{\pi}|X(e^{j\Omega})|^2 d\Omega$, and (vi) $\int_{-\pi}^{\pi}\left|\frac{dX(e^{j\Omega})}{d\Omega}\right|^2 d\Omega$ for each of the following:

(a) $x[n]=\begin{cases}-1, & n=-3,7\\ 1, & n=-1,1,3,5\\ 2, & n=0,4\\ 0, & \text{otherwise}\end{cases}$

(b) $x[n]=\begin{cases}-1, & n=-6,14\\ 1, & n=-2,2,6,10\\ 2, & n=0,8\\ 0, & \text{otherwise}\end{cases}$

10. Consider an LTI system with the impulse response $h[n]=(\frac{1}{3})^n u[n]+g[n]$. Suppose that the frequency response of the system is given by

$$H(e^{j\Omega})=\frac{-12+5e^{-j\Omega}}{12-7e^{-j\Omega}+e^{-j2\Omega}}.$$

Find $g[n]$.

11. Suppose that $x[n]=\sin(\pi n/6)-2\cos(\pi n/3)$. Using the Fourier transform and the inverse Fourier transform, find the output $y[n]$ of each of the following systems for the input $x[n]$:

(a) An LTI system with the impulse response $h[n]=\frac{\sin(\pi n/4)}{\pi n}+\frac{\sin(\pi n/2)}{\pi n}$.

(b) An LTI system with the impulse response $h[n]=\frac{\sin(\pi n/4)\sin(\pi n/2)}{\pi^2 n^2}$.

12. Consider an LTI system with the impulse response $h[n]$ such that

$$H(e^{j\pi/2})=1 \quad\text{and}\quad H(e^{j\Omega})=H(e^{j(\Omega-\pi)}).$$

Suppose that the input $x[n]=(\frac{1}{4})^n u[n]$ yields the output $y[n]=0$ for $n<0$ and $n\geq 2$. Note that the output $y[n]$ is not known for $n=0,1$. Find $h[n]$.

13. Compute the Fourier transform of each of the following in terms of the Fourier transform of $x[n]$ (i.e., $X(e^{j\Omega})$):

(a) $x[2-n]+x[-2-n]$

(b) $\frac{1}{3}(x^*[-n]+x[n])$

(c) $(n-2)^2 x[n+1]$

(d) $(nx[n])*x[n-1]$

14. Consider a causal and stable LTI system that yields the output $n(\frac{2}{3})^n u[n]$ for the input $(\frac{2}{3})^n u[n]$. Using the Fourier transform and the inverse Fourier transform, compute each of the following:

(a) The difference equation relating the input $x[n]$ and the output $y[n]$.

(b) The frequency response of the system, $H(e^{j\Omega})$.

(c) The impulse response of the system, $h[n]$.

15. Suppose that $x[n]$ is a signal with the Fourier transform

$$X(e^{j\Omega}) = \begin{cases} 1 + \frac{2\Omega}{\pi}, & -\frac{\pi}{2} < \Omega < 0 \\ 1 - \frac{2\Omega}{\pi}, & 0 < \Omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\Omega| < \pi \end{cases}, \quad \text{with } \Omega_0 = 2\pi.$$

Compute the Fourier transform of the output $y[n]$ of each of the following systems for the input $x[n]$:

(a) $y[n] = x[n]\sin(\pi n)$

(b) $y[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-k]$

(c) $y[n] = x[n]\cos(3\pi n/2)$

(d) $y[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-3k]$

Chapter 9

Sampling

We have alluded to sampling throughout this course and we are finally ready to discuss sampling in more detail. We will begin with a definition of what a bandlimited signal is.

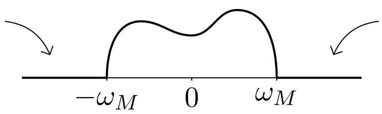
9.1 Bandwidth and Bandlimited Signals

A continuous time signal $x(t)$ is said to be bandlimited to $[-\omega_M, \omega_M]$ if

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M \tag{9.1}$$

An example of such an $X(j\omega)$ is shown below. Sometimes, the terminology is abused and we say $X(j\omega)$ is bandlimited. This should not cause too much of confusion since regardless of whether we say $x(t)$ is bandlimited or $X(j\omega)$ is bandlimited, mathematically what we mean is given in (9.1).

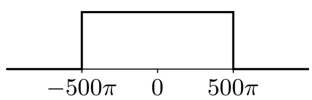
The FT of $x(t)$ is zero if $\omega < -\omega_M$



The FT of $x(t)$ is zero if $\omega > \omega_M$

Example 9.1.1.

$$x(t) = \frac{\sin 500\pi t}{\pi t} \leftrightarrow X(j\omega)$$



We say that $x(t)$ is bandlimited to $[-500\pi, 500\pi]$

Example 9.1.2. *Many signals that occur in nature are bandlimited or can be approximated very well by bandlimited signals. For example, human voice is bandlimited to [-20000 Hz, 20000 Hz] or $[-40000\pi, 40000\pi]$ rad/s.*

Example 9.1.3. $x(t) = \dfrac{\sin 3\pi t}{\pi t}$ *is bandlimited to* $[-3\pi, 3\pi)$.

Example 9.1.4. $x(t) = b_0 \dfrac{\sin 3\pi t}{\pi t} + b_1 \dfrac{\sin 3\pi(t-1)}{\pi(t-1)}$ *is also bandlimited to* $[-3\pi, 3\pi)$.

Example 9.1.5. *More generally,* $x(t) = \sum_{k=-\infty}^{\infty} b_k \frac{\sin(\omega_0(t-kT_s))}{\pi(t-1)}$ *is bandlimited to* $[-\omega_0, \omega_0)$ *for any* b_k *and* T_s.

9.2 Review of Fourier transform of periodic signals

We can compute the Fourier transform of periodic signals even though they do not have finite energy. In this case, the Fourier transform has impulses and hence, it is a generalized function. Recall that a generalized function is a function that is undefined for some values of t. The Fourier transform in this case is not defined at the points where the delta functions occur, but we can understand that there is non-zero energy in those frequencies.

Let $x(t)$ be a periodic signal with fundamental time period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Then $x(t)$ has a Fourier series representation. We can take the Fourier transform of both sides to get $X(j\omega)$, i.e.

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} && (9.2) \\ X(j\omega) &= \sum_{k=-\infty}^{\infty} X[k] 2\pi \delta(\omega - k\omega_0) && (9.3) \\ &= \sum_{k=-\infty}^{\infty} (2\pi X[k]) \delta(\omega - k\omega_0) && (9.4) \end{aligned}$$

To compute the Fourier transform a periodic signal, first compute the fundamental frequency ω_0 and the Fourier series coefficients, $X[k]$. Then, the Fourier transform is given by

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta(\omega - k\omega_0)$$

$X(j\omega)$ being a sum of delta functions means that the signal contains only discrete frequency components

Example 9.2.1. *Compute the Fourier transform of* $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

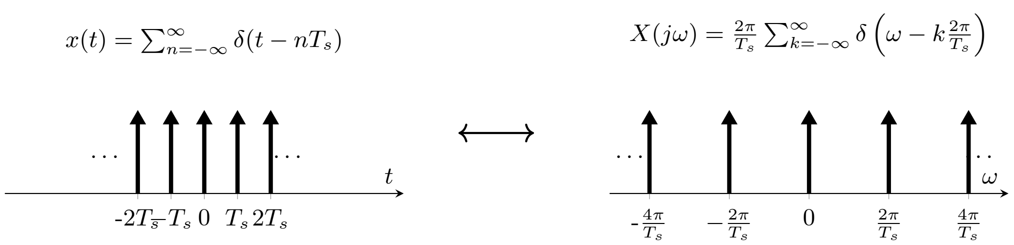


Figure 9.1:

Since the fundamental time period is T_s, the fundamental frequency is $\omega_0 = \frac{2\pi}{T_s}$. Now, we can compute $X[k]$ according to

$$X[k] = \frac{1}{T_s}\int_{\frac{-T_s}{2}}^{\frac{T_s}{2}} \delta(t)e^{-j\omega t}\,dt = \frac{1}{T_s}$$

Therefore,

$$X(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_0) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

9.3 Sampling

Sampling is the process of taking measurements or observations of $x(t)$ at periodic intervals. We observe the signal $x(t)$ only at $t = \ldots, -3T, -2T, 0, T, 2T, 3T, \ldots, nT, \ldots$ where

- $x(nT)$ is called the sample taken at time nT
- T is called the sampling time
- $f_s = \frac{1}{T}$ is called the sampling rate or sampling frequency in Hz
- $\omega_s = \frac{2\pi}{T}$ is called the sampling rate on sampling frequency in rad/s

In this section, we will study under what conditions we can recover the signal $x(t)$ given on the samples of $\{x(nT)\}$. Consider the signal $x_1(t)$ shown in Fig. 9.2. This signal is sampled at time instants $-T, 0, T, \ldots$. Is it possible to uniquely recover $x_1(t)$ from the samples? In general, the answer to this question is indeed no. In the figure, we can see that $x_2(t)$ and $x_3(t)$ also have exactly the same values as $x_1(t)$ at the sampling instants and hence, there is no way for the reconstruction algorithm to decide whether original signal was $x_1(t), x_2(t)$ or $x_3(t)$. What if we put some restriction on the class of signals that $x_1(t)$ must belong to for us to be able to uniquely reconstruct it? This is exactly where the class of bandlimited signals comes in and the main result of sampling is the Nyquist sampling theorem stated below.

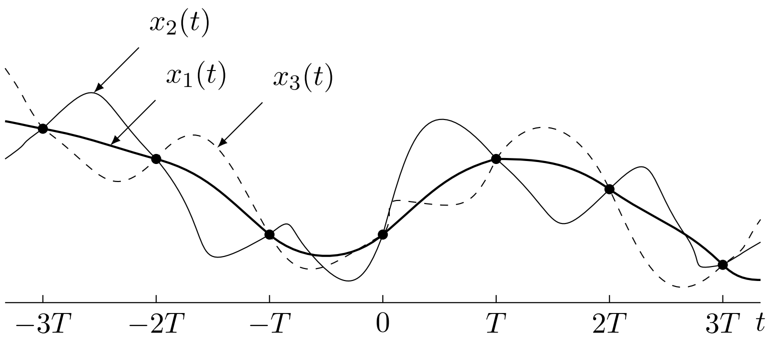


Figure 9.2: Three continuous-time signals with identical values at integer multiples of T.

Nyquist sampling theorem and Nyquist rate

Let $x(t)$ be a signal that is band-limited signal to $[-\omega_M, \omega_M)$ i.e., $X(j\omega) = 0$ for $|\omega| > \omega_M$, Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M, where\ \omega_s = \frac{2\pi}{T}.$$

Notice that $\omega_s > 2\omega_M \Rightarrow \frac{2\pi}{T} > 2\omega_M \ \Rightarrow T < \frac{\pi}{\omega_M}$.
$2\omega_M$ is called the Nyquist sampling rate. This theorem goes by several names - Shannon-Nyquist sampling theorem, Shannon-Nyquist-Whitaker sampling theorem, Shannon-Nyquist-Kotelnikov theorem etc.

Example 9.3.1. *Suppose $x(t)$ is bandlimited to [−100 Hz, 100 Hz], $\omega_M = 200\pi$ rad/s*

$$T_s = \frac{2\pi}{2\omega_M} = \frac{\pi}{\omega_M} = \frac{1}{200} \ seconds.$$

We say that the Nyquist rate is 400π rad/s. What this means is that the minimum sampling rate required to reconsturct $x(t)$ from its samples is 400π rad/s and the maximum sampling time we can tolerate is $1/200$ seconds.

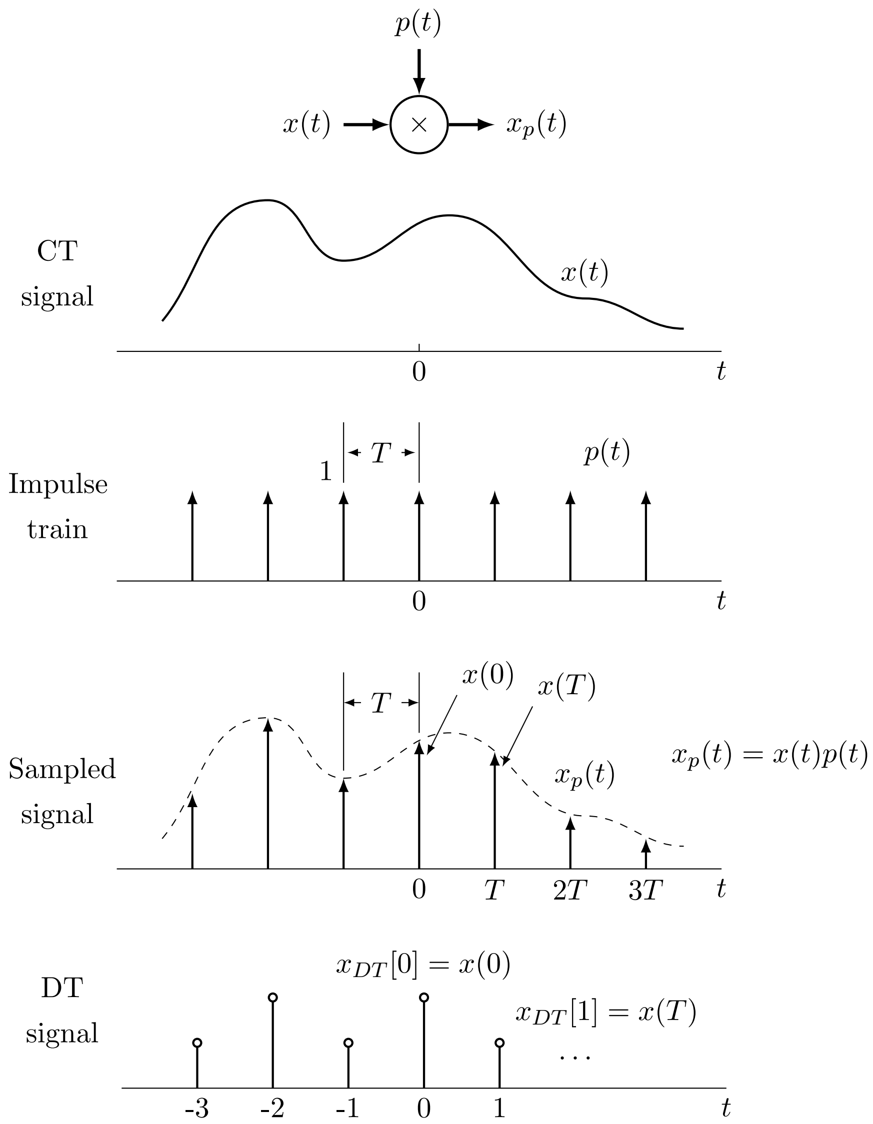
Example 9.3.2. *Human's can hear frequencies only upto 20 KHz or 40000π rds/s. All music and voice intended for human consumption can be low pass filtered to 40000π rads/s and hence, are strictly bandlimited. When music CDs are made, the continuous-time music waveform is sampled at 44.1 KHz to produce 44,100 samples/s. Why does it suffice to sample it at 44.1 KHz? is the main question we will try to answer in this section.*

Example 9.3.3. *How much information can a CD hold?*

How to reconstruct a CT signal from its samples?

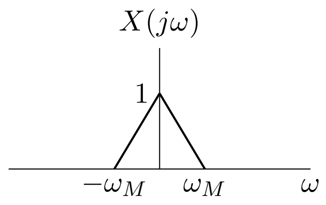
Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$ - for example $\omega_s/2$. The resulting output signal will exactly equal $x(t)$.

We now discuss a proof of the Nyquist theorem and how to reconstruct $x(t)$ from its samples.



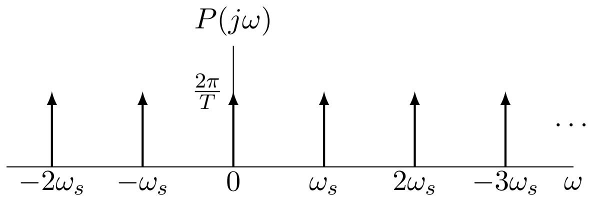
Let us recall a few preliminaries before we discuss the proof

- $X(j\omega) * \delta(\omega - \omega_s) = X(j(\omega - \omega_s))$
- $X(j\omega) * [\cdots + \delta(\omega + \omega_s) + \delta(\omega) + \delta(\omega - \omega_s) + \dots] = \cdots + X(j(\omega + \omega_s)) + X(j\omega) + S(j(\omega - \omega_s)) + \dots$
- Reconstructing $x(t)$ from the samples is the same as reconstructing $X(jw)$



$x_p(t) = x(t)p(t)$

$Xp(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega)$



We begin by noting that since $x_p(t) = x(t)p(t)$,

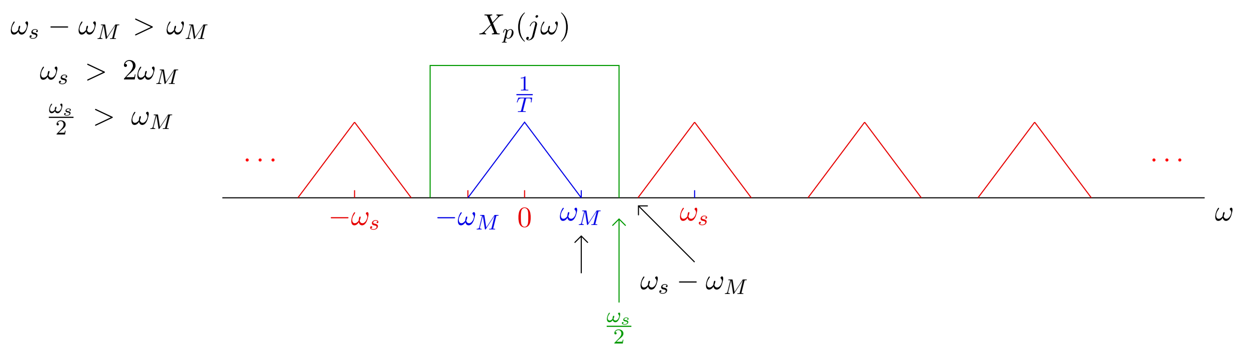
$$X_p(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega) = \frac{1}{2\pi}X(j\omega) * \frac{1}{T}\sum_{l=-\infty}^{\infty}\delta(\omega - l\omega_s) = \sum_{l=-\infty}^{\infty}X(j(\omega - l\omega_s))$$

Thus, $X_p(j\omega)$ is the sum of many shifted replicas of $X(j\omega)$. We consider two cases separately to get more insight.

Case 1

$$\omega_s - \omega_M > \omega_M \quad \Rightarrow \quad \boxed{\omega_s > 2\omega_M} \quad \leftarrow \quad \text{Nyquist frequency}$$

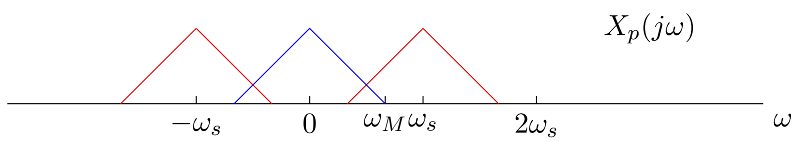
In this case, there is no overlap between the replicas. We can recover $X(j\omega)$



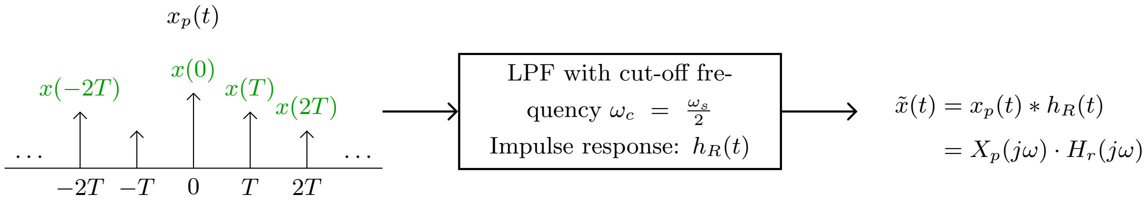
Case 2

$$\omega_s - \omega_M < \omega_M \quad \Rightarrow \quad \omega_s < 2\omega_M$$

There will be overlap between the replicas, we cannot recover $X(j\omega)$



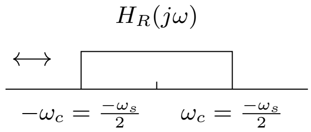
Reconstruction - the frequency-domain view To reconstruct $x(t)$, we reconstruct $X(j\omega)$ from $X_p(j\omega)$. This can be performed by low pass filtering $x_p(t)$ with a low pass filter with cut-off frequency $\omega_s/2$ and hence, with an impulse response $h_T(t) = \text{sinc}\left(\frac{t}{T}\right)$.



$$h_R(t) = T \cdot \frac{\sin \frac{\omega_s}{2} t}{\pi t}$$

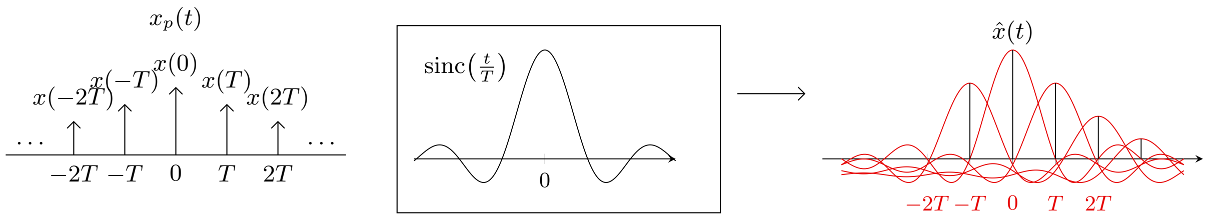
$$= \frac{T\omega_s}{\pi \cdot 2} \frac{\sin \frac{\omega_s}{2} t}{\frac{\omega_s}{2} t}$$

$$h_R(t) = \text{sinc}\left(\frac{\omega_s t}{2\pi}\right)$$

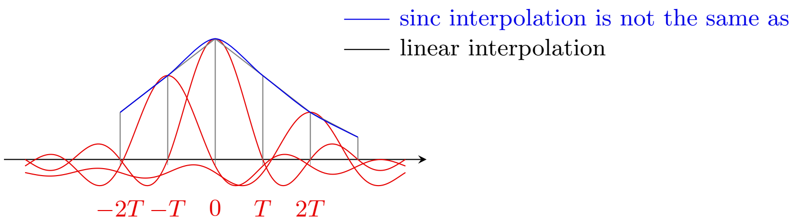


Notice that $\omega_s = \dfrac{2\pi}{T}$ and therefore, $\dfrac{\omega_s}{2\pi} = \dfrac{1}{T}$. Hence, $\boxed{h_R(t) = \text{sinc}\left(\dfrac{t}{T}\right) = \text{sinc}\left(\dfrac{\omega_s t}{2\pi}\right)}$

Reconstruction as interpolation: the time-domain view The operations to be performed in the time domain for reconstructing the signal $x(t)$ can be obtained by realizing that low pass filtering a signal is the same as convolving the signal with the impulse response of a low pass filter, which is a sinc function as shown below.



The operation of convolution is equivalent to interpolating the sampled signal using a sinc function as shown below. This tells us a very interesting fact about interpolating the discrete samples to reconstruct the original signal. If the original signal is bandlimited, the correct way to interpolate between the samples is to use a sinc function. Would you have guessed that before taking this course?



$$\tilde{x}(t) = \left(\sum_{n=\infty}^{\infty} x(nT)\delta(t-nT)\right) * \operatorname{sinc}\left(\frac{w_c t}{\pi}\right) = \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}\left(\frac{w_c(t-nT)}{\pi}\right)$$

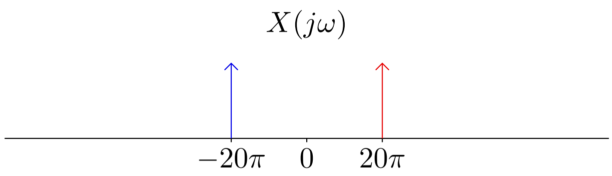
Since $w_c = \frac{w_s}{2} = \frac{\pi}{T}$ or, equivalently, $\frac{w_c T}{\pi} = 1$, therefore

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}\left(\frac{t}{T} - n\right) = \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}\left(\frac{t-nT}{T}\right)$$

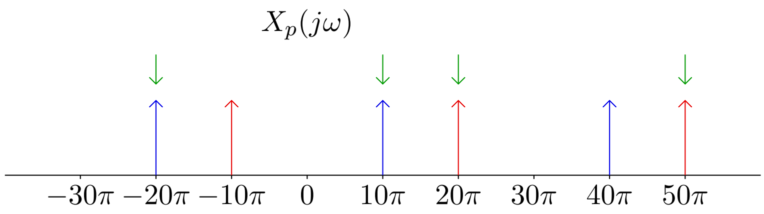
9.4 Aliasing

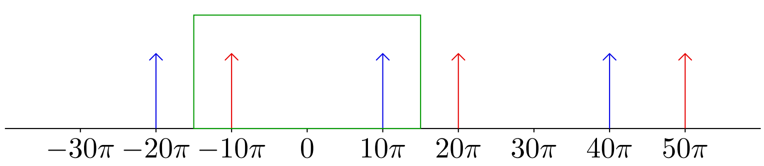
Question What happens if $\omega_s, 2\omega_M$? But we didn't know this and we tried to reconstruct $x(t)$ by passing $x_p(t)$ through an ideal LPF with cutoff frequency $\frac{\omega_s}{2}$? How would our reconstructed signal $\hat{x}(t)$ be related to $x(t)$?

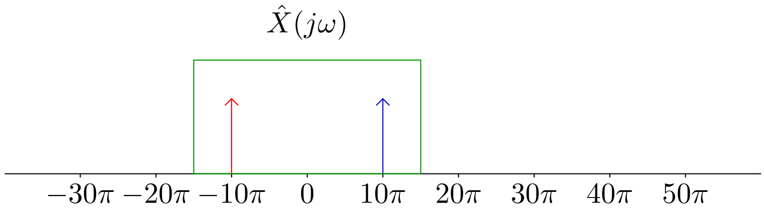
To understand this, let us consider a signal $x(t) = \cos(20\pi t)$ with $\omega_M = 20\pi$. The Fourier transform of the signal is given below.



Nyquist theorem tells us that ω_s must be $> 2\omega_M = 40\pi$. Suppose we violate the theorem and let us say we choose $\omega_s = 30\pi$ and reconstruct with a low pass filter with cut off frequency, $w_c = w_s/2 = 15\pi$. Then $X_p(j\omega)$ and $\hat{X}(j\omega)$ are given below.

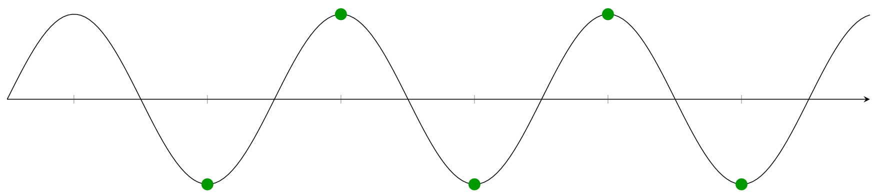




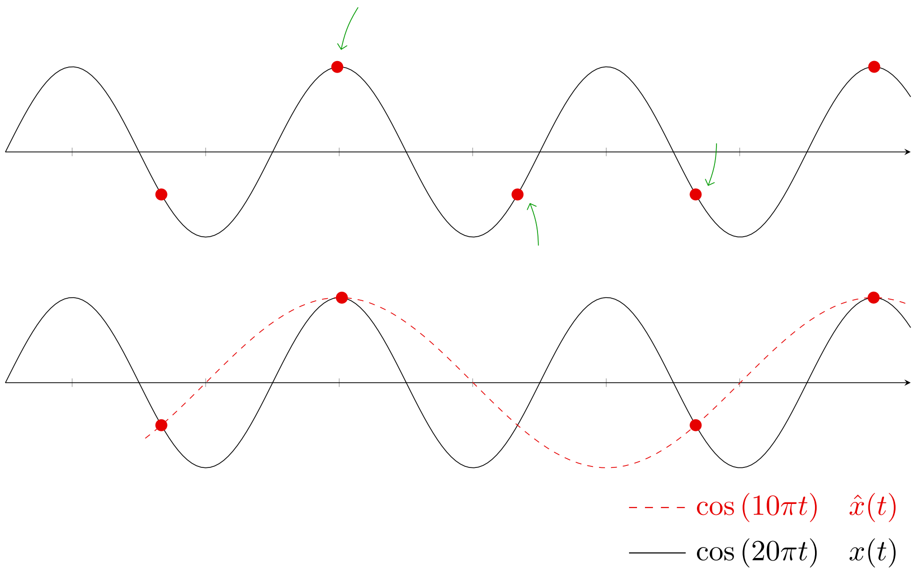


Therefore, our reconstruction $\hat{x}(t) = \cos(10\pi t)$. Thus, when under sampled, the cosine signal with frequency 20π appears or aliases itself as a cosine signal with frequency 10π!

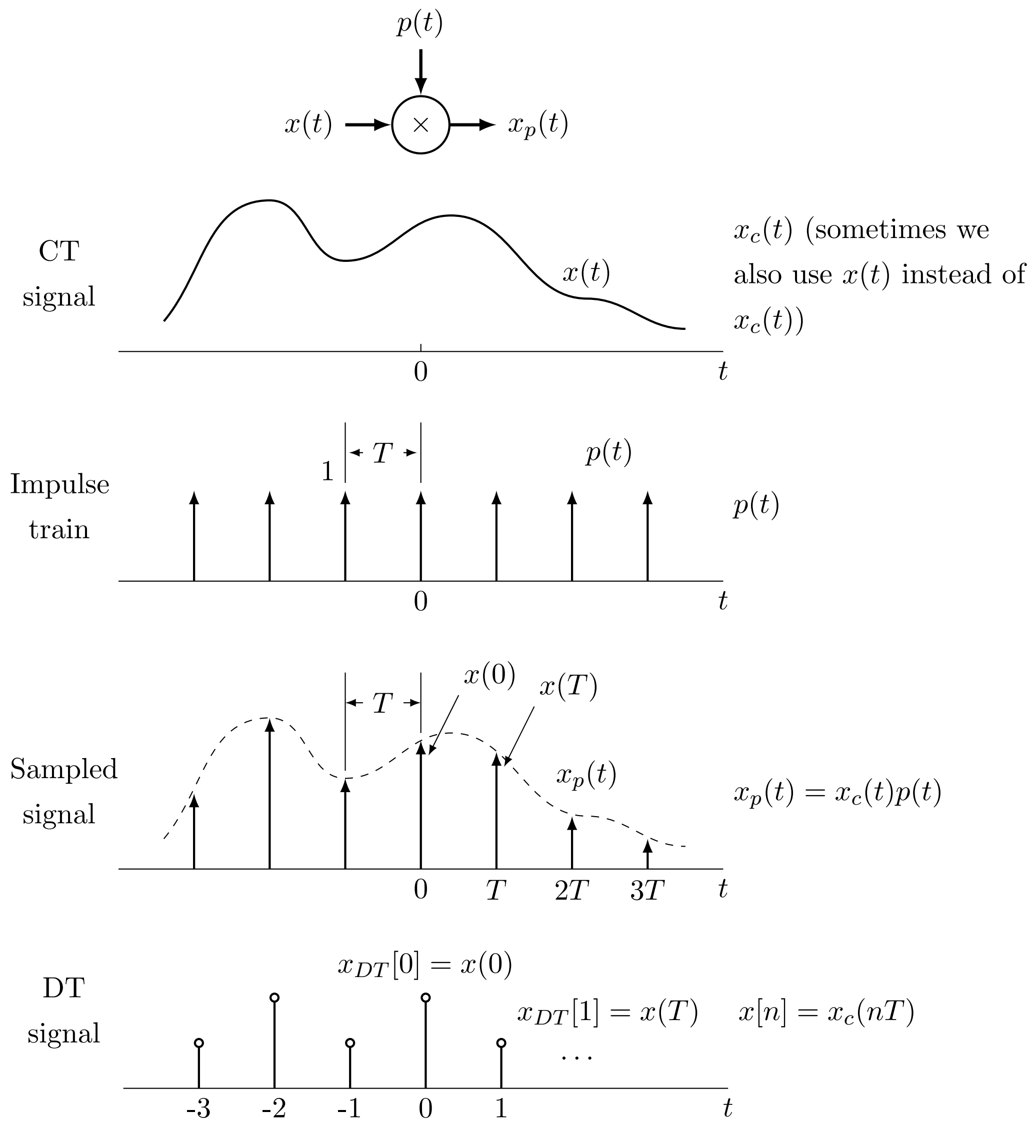
It is instructive to understand this phenomenon from the time domain as well. If $\omega_s = 2\omega_M$, then we are taking 2 samples per cycle



If $\omega_s < 2\omega_M$, say $\omega_s = \frac{3}{2}\omega_M$, we are taking only 1.5 samples per cycle.



9.5 Discrete-time processing of continuous-time signals



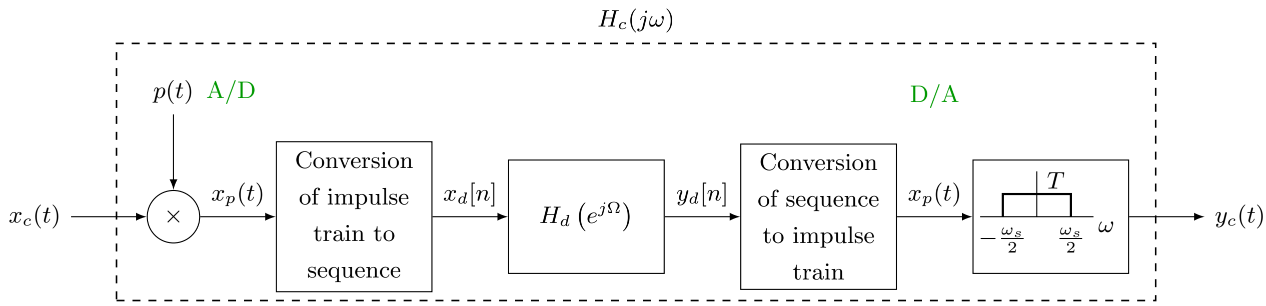


Figure 9.3: Overall system for filtering a continuous-time signal using a discrete time filter.

The relationship between between frequency ω in the CTFT and the frequency Ω in the DTFT of the DT signal obtained by sampling is important to understand. Notice that ω_s in the CTFT gets mapped to 2π in the DTFT and more generally frequency ω in the CTFT gets mapped to $\Omega = \frac{2\pi}{\omega_s}\omega$ in the DTFT.

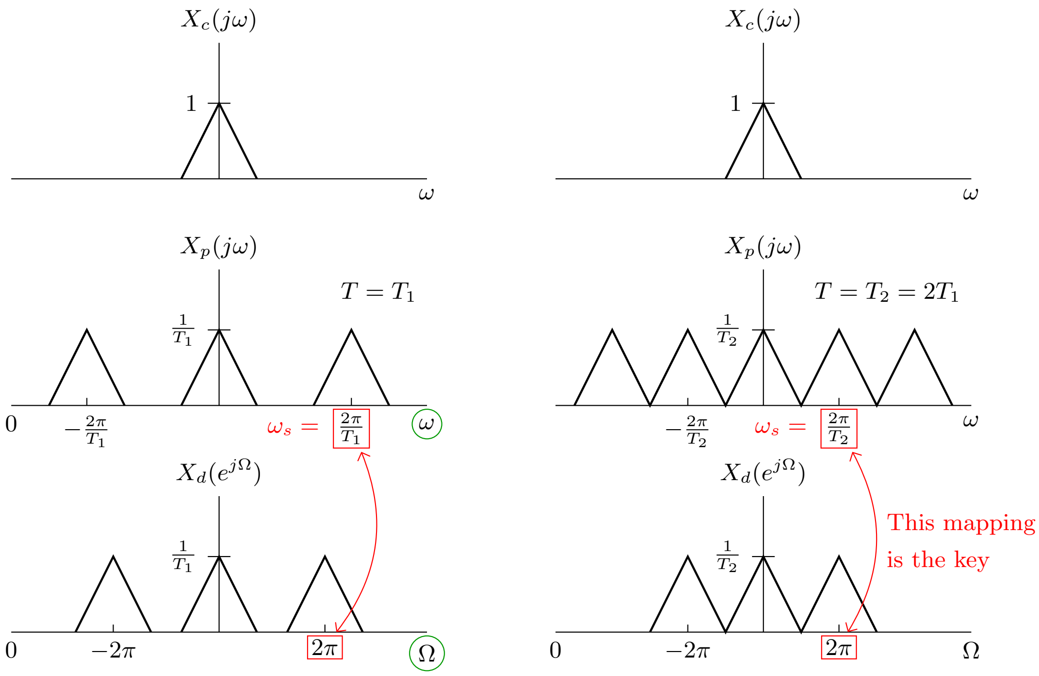
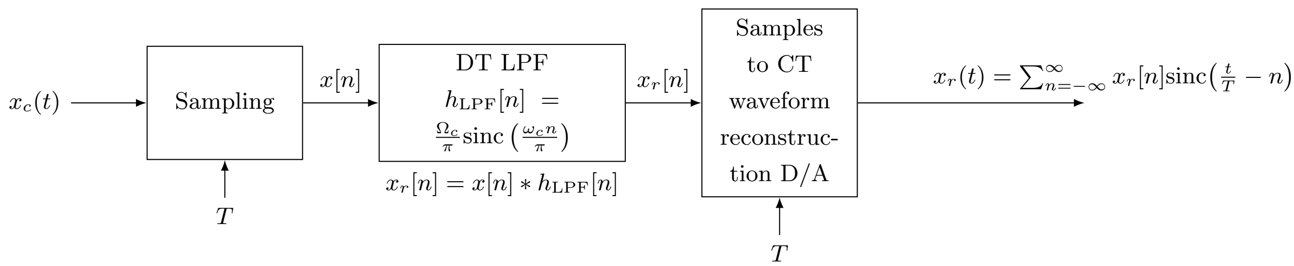


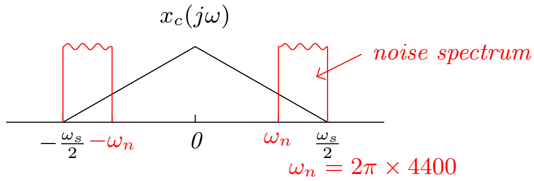
Figure 9.4: Relationship between $X_c(j\omega), X_p(j\omega), and X_d(e^{j\Omega})$ for two different sampling rates.

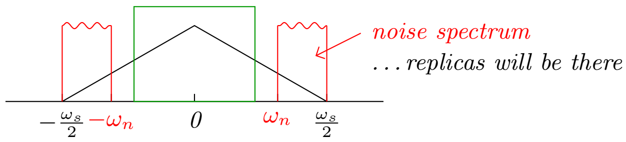
If we want to design a DT LPF with cutoff frequency Ω_0 to archive the effect of low-pass filtering the CT signal with a cutoff frequency of ω_c, we must choose $\boxed{\Omega_c = \frac{2\pi}{\omega_s}\omega_c}$ and the impulse response of such a filter will be given by

$$h_{\mathrm{LPF}}[n] = \frac{\sin \Omega_c n}{\pi n} = \frac{\Omega_c}{\pi}\mathrm{sinc}\left(\frac{\Omega_c n}{\pi}\right)$$



Example 9.5.1. *Suppose $x_c(t)$ is an audio recording which is sampled at 11025 Hz and hence, $T = \frac{1}{11025}s$ $\quad \omega_s = 2\pi \times 11025$ rad/s $f_s = 11025$ Hz. Suppose some noise gets added to the recording but the noise occupies only the part of the spectrum above 4400Hz and less than -4400Hz. One way to filter the noise is to pass the noisy signal through a low pass filter with cut-off frequency 4400 Hz. We wish to filter the sampled DT signal instead of filtering the CT signal. What should be the cut-off frequency of the DT filter?*





$$\Omega_c = \frac{2\pi}{\omega_s} \times \omega_c = 2\pi \times \frac{4000}{11025}.$$