Week 3 Problem Set 2

Justin Hink

Step 1: Determine Eigenvalues

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

$$det(\lambda \mathrm{I} - \mathrm{A}) = 0$$
$$Where\ I\ is\ the\ identity\ matrix$$

$$det\left(\begin{bmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & 5 \\ 0 & 0 & \lambda - 6 \end{bmatrix}\right) = 0$$

The 3x3 determinant can be rewritten as:

$$(\lambda - 1)det\left(\begin{bmatrix} \lambda - 4 & -5 \\ 0 & \lambda - 6 \end{bmatrix}\right) - (-2)det\left(\begin{bmatrix} 0 & -5 \\ 0 & \lambda - 6 \end{bmatrix}\right) + (-3)det\left(\begin{bmatrix} 0 & \lambda - 4 \\ 0 & 0 \end{bmatrix}\right) = 0$$

$$(\lambda - 1)[(\lambda - 4)(\lambda - 6) - 0] + 2(0) - 3(0) = 0$$

Expanding this out, we get our characteristic polynomial:

$$p(\lambda) = \lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0$$

Factoring this (trivial from the preceding step), we can easily obtain our eigenvalues.

$$(\lambda - 1)(\lambda - 4)(\lambda - 6) = 0$$

$$\lambda = 1$$
$$\lambda = 4$$
$$\lambda = 6$$

Step 2: Determine Eigenvalues

Generally speaking:

$$(\lambda I - A)\vec{v} = \vec{0}$$

We have 3 eigenvalues that each correspond to an eigenvector.

Case 1: $\lambda = 1$

$$\begin{bmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & -5 \\ 0 & 0 & \lambda - 6 \end{bmatrix} \vec{v} = 0$$

$$\begin{bmatrix} 0 & -2 & -3 \\ 0 & -3 & -5 \\ 0 & 0 & -5 \end{bmatrix} \vec{v} = 0$$

Convert to reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$v_1 = t$$
$$v_2 = 0$$
$$v_3 = 0$$

Our eigenvector for $\lambda = 1$ is:

$$V = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad t \in R \right\}$$

Case 1: $\lambda = 4$

$$\begin{bmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & -5 \\ 0 & 0 & \lambda - 6 \end{bmatrix} \vec{v} = 0$$

$$\begin{bmatrix} 3 & -2 & -3 \\ 0 & 0 & -5 \\ 0 & 0 & -2 \end{bmatrix} \vec{v} = 0$$

Convert to reduced row echelon form:

$$\begin{bmatrix} 1 & {}^{2}/_{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$v_1 + {}^{2}/_{3}\, v_2 = 0 \rightarrow v_1 = -{}^{2}/_{3}\, v_2$$
$$v_3 = 0$$

Our eigenvector for $\lambda = 4$ is:

$$V = \left\{ v_2 \begin{bmatrix} -{}^{2}/_{3} \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let $v_2 = 1$:

$$V = \begin{bmatrix} -{}^{2}/_{3} \\ 1 \\ 0 \end{bmatrix}$$

Case 1: $\lambda = 6$

$$\begin{bmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & -5 \\ 0 & 0 & \lambda - 6 \end{bmatrix} \vec{v} = 0$$

$$\begin{bmatrix} 5 & -2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = 0$$

Convert to reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & {}^{8}\!/_{5} \\ 0 & 1 & -{}^{5}\!/_{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$v_1 + {}^{8}\!/_{5}\, v_3 = 0 \;\rightarrow\; v_1 = -{}^{8}\!/_{5}\, v_3$$
$$v_2 - {}^{5}\!/_{2}\, v_3 = 0 \;\rightarrow\; v_2 = {}^{5}\!/_{2}\, v_3$$

Our eigenvector for $\lambda = 6$ is:

$$V = \left\{ v_3 \begin{bmatrix} -{}^{8}\!/_{5} \\ {}^{5}\!/_{2} \\ 0 \end{bmatrix} \right\}$$

Let $v_3 = 1$:

$$V = \begin{bmatrix} -{}^{8}\!/_{5} \\ {}^{5}\!/_{2} \\ 0 \end{bmatrix}$$