Exercise 2

Matrix	Basis	Mapping	Dimension	Transformation
U	XX^T	$\tilde{W} = U$	$n \times n$	rotation
Σ	X^TX, XX^T	$D = \frac{\Sigma^2}{n-1}$	$n \times d$	scaling
V	X^TX	$W = V$	$d \times d$	rotation

1. a) $X = U\Sigma V^T$ — (i)

$$C = \frac{1}{n-1} X^T X = WDW^T \quad \text{— (ii)}$$

$$\tilde{C} = \frac{1}{n-1} X X^T = \tilde{W}\tilde{D}\tilde{W}^T \quad \text{— (iii)}$$

~~$U = XX^T$~~

U has eigenvectors of XX^T

$$\therefore \tilde{C} = \frac{1}{n-1} XX^T = \frac{1}{n-1} U\Sigma V^T \cdot (U\Sigma V^T)^T \quad \text{(Using equation (i))}$$

$$= \frac{1}{n-1} U\Sigma V^T \cdot V \Sigma^T U^T = \frac{1}{n-1} U \Sigma^2 \cdot U^T \quad \text{(Since } V \text{ is orthogonal} \therefore V^T V = 1)$$

Since Σ is a digonal matrix. $\therefore \Sigma\Sigma^T = \Sigma^2$

$$= U \frac{\Sigma^2}{n-1} \cdot U^T = \tilde{W}\tilde{D}\tilde{W}^T$$

$\left(\text{Given } \tilde{D} = \frac{\Sigma^2}{n-1}\right) \quad \therefore \tilde{W} = U$

V has eigenvectors of X^TX

$$C = \frac{1}{n} X^T X = \frac{1}{n-1} (U\Sigma V^T)^T \cdot U\Sigma V^T \quad \text{(Using eq. (i))}$$

$$= \frac{1}{n-1} V \Sigma^T U^T U \Sigma V^T = \frac{1}{n-1} V \cdot \Sigma^2 \cdot V^T$$

(Same reason as before, U is orthogonal and Σ is digonal)

$$= V \cdot \frac{\Sigma^2}{n-1} \cdot V^T = WDW^T$$

$\therefore W = V$

b) $D = \frac{\Sigma^2}{n-1}$ (D has eigen ~~vectors~~ values & Σ has singular values ~~& (diagonal matrix)~~)

$$\Sigma = \sqrt{n-1\, D}$$

$\therefore s_i = \sqrt{(n-1)\lambda_i}$ (Since D & Σ are digonal matrix)

c) $U: n \times n$ $V = d \times d$

$\Sigma: n \times d$

d) U and V are orthogonal hence, U & V has relation as affine transformation.
Σ is a digonal matrix, thus scaling is its affine transformation.

Exercise 3:

6.

Technique	Supervision	Focus	Projection dim
PCA	unsupervised	variance within data	d
LDA	supervised	discriminate classes	c − 1 (where c is class label)