LectureNote2: CS221 (FALL2021, Stanford)

Linear Regression

The discovery of Ceres

- 1801: astronomer Piazzi discovered Ceres, made 19 observations of location before it was obscured by the sun
- September 1801: Gauss took Piazzi's data and created a model of Ceres's orbit, makes prediction
- December 7, 1801: Ceres located within 1/2 degree of Gauss's prediction, much more accurate than other astronomers
  - METHOD: Least squares linear regression

Linear Regression Framework

$D_{train} - learningalgorithm \rightarrow prediction[f]$

x	y
1	1
2	3
4	3

- Design decisions:
  1. Which predictors are pssible? *hypothesis class*
  2. How good is a predictor? *loss function*
  3. How do we compute the the best predictor? *optimization algorithm*

- Hypothesis Class: which predictors?
  - $f(x) = w_1 + w_2 x$
  - Vector Notation
    $\vec{w}_{weight\ vector} = [w_1, w_2]$
    $\phi_{feature\ extractor} = [1, x]_{feature\ vector}$
    $f_{\vec{w}}(x) = \vec{w} \cdot \phi(x)$

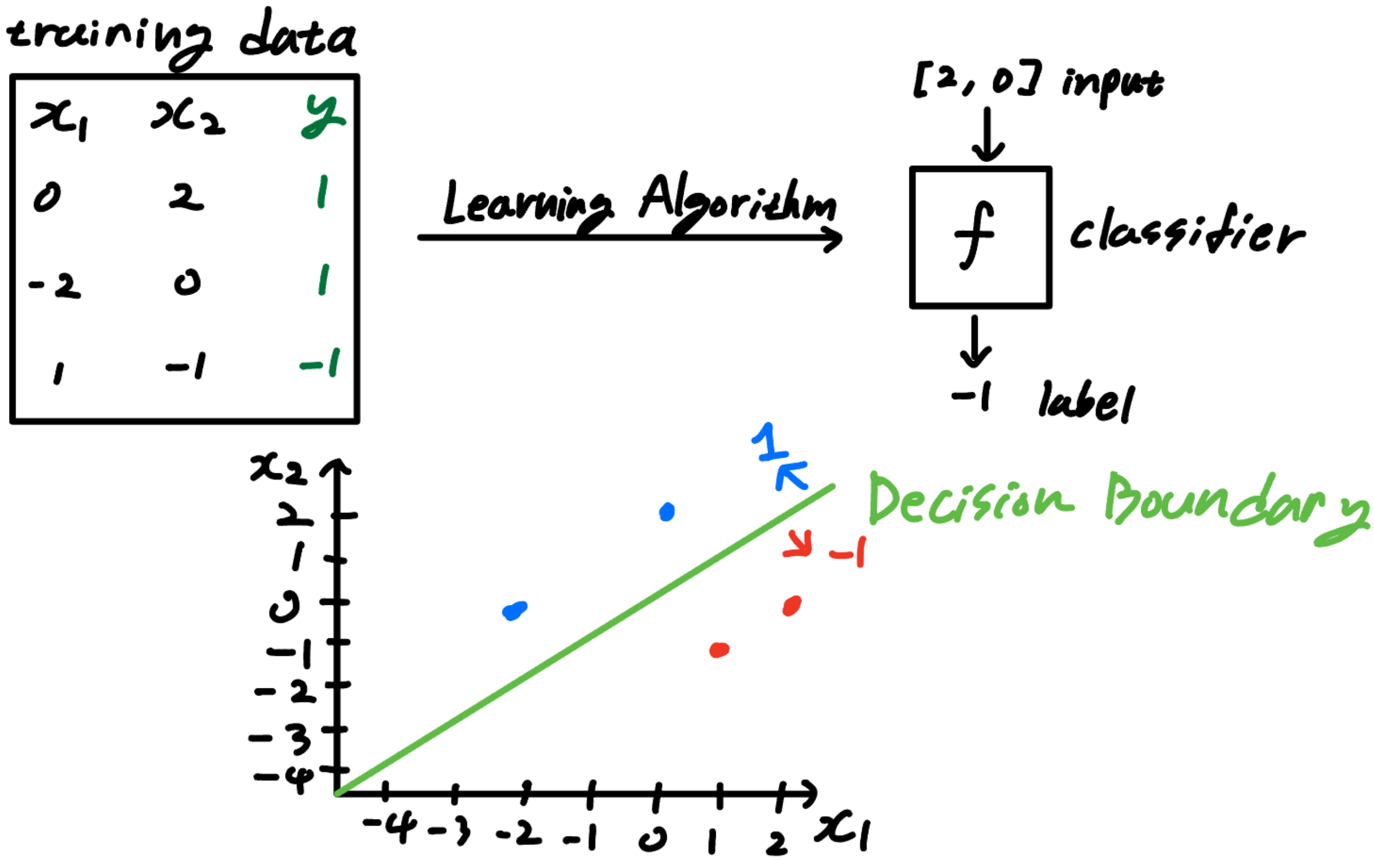
- Hypothesis Class: $F = \{\ f_{\vec{w}}(x) : \vec{w} \in \mathbb{R}^2\ \}$

- Loss Function: how good is a predictor?
  - $Loss(x, y, \vec{w}) = (f_{\vec{w}}(x) - y)^2{}_{\ squared\ loss}$
    - residual: $|f_{\vec{w}}(x) - y|$
  - $TrainLoss(\vec{w}) = \frac{1}{|D_{train}|} \sum_{(x,y) \in D_{train}} Loss(x, y, \vec{w})$

- Optimization Algorithm: how to compute best?
  - Goal: $min_{\vec{w}}\ TrainLoss(\vec{w})$
  - Definition: gradient
    - The gradient $\nabla_{\vec{w}} TrainLoss(\vec{w})$ is the direction that increases the training loss the most
  - Algorithm: gradient descent
    - Initialize $\vec{w}$ = [0, ... , 0]
    - For t = 1, ..., T: epochs
      - $\vec{w} \leftarrow \vec{w} - \eta_{\ step\ size} \nabla_{\vec{w}} TrainLoss(\vec{w})$
    - initial weight = 0
    - we want to find the weight vector that minimizes the value of TrainLoss function
    - update weghit vector by going to the direction that decreases the TrainLoss function's value by substracting gradient

- Computing the gradient
  - $\nabla_{\vec{w}} TrainLoss(\vec{w}) = \frac{1}{|D_{train}|} \sum_{(x,y) \in D_{train}} 2(\vec{w} \cdot \phi(x) - y)\ \phi(x)$ = $\frac{1}{|D_{train}|} \sum_{(x,y) \in D_{train}} 2(residual)\ \phi(x)$
  - if (prediction = target) $\longrightarrow$ gradient = 0
  - if gradient at the end of iteration be 0, it means the gradient descent has converged

Linear Classification



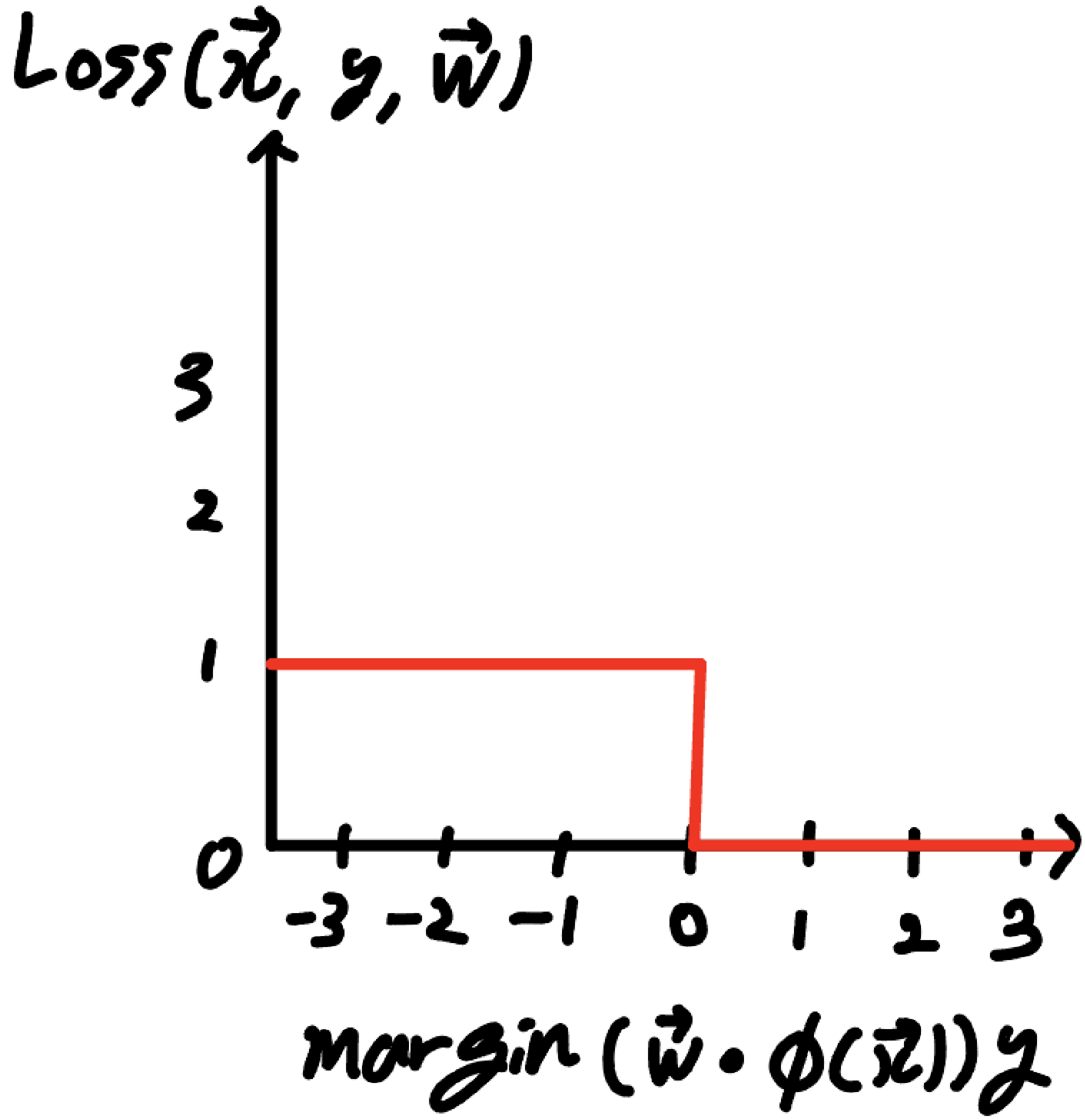
- Design decisions:
    i. Which classifiers are possible? *hypothesis class*
    ii. How good is a classfier? *loss function*
    iii. How do we compute the best classifier? *optimization algorithm*
- An example linear classifier
    - $f(\vec{x})$ = sign$([-0.6, 0.6] \cdot [x_1, x_2]\ )$ = sign($\vec{w} \cdot \phi(\vec{x})$)
    - sign(z) = $\begin{cases} +1 & (z > 0) \\ -1 & (z < 0) \\ 0 & (z = 0) \end{cases}$
    - if the angle between $\vec{w}$ and $\phi(\vec{x})$ is **acute**, the sign of the dot product of these two is +
    - if the angle between $\vec{w}$ and $\phi(\vec{x})$ is **obtuse**, the sign of the dot product of these two is -
    - if the angle between $\vec{w}$ and $\phi(\vec{x})$ is **perpendicular**, the dot product of these two is 0 $\rightarrow$ **Decision Boundary**
    - $\therefore$ linear classifier labels the inputs according to the angle of $\phi(\vec{x})_{input}$ with respect to $\vec{w}_{weight\ vector}$

- Hypothesis Class: which classifiers?
  - General binary classifier: $f_{\vec{w}}(\vec{x})$ = sign($\vec{w} \cdot \phi(x)$)
  - Hypothesis class: $F = \{f_{\vec{w}}(\vec{x}) : \vec{w} \in \mathbb{R}^2\}$

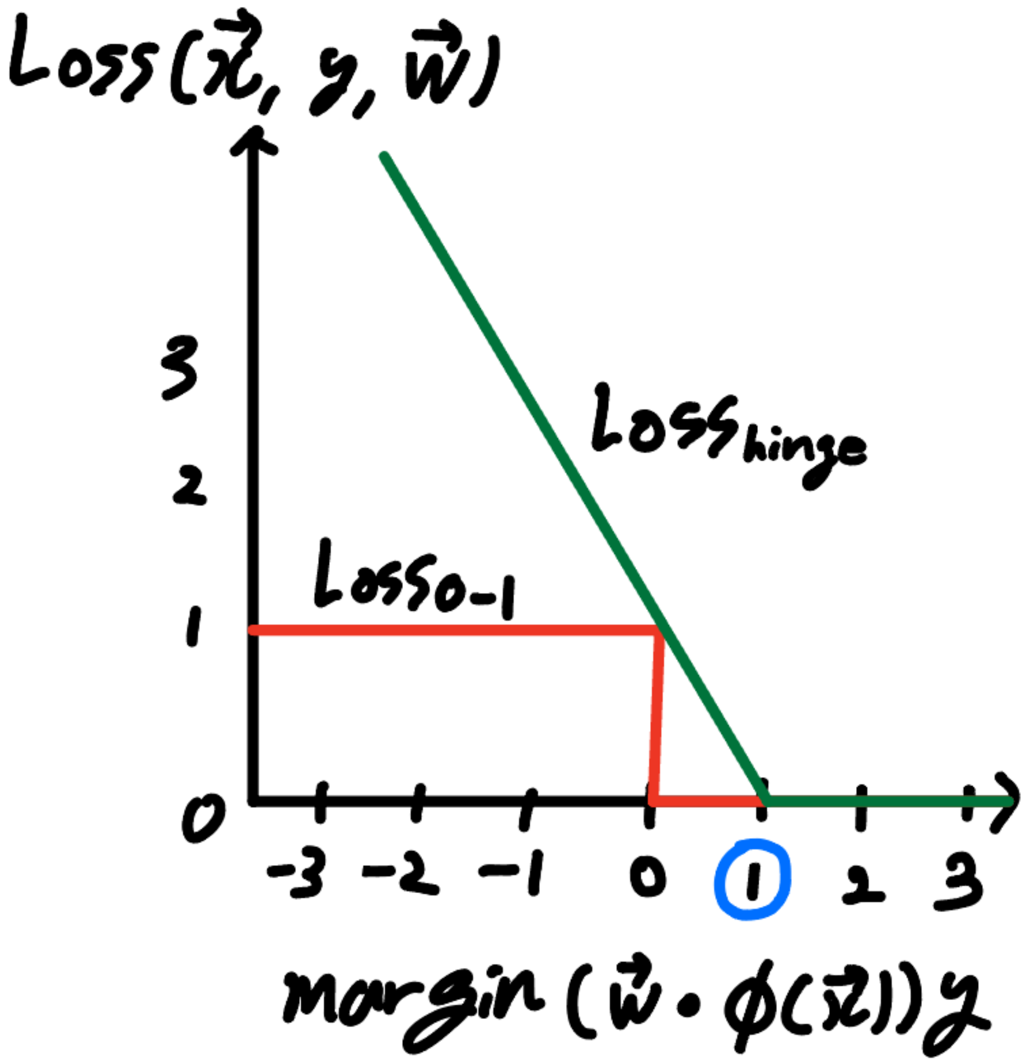
- Loss Function: how good is a classifier?
  - $Loss_{0-1}(\vec{x}, y, \vec{w}) = 1[f_{\vec{w}}(\vec{x}) \neq y]$ $_{zero-one\ loss}$
    - $Loss_{0-1}(\vec{x}, y, \vec{w})$ = $\begin{cases} 1 & (f_{\vec{w}}(\vec{x}) \neq y) \\ 0 & (f_{\vec{w}}(\vec{x}) = y) \end{cases}$

- Score and Margin
  - predicted label: $f_{\vec{w}}(\vec{x})$ = sign($\vec{w} \cdot \phi(\vec{x})$)
  - target label: y
  - Defition: score
    - The score on an example (x, y) is $\vec{w} \cdot \phi(\vec{x})$, how **confident** we are in predicting
    - *raw output of the model before applying a decision threshhold*
  - Definition: margin
    - The margin on an example (x, y) is $(\vec{w} \cdot \phi(\vec{x}))y$, how **correct** our prediction is
    - $positive \rightarrow correct$
    - $negative \rightarrow incorrect$
    - |margin| = distance from the decision boundary
  - $Loss_{0-1}(\vec{x}, y, \vec{w}) = 1[f_{\vec{w}}(\vec{x}) \neq y]$ = $1[margin \leq\ 0]$ = $\begin{cases} 1 & (margin \leq\ 0) \\ 0 & (margin >\ 0) \end{cases}$

  - Optimization Algorithm: how to compute best?
    - $\nabla_{\vec{w}} TrainLoss(\vec{w}) = \sum_{(x,y) \in D_{train}} \nabla Loss_{0-1}(\vec{x}, y, \vec{w})$
    - $\nabla Loss_{0-1}(\vec{x}, y, \vec{w})$ = $\nabla[(\vec{w} \cdot \phi(\vec{x}))\ y \leq\ 0]$
    - $\nabla Loss_{0-1}(\vec{x}, y, \vec{w})$ is zero almost everywhere! $\rightarrow$ we can't use gradient descent

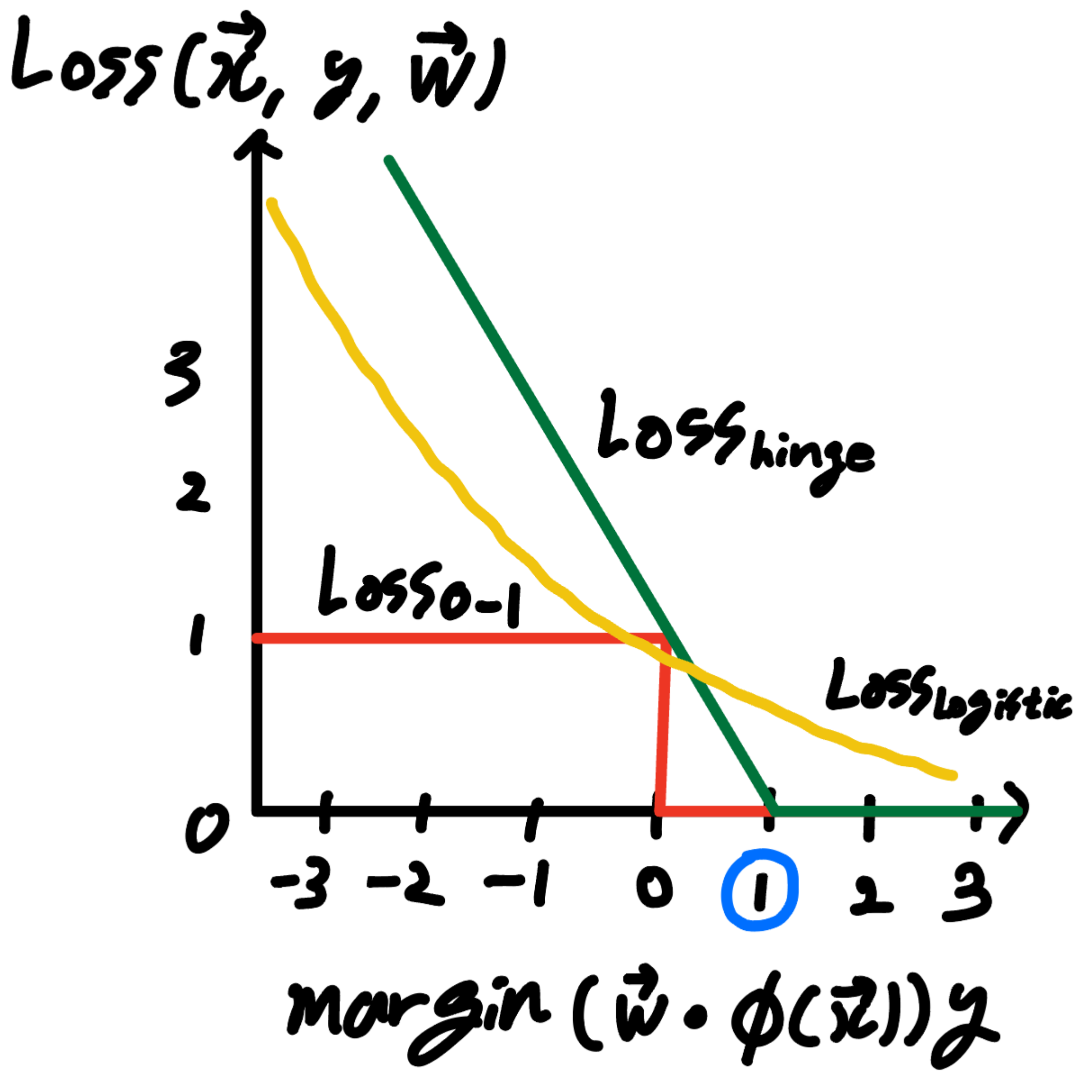


- Solutions
  a. Hinge loss
     - $Loss_{hinge}(\vec{x}, y, \vec{w}) = max\{1 - margin, 0\}$



- The 1 can be any positive number, and the magnitude determines the regularization strength
- Hinge loss is the upper bound on the zero-one loss
- $\nabla_{\vec{w}} Loss_{hinge}(\vec{x}, y, \vec{w})$ = $\begin{cases} -\phi(\vec{x})\ y & (1 > margin) \\ 0 & (otherwise) \end{cases}$
- *Even if an input is classified correctly, it can still generates loss if it didn't meet the threshhold (in this case, 1)*

b. Logistic Regression
- $Loss_{hinge}(\vec{x}, y, \vec{w}) = \log(1 + e^{-margin})$



- Intuition: try to increase margin even when it already exceeds 1

Summary

score: $\vec{w} \cdot \phi(x)$	Regression	Classification
Prediction $f_{\vec{w}}(x)$	score	sign of score
Relate to target y	residual (score $-\ y$)	margin (score $*\ y$)
Loss functions	Squared Absolute deviation	zero-one hinge logistic
Algorithm	gradient descent	