Dispersive modeling of breaking waves on a slope

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Abstract

The nonlinear shallow water model is widely used in the study of tsunami propagations, but an increasing number of studies are dedicated to the dispersion dynamics of tsunamis. If the wave dispersion becomes important, Boussinesq-type models are often used. In this work, an operational Boussinesq solver, BoussClaw, is introduced for modeling fully non-linear dispersive tsunami propagation, taking into account inundation. In the BoussClaw model, Boussinesq equations from the literature, based on the depth-averaged velocity and with enhanced dispersion properties, are implemented using a hybrid of the finite volume and finite difference methods. In order to validate BoussClaw, numerical results are compared to the analytical solutions and laboratory experiments. Furthermore, the wave steepening and breaking motion is carefully scrutinized, and we demonstrate that the point of wave breaking may be wrongly identified in many of the commonly used Boussinesq models.

Keywords: Breaking wave, Boussinesq equation, finite volume method

1. Introduction

Tsunamis are often considered as long waves compared to the water depth, and long-wave models are consequently widely used in the study of their propagation and inundation. Through the use of numerical shock capturing techniques for modeling the nearshore bore formation of the tsunami, *nonlinear shallow water* (NLSW) models did become the standard model for modeling tsunami

propagation and run-up, see e.g. (Titov and Synolakis, 1995; Imamura, 1996; Harig et al., 2008; Berger et al., 2011).

The NLSW models do not incorporate frequency dispersion, which may be included by ascending in the hierarchy of long wave expansion to the Boussinesq type equations. Numerical models based on Boussinesq type equations have been used for idealized studies of wave processes since 1966 (Peregrine, 1966) and additionally to simpler problems in coastal engineering in the following decades (Brocchini, 2013). However, the accumulated effect of the frequency dispersion for the wave propagation over the open sea is a function of propagation time and the shape of the disturbance (Glimsdal et al., 2013), and may become important for some tsunamis, in particular for landslide sources (Løvholt et al., 2015). Dispersion may further be of importance, in combination with non-linear effects, for the evolution of undular bores for tsunamis (Glimsdal et al., 2013; Grue et al., 2008; Løvholt et al., 2008; Behrens and Dias, 2015). In the last decades we have seen a development on long wave expansions and their numerical formulations. In the 1990s the modeling with Boussinesq type equations were vitalized by new formulations, in particular those of Madsen and Sørensen (1992) and Nwogu (1993) which displayed improved dispersion properties in comparison to the standard formulation of Peregrine (1967). **A few more lines here**

Boussinesq-type equations differ in mathematical structure from the NLSW equations and do not inherit characteristics in the same simple form. Hence, other stratergies have been attempted for inclusion of wave breaking and post-breaking motion in Boussinesq models. Schäffer et al. (1993) employed the concept of the *surface roller*, first proposed by Svendsen (1984), which is volume of water passively riding at the bore front. Another way of incorporating the breaking that was suggested by Kennedy et al. (2000) who included diffusive terms in the momentum equation which was activated and deactivated as a steepness measure crosses thresholds. The original steepness measure was the temporal rate of surface elevation corresponding to a very steep solitary wave. Later, Lynett (2006) investigated a variety of steepness measures and

then identified that the surface steepness provides the least sensitive breaking threshold. Løvholt et al. (2013) similarly employed a diffusive model including transport terms, but pointed out that breaking wave Boussinesq models were prone to instabilities. Tissier et al. (2012) suggested a breaking decision model based on the surface roller, the maximal front angle and the Froude number. An alternative non-linear diffusive ad-hoc breaking term was suggested Matsuyama et al. (2007), based on their large scale experiments of the wave propagation of undular bores on various slope angles.

However, there is a natural desire to exploit the effecient and well established shock capturing framework for NLSW models also in a dispersive context. Hence, recently developed of operational Boussinesq models are often based on hybrid numerical techniques such as use of approximate Riemann solvers combined with TVD limiters for the conserved variables, and finite differences for additional higher order terms (Erduran et al., 2005; Kim et al., 2009; Shi et al., 2012). Among other models, this have led to the popular FUNWAVE-TVD and COULWAVE-TVD applications. For instance in Shi et al. (2012), Boussinesq terms are switched off the near-shore region where large amplitude-to-depth-ratios occur, implying that only NLSW terms remain in the shallowest region. This allows for a relatively robust treatment of the modeling of the post breaking phase. To this end, the $A/h = 0.8$ threshold suggested by (Shi et al., 2012) based on the Froude similarity analysis by Tonelli and Petti (2009), have in many ways become the standard for incorporating breaking in a feasible and practical way.

In this paper, we present a new hybrid Boussinesq type model BOUSSCLAW, of similiar mould as FUNWAVE-TVD and COULWAVE-TVD. The emphasis is twofold. First, to present a careful validation of the model, both towards laboratory experiments and more general reference models. Second, we use the new model in comparison with a full potential reference model to explore how accurate the Boussinesq models can represent the wave evolution until the point of breaking. In the present example, we are finally able to demonstrate that Boussinesq models may be employed to accurately model the near shore tsunami

propagation beyond the standard $A/h = 0.8$ threshold depth. Conversely, we find that the use of the standard $A/h = 0.8$ threshold depth invokes a too early formation of a breaking bore. This points out that that the breaking criteria employed so far lacks generality.

This paper is organized as follows: In Section 2, the base model for the wave equations is given and the numerical scheme is outlined, including a von Neuman stability analysis. Sections 3 compares results from the BoussClaw with the analytic solutions and laboratory experiments. Section 3.4 discusses the wave steepening and using a Boundary Integral Method (BIM) for solving the full potential equations. In Section 4.1, we compare these results of pre-breaking evolution with Boussinesq type models.

2. Model Description

Boussinesq-type equations include non-hydrotstic pressure in an approximate way treat short waves more accurately than the shallow water equations. The Boussinesq-type wave models have been derived on the assumption that $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\mu)$ terms are small, where ϵ and μ denote the ratio of wave amplitude to depth and the ratio of depth to wavelength respectively. Various models have been suggested, and the papers of Peregrine (1967), Madsen and Sørensen (1992), Nwogu (1993), Lynett et al. (2002), and Wei and Kirby (1995) are representative examples.

In this work, a new numerical model, called BoussClaw, is introduced. It is an extension of Geoclaw (Clawpack Development Team, 2016), and solves the Boussinesq-type equations derived by Schäffer et al. (1993). The BoussClaw model is a hybrid of the finite volume and finite difference solvers with the fractional step technique. The Geoclaw software is a part of Clawpack (Clawpack Development Team, 2016) developed mainly by LeVeque (1997), George (2008) and Berger et al. (2011), which is designed to solve the nonlinear shallow water equations.

2.1. BoussClaw - *a new long wave model for tsunami propagation and run-up*

2.1.1. *Boussinesq-type equations*

Schäffer et al. (1993) derived Boussinesq-type equations with an addition of a Padé approximation of the linear dispersion relation. The equations read

$$H_t + (Hu)_x = 0, \tag{1}$$

$$(1-D)\left[(Hu)_t\right] + \left(Hu^2 + \frac{g}{2}H^2\right)_x - gHh_x - Bgh^2\left(h\eta_x\right)_{xx} = 0, \tag{2}$$

where the operator D is defined as

$$D(w) = \left(B + \frac{1}{2}\right)h^2 w_{xx} - \frac{1}{6}h^3\left(\frac{w}{h}\right)_{xx}, \tag{3}$$

for any $w(x,t)$. In the above equations $H(x,t)$ and $u(x,t)$ are the total flow depth and the depth averaged velocity of the water, respectively, $h(x)$ is the still water depth, $\eta(x,t)$ is the surface elevation, and thus $H(x,t) = h(x) + \eta(x,t)$. Moreover, g is the acceleration of gravity, and B is a dispersion parameter. Madsen and Sørensen (1992) have chosen the parameter $B = 1/15$ from a Padé expansion of the linear dispersion analysis. When $B = 0$, this set of the Boussinesq-type equations approximately reduces to that of Peregrine (1967) as the linear dispersion relations are identical. However, unlike Peregrine's momentum equation the hydrostatic parts of (2) are written in a conservative form. Moreover, some nonlinearity is introduced in the dispersion term for convenience. Even though (1), (2) and (3) do not constitute a fully nonlinear set of Boussinesq equations, they do describe shoaling of solitary waves markedly better than, for instance, the Peregrine equations, as will be demonstrated in section 4.1.

The BoussClaw model solves the Boussinesq-type equations (1) and (2) numerically with a hybrid combination of the finite volume and finite difference methods that will be explained in a moment. There have been several studies of this type of hybrid schemes. For example, see Tissier et al. (2011), Shi et al. (2012) and Dutykh et al. (2013).

To facilitate a fractional step method, as outlined below, we move the hydrostatic terms of (2) inside the $(1-D)$ operator, while balancing with extra

terms in the Ψ, to obtain

$$(1-D)\left[(Hu)_t + \left(Hu^2 + \frac{g}{2}H^2\right)_x - gHh_x\right] = -\Psi(x,t), \tag{4}$$

where

$$\Psi(x,t) = \left(B + \frac{1}{2}\right) h^2 \left((Hu^2)_x + gH\eta_x\right)_{xx} - \frac{1}{6}h^3 \left(\frac{(Hu^2)_x + gH\eta_x}{h}\right)_{xx} - Bgh^2 \left(h\eta_x\right)_{xx}. \tag{5}$$

2.1.2. Numerical scheme

The equations (1) and (4) are written in a form that conserves momentum to leading order in μ, but with the Ψ term as a pseudo source. Such equations may be solved by a *fractional step method* as described in LeVeque (2002), for instance. First, it is observed that (4) may be formally rewritten as

$$(Hu)_t = -\left\{\left(Hu^2 + \frac{g}{2}H^2\right)_x - gHh_x\right\} - (1-D)^{-1}\Psi(x,t), \tag{6}$$

At the first stage of the hybrid scheme, we integrate Hu over a time step taking into account all hydrostatic terms, namely those within the braces on the right hand side, and omitting the source terms involving Ψ. When this is combined with the continuity equation (1) this simply corresponds to advancing the shallow water equations one time step forward. To this end we employ GEOCLAW, a high-order accurate finite volume solver for the shallow water equations with adaptive mesh refinements.

In the second stage, we retain the new H value, but integrate Hu (essentially being the momentum density) further from the first stage by solving

$$(1-D)\left[(Hu)_t\right] = -\Psi. \tag{7}$$

Since the differential operator D contains spatial derivatives, a systems of difference equations must then be solved.

The spatial and time discretization should be carefully chosen for the stability of the second stage. In our numerical scheme, the second order centered scheme is used for the spatial discretization, and a four stage Runge-Kutta

method is used for the time integration. The von Neumann stability analysis of this numerical scheme is outlined in Appendix A.

Suppose the spatial domain is divided into n grid cells with the spatial grid size Δx. Arrays of nodal values for flow depth and Hu, respectively, are defined as

$$\mathbf{H} = (H_1, H_2, \ldots, H_n)^T,$$
$$\mathbf{M} = (H_1u_1, H_2u_2, \ldots, H_nu_n)^T.$$

With time increment Δt the fourth order Runge-Kutta scheme can be written as follows,

$$\mathbf{M}^1 = \mathbf{M}, \quad \mathbf{M}^2 = \mathbf{M} + \frac{\Delta t}{2}\mathbf{S}^1, \quad \mathbf{M}^3 = \mathbf{M} + \frac{\Delta t}{2}\mathbf{S}^2, \quad \mathbf{M}^4 = \mathbf{M} + \Delta t\mathbf{S}^3, \quad (8)$$

where $\mathbf{M}^k$ are intermediate value arrays and $\mathbf{S}^k$ are correpondingly arrays for the time derivatives of Hu, obtained by solving

$$(I - \bar{D})\mathbf{S}^k = -\bar{\Psi}(\mathbf{H}, \mathbf{M}^k), \quad \text{for } k = 1, \ldots, 4. \quad (9)$$

Here $\bar{\Psi}$ and $\bar{D}$ represent centered spatial discretizations for the term Ψ and the operator D, respectively. These are given explitly below. Finally the value of $\mathbf{M}$ at the new time level is obtained by

$$\mathbf{M}^+ = \mathbf{M} + \frac{\Delta t}{6}\left[\mathbf{S}^1 + 2\mathbf{S}^2 + 2\mathbf{S}^3 + \mathbf{S}^4\right]. \quad (10)$$

In (9), $\bar{D}$ is a tri-diagonal $n \times n$ matrix with elements

$$\bar{D}_{i,i-1} = \frac{1}{\Delta x^2}\left[\left(B + \frac{1}{2}\right)h_i^2 - \frac{1}{6}\frac{h_i^3}{h_{i-1}}\right],$$
$$\bar{D}_{i,i} = \frac{1}{\Delta x^2}\left(-2B - \frac{2}{3}\right)h_i^2,$$
$$\bar{D}_{i,i+1} = \frac{1}{\Delta x^2}\left[\left(B + \frac{1}{2}\right)h_i^2 - \frac{1}{6}\frac{h_i^3}{h_{i+1}}\right].$$

Correspondingly, the i-th element of $\Psi(\bar{\mathbf{H}}, \mathbf{q})$ is

$$\begin{aligned}
\bar{\Psi}_i = & \left(B + \frac{1}{2}\right) \frac{h_i^2}{2\Delta x^3} \left[\left(\frac{M_{i+2}^2}{H_{i+2}} - 2\frac{M_{i+1}^2}{H_{i+1}} + 2\frac{M_{i-1}^2}{H_{i-1}} - \frac{M_{i-2}^2}{H_{i-2}} \right) \right. \\
& \left. + g \left(H_{i+1} \left(\eta_{i+2} - \eta_i \right) - 2H_i \left(\eta_{i+1} - \eta_{i-1} \right) + H_{i-1} \left(\eta_i - \eta_{i-2} \right) \right) \right] \\
& - \frac{1}{6} \frac{h_i^3}{2\Delta x^3} \left[\frac{M_{i+2}^2/H_{i+2} - M_i^2/H_i}{H_{i+1}} - 2\frac{M_{i+1}^2/H_{i+1} - M_{i-1}^2/H_{i-1}}{h_i} \right. \\
& + \frac{M_i^2/H_i - M_{i-2}^2/H_{i-2}}{H_{i-1}} \\
& \left. + g \left(\frac{H_{i+1}(\eta_{i+2} - \eta_i)}{H_{i+1}} - 2\frac{H_i(\eta_{i+1} - \eta_{i-1})}{h_i} + \frac{H_{i-1}(\eta_i - \eta_{i-2})}{H_{i-1}} \right) \right] \\
& - \frac{Bgh_i^2}{2\Delta x^3} \left(H_{i+1} \left(\eta_{i+2} - \eta_i \right) - 2H_i \left(\eta_{i+1} - \eta_{i-1} \right) + H_{i-1} \left(\eta_i - \eta_{i-2} \right) \right),
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

2.1.3. Additional numerical features

Following Shi et al. (2012) we may catch wave breaking in a heuristic fashion through invokation of a threshold $\epsilon_B := \eta/h = 0.8$ in BoussClaw . When the threshold is reached, the wave breaking is supposed to be initiated, and the dispersive terms are suppressed. At the breaking, the set of equations is locally switched to the shallow water equations for the corresponding wave packet, and the trailing waves are solved with the Boussinesq equations if the thresholds are not reached.

G: wave packet is too loose. Must specify.

When a wave reaches the coastline, the coastline changes in time as the wave runs up the slope and recedes eventually. For the numerical simulations, it has been a challenge to compute the inundation correctly. At the coastline, BoussClaw switches to the NLSW solver of the Geoclaw software, which can handle wet and dry states with the depth positivity property. Details can be found in George (2008).

RJL: I modified this sentence, is this correct?

Friction terms are important for inundation on gentle slopes. For example, the Figure 7 shows that the run-up height is much reduced by the friction terms.

BoussClaw uses the Manning-type frictions as follows,

$$f_D = -\frac{gC_d^2 u|u|}{H^{5/3}},$$

where C_d is a Manning drag coefficient. Another parameter d^* needs to be chosen so that the water depth is set to $H = 0$ if the water depth is smaller than this threshold, $H < d^*$. Antuono et al. (2012) derived an approximate solution of the shoreline for the shallow water equations with the Chezy friction, and studied the effect of the frictional coefficient and the water depth tolerance. In this work, the parameter d^* is set to 10^{-4}.

2.2. Models for comparison

The performance of the Boussinesq model presented here is assessed by comparison with numerical results from a full potential flow model which is described in Løvholt et al. (2013) and references therein. The model is based on a boundary integral technique and is run with fully nonlinear solitary wave solutions as initial conditions. During shoaling and breaking this model can describe the evolution of a plunger, but breaks down when the plunger reaches the free surface. Hence, the potential flow results are used to determine the point of breaking due to shoaling and to evaluate the evolution of amplitude and wave shape of the current model until this point. Below we refer to the full potential model as the BIM (Boundary Integral Method) model.

Comparison with a pre-existing, fully nonlinear Boussinesq model is facilitated by the application of a Lagrangian model, described in Løvholt et al. (2013). Apart from the use of Lagrangian coordinates the equations employed in this model are similar to (1) and (2). They differ only concerning the nonlinearities in the dispersion terms and that the dispersion optimization terms are added in a fully nonlinear fashion. Presently, the Lagrangian model has no established bore capturing facility and is hence valid only to the point of breaking. Results from this model will be referred to as 'Serre', even though the dispersion enhancement is invoked.

Results for the Peregrine-type Boussinesq equations are obtained by the GloBouss model. This is a dispersive tsunami propagation model which is based on Peregrine-type equations and discretization on a staggered grid. Further details are found in Løvholt et al. (2008).

For comparison also the version 2.1 of the FUNWAVE-TVD model by Shi et al. (2012) is used. The FUNWAVE-TVD model shares important features with BOUSSCLAW, employing a hybrid of the finite volume and finite difference scheme to solve the fully non-linear higher order dispersive Boussinesq model numerically. While we refer to Shi et al. (2012) for details, we briefly note that FUNWAVE-TVD is based the fully nonlinear Boussinesq equations of Chen (2006). The numerical spatial representation in FUNWAVE-TVD is MUSCL TVD scheme to discretize for the flux and first order terms, whereas a central finite difference scheme Wei et al. (1995a) is utilized for the higher order momentum terms. A Runge-Kutta scheme is employed for the time stepping.

3. Numerical Tests

3.1. Solitary wave propagation

In order to validate the numerical approach a solitary wave propagation is tested on a constant water depth. For the initial conditions, the analytic solitary wave solution of the Serre's equations is used since analytic solutions are unknown for the set (1) and (2). Solitary wave solutions to Serre's equations are given as

$$\begin{aligned} \eta(x,t) &= A\,\mathrm{sech}^2\left(\kappa(x-ct)\right), \\ u(x,t) &= c\frac{\eta(x,t)}{h}, \end{aligned} \tag{11}$$

where

$$\kappa = \frac{\sqrt{3h}}{2A\sqrt{A+h}}, \quad \text{and} \quad c = \sqrt{g\,(A+h)}. \tag{12}$$

In this expression, A and h are constants which represent the wave amplitude and the undisturbed water depth respectively.

In Figure 1, snapshots from the BoussClaw simulation are shown at $t = 0, 4, 8$ and 12 with $\Delta x = 0.1$. For the initial conditions, the solution (11) is used with $A = 0.2$, $h = 1$ and $g = 9.81$. The computational results are in good
195 agreement with the analytic solutions concerning height, shape and propagation speed. The amplitudes decreases very gently as the wave propagates.

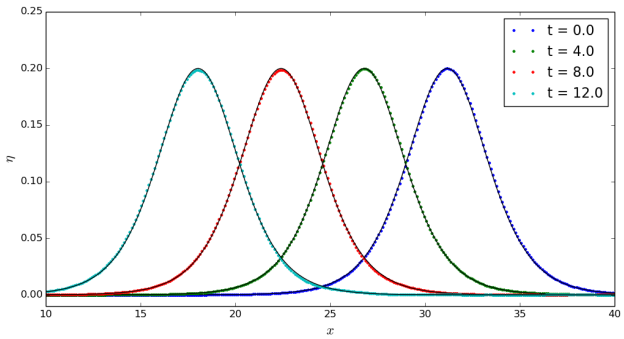

Figure 1: Snapshot of the analytic and computed solitary wave at $t = 0, 4, 8$ and 12 with $A/h = 0.2$. The wave propagates from right to left, and the analytic solutions are black solid lines.

The wave energies for the shallow water equations and the Boussinesq equations are E_0 and $E_0 + E_1$, respectively, where

$$E_0 = \frac{1}{2}\left(g\eta^2 + H\bar{u}^2\right), \tag{13}$$

$$E_1 = \frac{1}{6}H^3\bar{u}_x^2 + \frac{1}{2}H^2 h_x \bar{u}\bar{u}_x + \frac{1}{2}Hh_x^2\bar{u}^2. \tag{14}$$

Details are given in Madsen et al. (1997) and Appendix B.

In Figure 2a, the energy of the solitary wave is shown with $A/h = 0.2$ and $\Delta x = 0.2$. There are fluctuations both in the potential and kinetic energy that is evident when we zoom in, and the total energy decreases showing that the numerical procedure has dissipation. In Figure 2b, the relative error of the energy at $t = 10$,

$$Error = \frac{|E_{t=0} - E_{t=10}|}{|E_{t=0}|},$$

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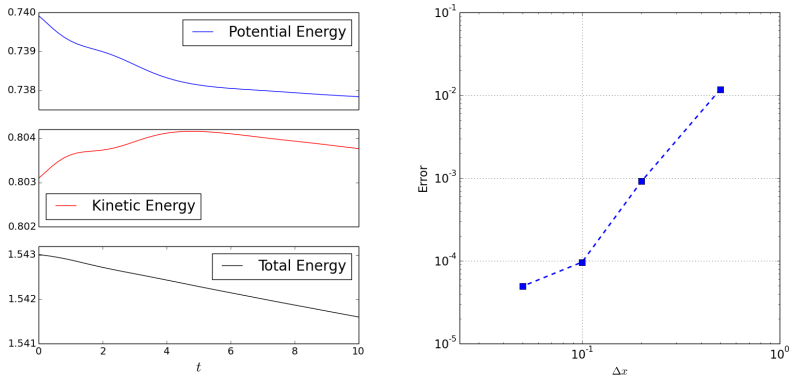


(a) Energy (b) Relative error of energy

Figure 2: The energy of a solitary wave with $\Delta x = 0.2$ (left), and log-log plot of relative error at $t = 10$ for $\Delta x = 0.05, 0.1, 0.2$ and 0.5 (right).

is shown for different Δx. For a solitary wave on a constant depth, the energy dissipation decreases with the grid increments.

3.2. Waves on a composite slope

A physical model was constructed at the Coastal Hydraulic Laboratory of the U.S. Army Corps of Engineers in order to address beach erosion and severe flooding problems Briggs et al. (1995). The model beach consists of three piecewise linear slopes of 1:53, 1:150, and 1:13 with a vertical wall at the shoreline as shown in Figure 3. In the laboratory, the wavemaker was located at 23.23 m. The gauge data from three cases are provided where the ratio A/h is equal to 0.038, 0.259 and 0.681 with $h = 21.8$ cm.

The second case with $A/h = 0.259$ has been compared with the numerical tests which employed 400 grid points. To specify the incoming wave from the left boundary, the data at Gauge 4 were used for the wave height, and the corresponding velocity (11) was applied.

In Figure 4, water surface elevations at gauges 5, 7 and 8 are shown. The simulated waves are in good agreement with the laboratory measurements. For the reflected waves, larger discrepancies are observed. The increased discrep-

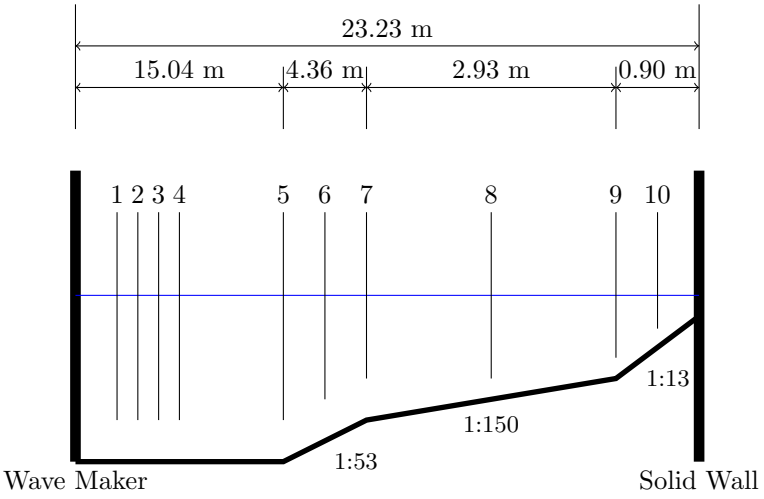


Figure 3: A sketch of the water tank

ancy occurs because the full interaction between the wave and the wall at the right boundary is less accurately captured. Friction forces influence the wave evolution along the shallow region near the right wall, but we have not included these in the present numerical simulation. A better fit may possibly be obtained by incorporating friction, however, tuning the friction models is not the scope of this work.

3.3. Runup on a steep slope

On a 10° slope an incident solitary wave of amplitude $A/h = 0.3$ will not break until the end of the draw-down phase Grilli et al. (1997). Still, this may be a challenging task for Boussinesq type models (Løvholt et al., 2013). Runup on a 10° was investigated experimentally by Pedersen et al. (2013) who found a theoretical overshoot of roughly 20% in the maximum runup height. This was alloted to the viscous boundary layer on the beach and capillary effects. Moreover, the measurements showed that the boundary layer flow during runup was mostly laminar, albeit indications of transition was observed in the upper part of the swash tongue close to flow reversal. Hence, it is not appropriate to employ a manning friction term and we compare the models without any bed

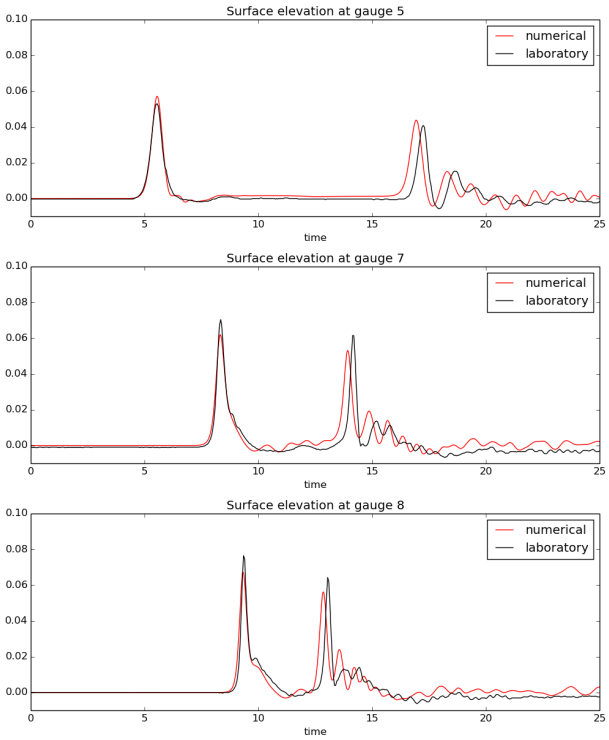


Figure 4: Water surface elevation at gauges 5,7 and 8 for $A/h = 0.269$ case.

friction, while leaving the experiments out.

3.4. Comparsion with experiments on a breaking wave

Synolakis (1987) performed a series of laboratory experiments for the run-up of solitary waves on uniform slopes. Here, we are interested in the breaking cases. One such example in Synolakis (1987) is a solitary wave of amplitude $A/h = 0.28$ approaching a slope of $1 : 19.85$. In Figure 5, the initial set-up for a test is shown.

The set-up of the wave tank in the simulations follows the laboratory experiments by Synolakis (1987). The bathymetry of the wave tank is composed of a horizontal bottom and a uniform slope as shown in Figure 5. A solitary wave of amplitude $A/h = 0.28$ is generated at the right end of the tank and propagates leftwards to the beach.

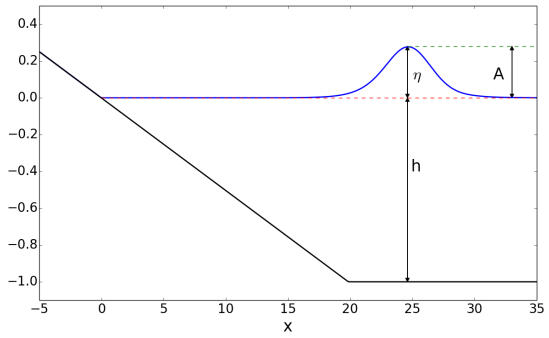


Figure 5: Set-up of a numerical test for Synolakis' experiments.

We present the results using the non-dimensional time $t^* = t\sqrt{h/g}$ and non-dimensional space $x^* = x/h$. In the following, we drop the asterisks in the presentation of the results. In Synolakis (1987), $t = 0$ was defined as when the wave crest was a non-dimenional distance, L, from the toe of the slope, where

$$L = \sqrt{\frac{4A}{3h}}\operatorname{arccosh}\left(\frac{1}{0.05}\right).$$

However, at $t = 0$, the solitary wave has an elevation of 5% of it maximum at the toe of the beach, meaning that the slope has started to interact with the solitary wave. To avoid any such interaction obscuring our analysis, we instead place the initial solitary wave using equation (11) at $L + 5c$, where c is the shallow water wave celerity. In this way, the initial solitary wave has a negligible interaction with the slope when initialized.

In Figure 6, the laboratory measurements are shown with the computational results from the BoussClaw (in Boussinesq and NLSW mode) and the BIM models for $A/h = 0.28$ and a $1 : 19.85$ slope at $t = 15$. This is before the w3ave breaks and both BoussClaw and the BIM model are in good agreement with the experiments.

Also include the BoussClaw results and NLSW in fig. 6

The ratio of amplitude to depth, A/h, is about 2.01 at the break point. The potential flow model cannot be run much beyond the breaking points (until the attachment of the plunger only) and gives no information on the following bore propagation. In figure 7 we have compared the BoussClaw model, with and

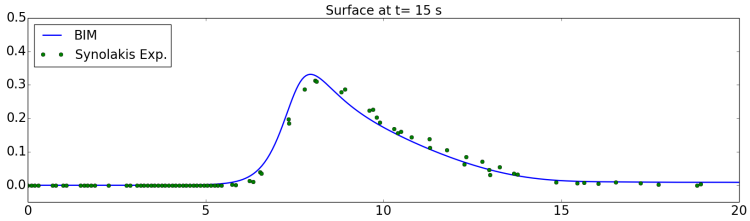


Figure 6: Comparison of the laboratory experiments and BIM at t=15 with $A/h = 0.28$ on a slope of 1 : 19.85.

without a Manning friction, with the experimental data. The agreement is good and the introduction of bed-friction even seem to match the truncated swash tongue of the experiments well. However, this may be a coincidence. Even though the wave has broken and some irregular flow features are introduced thereby, we have no evidence of the flow state being anywhere near turbulent, which is required for a quadratic bottom resistance to be appropriate. Capillary effects and experimental errors may also affect the comparison.

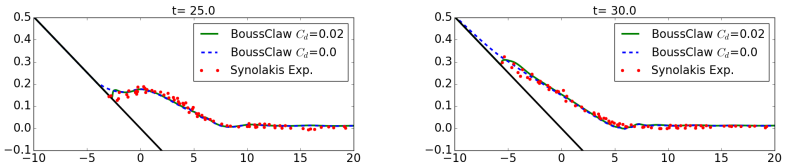


Figure 7: Comparison of experiments and BoussClaw results at t=25 and 30.

4. Shoaling and breaking phenomena

4.1. Pre-breaking

Wei et al. (1995b) made computation of pre-breaking solitary wave shoaling with their fully nonlinear extension of Nwogu's model with full potential theory and the weakly nonlinear version of Nwogu's model. They found that the fully nonlinear Boussinesq equations were superior to those of Nwogu in the later stages of the shoaling. In this subsection we will do a similar comparison for our models on the 1 : 19.85 slope which was not included in the reference.

We use the set-up described in section 3.4 for the Boussinesq modeling of solitary waves on a slope. The BoussClaw simulations are compared with those of other Boussinesq solvers, namely Funwave (Shi et al., 2012), GloBouss (Løvholt et al., 2010) and the Serre type formulation (Løvholt et al., 2013). As noted above, the original Serre's equations are enhanced by adding the Schäffer et al. (1993) terms.

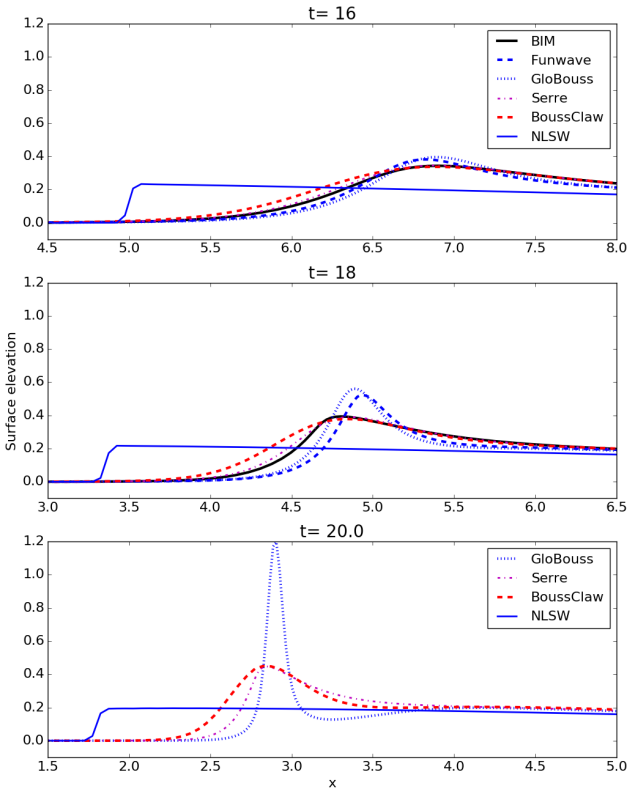


Figure 8: Snapshots of BIM, Serre, GloBouss, BoussClaw and Funwave at t = 16, 18 and 20. The BoussClaw is used with B=1/15, and the Peregrine's form is used for GloBouss.

In Figure 8, snapshots from different numerical models are shown at $t = 16$, 18 and 20. At the $t = 20$ there are no results from the BIM model as the wave

has broken. For the BoussClaw, the parameter $B = 1/15$ is used, but the computational results are rather similar with $B = 0$ for this case. At t=16, the computational results from the Boussinesq-type equations show similar results. The NLSW model, on the other hand, has yieldes a premature breaking causing a too low amplitude. Moreover, the wave celerity is also over-estimated by the hydrostatic model. At t=18, some discrepancies are observed that can be split into two groups, and GloBouss and Funwave are similar while the BoussClaw and the Serre results are similar. The wave amplitudes computed by the GloBouss and Funwave models, are more than 20 %larger than those from the BIM model. The wave amplitude continues to increase with GloBouss simulations, and the difference from the BoussClaw result becomes larger at t = 20. The results from the Serre and BoussClaw models clearly more similar to those of the BIM model. Especially, the wave amplitudes are correctly determined by these models. Our observations are in line with those of Wei et al. (1995b).

RJL: Why no BIM solution at t = 20? G: could we include t=19.1 instead, with the BIM in place ?

Is the figure 20% OK. The old 10 was far too small

4.2. Wave breaking and run-up

In the BIM model we may identify the onset of breaking according when and where we first observe a vertical slope at the wave front. For an incident amplitude of $A/h = 0.28$ on a $1 : 19.85$ slope breaking then starts at $x = 4.09$. When the crest in the BoussClaw simulation reaches $x = 4.09$ we find $A/h = 1.97$, the ratio of wave speed to celerity $(u/\sqrt{gH})$ is 1.034 and maximum surface slope angle of 39.1°.

What are the height, time, etc for the BIM simulation

For an incident amplitude of $A/h = 0.28$ and a $1 : 19.85$ slope the BoussClaw model, the treshold $\epsilon_B = 0.8$ (se sec. 2.1.3) is reached at $t = 14.9$ when the peak of the wave is at $x = 8.03227$. In the following, we explore the wave evolution with and without the application of this threshold.

In Figure 9, snapshots are shown at t=20, 25, and 30 of the solutions from BoussClaw and NLSW with the Manning coefficient $C_d = 0.02$. We compared the NLSW and BoussClaw with $\epsilon_B = 0.8$ and without the threshold. At $t = 20$ the simulation with $\epsilon_B = 0.8$ has already been in NLSW mode for 5 time units

and the difference in the wave height from the full Boussinesq simulation is significant. In fact, the treshold solution is closer to the NLSW solution. is clearly observed.

315 At $t = 25$ and $t = 30$, the wave is running up the slope, and the difference in the swash tongue is relatively small. On the other hand, while BoussClaw simulation with $\epsilon_B = 0.8$ is smooth at $t = 20$ irregularities are observed around $x = 8$ due to the accumulated errors generated by the splicing of Boussinesq and NLSW equations.

How does this eveolve? Is it seen in other paper ?

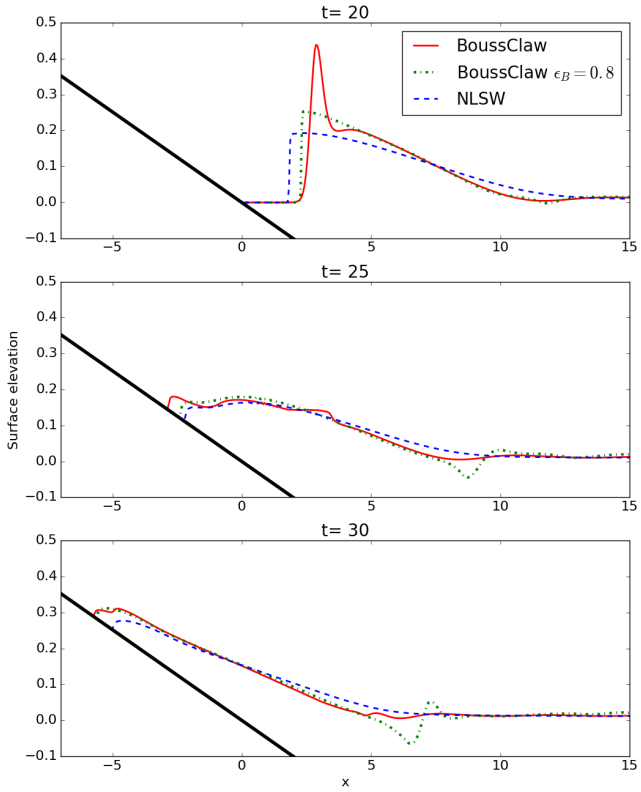


Figure 9: Comparison of BoussClaw and NLSW with $\epsilon_B = 0.8$ at t =20, 25 and 30. Fiction forces have been added with $C_d = 0.02$.

Figure 10 shows the amplitude to depth ratio, ϵ_B, as function of the crest location.

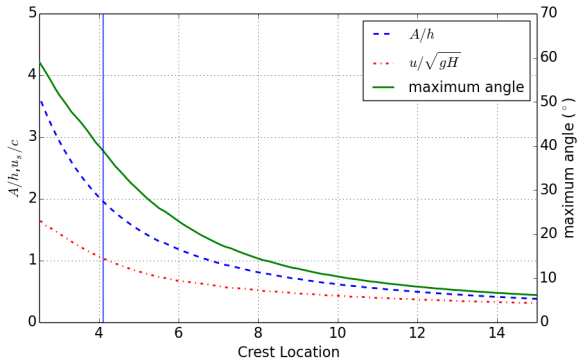
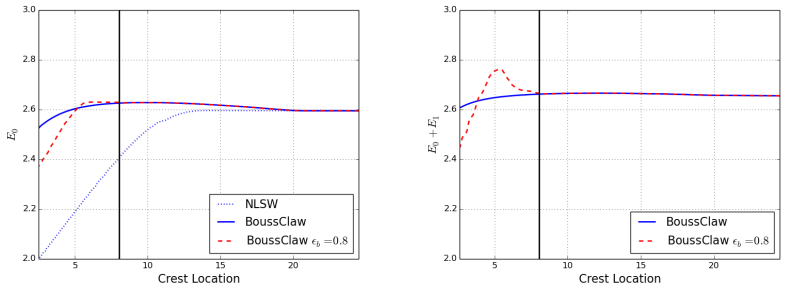


Figure 10: Plot of A/η, u_s/c and maximum angle of waves vs. crest location. BIM shows the wave break at $x = 4.09$.

4.3. Wave Energy

The wave energy is nearly conserved when the wave is smooth, but decreases as the wave breaks. The wave energy expressions for the shallow water equations and Boussinesq equations are E_0 and $E_0 + E_1$ respectively, which are given in (13) and (14).



(a) E_0 (b) $E_0 + E_1$

Figure 11: Energy plots of NLSW and BoussClaw . The vertical line is at $x = 8.03227$ m where $\epsilon_B = 0.8$, and the governing equations are switched to the NLSW from the Boussinesq equations.

In Figure 11, the wave energy E_0 and $E_0 + E_1$ are shown. The energy denotes the aggregate of the wave energy in the entire computational domain and is shown as function of the crest location. In the left panel, the energy E_0 is well-preserved for the shallow water equations before the wave forms a shock, and then the energy dissipates after a shock is formed. In the right panel, $E_0 + E_1$ is well-preserved for the BoussClaw simulation during deep water propagation, but the value decreases slightly as the wave steepens near the shoreline. When the threshold $\epsilon_B = 0.8$ is used, the energy $E_0 + E_1$ is well-preserved as shown on the right figure before the threshold is reached. When the crest is located at $x = 8.03227$ with $\epsilon_B = 0.8$, the wave energy E_0 on the left figure does not decrease immediately. The energy E_0 is preserved for a while, then starts to decrease as a shock is formed.

5. Conclusion

In the present paper we have presented a new operational Boussinesq type model, BoussClaw, for modeling fully non-linear dispersive tsunami propagation, taking also into account drying-wetting during inundation and withdrawal on the beach. BoussClaw resembles much used operational models such as Funwave-TVD and Coulwave-TVD, but is based on a slightly simpler and more transparent set of governing equations, and has a slightly different numerical scheme. We have tested numerical implementation towards analytical solitary wave expressions as well as laboratory experiments.

RJL: Here and in the abstract, is "operational" the right word? To me this means a code that is used in practical work such at tsunami warning centers.

Making use of the experiments of Synolakis (1987) enabled us to constrain a set of different long wave models, including BoussClaw, as well as a full potential BIM reference model. Using the BIM, we were able to explore in detail the post-breaking behaviour, and to identify the point of breaking accurately. This was useful for determining the validity of the respective long wave models. First, we found that by using standard NLSW models, the point of breaking will be located too far offshore. Boussinesq models provide the opportunity of providing a more accurate description of the near shore propagation and

shoaling. However, in current practise the Boussinesq terms are often omitted near shore through the $A/h > 0.8$ threshold criteria. As a consequence, the point of breaking may be misinterpreted also in Boussinesq type models.

In the present example, we investigated the near shore propagation over a relatively gentle shelf of 1/19.85 slope, and in this case the actual onset of breaking occurred for $A/h \approx 2$, which is significantly later than what would be predicted in any standard approach (NLSW or Boussinesq). As demonstrated in this paper, the combined effects of non-linearities and dispersion influence the solution markedly, when accumulated to the point of breaking. It is noted that the artificial effect discovered would depend on the slope, and the $A/h = 0.8$ limited may well work better on a much gentler slope as it is primarly derived based on solitary wave evolution on constant depth. On the other hand, 1/19.85 slope is already quite gentle, and the offset between the reference solution and Boussinesq models using this criteria may be even more pronounced for steeper slopes.

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Appendix A. Stability of the hybrid scheme

It is difficult to analyze the numerical stability for our full Boussinesq equations. To obtain some insight in the stability of the proposed hybrid numerical

scheme, we thus consider a closely related, but simpler, equation, namely the linearized Benjamin-Bona-Mahony (BBM) equation (Benjamin et al. (1972))

$$u_t + cu_x = \frac{h^2}{6} u_{txx}, \tag{A.1}$$

where $c = \sqrt{gh}$. This equation describes weakly dispersive, uni-directional waves in constant depth. The equation replaces the momentum equation, whereas no separate continuity equation is involved.

Following the steps of section 2.1.2, we rearrange the equation (A.1) as

$$(I - D)(u_t + cu_x) + Du_x = 0, \tag{A.2}$$

where $D = \frac{h^2}{6}\partial_x^2$. The first step of hybrid scheme for this equation is integration of the advection equation

$$u_t + cu_x = 0, \tag{A.3}$$

by the finite volume method. Then the Runge-Kutta method is applied to,

$$(1 - D)u_t + cDu_x = 0. \tag{A.4}$$

which is the counterpart to (7).

If we use the centered spatial difference approximation of $O(\Delta x^2)$ accuracy on a regular grid we may employ a standard von Neumann analysis where we calculate the growth of an harmonic mode over a single time step. Expressing the coefficients of the velocity array before the time step as $u_j = e^{i\xi j\Delta x}$ we then replace the coefficient of $\mathbf{M}^q$, defined in section 2.1.2, by $M_j^q = U_j^q = g^q e^{i\xi j\Delta x}$, where q is 1, 2, 3, 4 or +. Correspondingly, the coefficients of the $\mathbf{S}^k$ array, which contains auxiliary, nodal values for u_t, is expressed $(S_j^k) = s^k e^{i\xi j\Delta x}$.

The stability of the first step, (A.3), is assured by the standard CFL criterion

$$\frac{c\Delta t}{\Delta x} < 1.$$

If we instead solve the NLSW equations, as in BoussClaw, c must be replaced by the nonlinear characteristic velocity, which may lead to a more strict criterion. However, the method employed in the first step is not suited for a von Neumann

stability analysis and we thus apply this technique to the second step only. Hence, we may put g^1 to unity, but it is preferable to retain it in the calculations. The Runge-Kutta scheme for time stepping, (8), may now be expressed as

$$g^2 = g^1 + \frac{\Delta t}{2} s^1, \quad g^3 = g^1 + \frac{\Delta t}{2} s^2, \quad g^4 = g^1 + \Delta t s^3, \tag{A.5}$$

The discrete version of (A.4), which is the counterpart to (9) for the BBM equation reads

$$S_j^k - \frac{h^2}{6} \frac{S_{j+1}^k - 2S_j^k + S_{j-1}^k}{\Delta x^2} = -\frac{ch^2}{6} \frac{U_{j+2}^k - 2U_{j+1}^k + 2U_{j-1}^k - U_{j-2}^k}{2\Delta x^3},$$

which, inserted the harmonic expressions, implies

$$s^k = i\frac{\gamma}{\Delta t} g^k, \quad \gamma = c\Delta t \frac{2\sin(\xi\Delta x)(1-\cos(\xi\Delta x))}{6\Delta x^3 h^{-2} + 2\Delta x(1-\cos(\xi\Delta x))}, \tag{A.6}$$

where the Δt factors are included for convenience. The assembling of the intermediate values in the Runge-Kutta procedure, (10), now yields

$$g^+ = g^1 + \frac{\Delta t}{6}\left[s^1 + 2s^2 + 2s^3 + s^4\right]. \tag{A.7}$$

By combination of (A.5) and (A.6) s^k and g^k, $k = 1..4$ can be calculated successively and combined in (A.7) to provide the value of g^+,

$$\begin{aligned} g^+(\gamma) &= \left(1 - \frac{1}{2}\gamma^2 + \frac{\gamma^4}{24} + \left(\frac{\gamma^3}{6} - \gamma\right) i\right) g^1 \\ |g^+(\gamma)|^2 &= \left(1 + \frac{1}{4}\gamma^4 + \frac{\gamma^8}{24^2} - \gamma^2 + \frac{\gamma^4}{12} - \frac{\gamma^6}{24} + \gamma^2 + \frac{\gamma^6}{36} - \frac{\gamma^4}{3}\right) |g^1|^2 \\ &= \left(1 - \frac{1}{72}\gamma^6 + \frac{1}{576}\gamma^8\right) |g^1|^2. \end{aligned}$$

Stability requires $|g^+(\gamma)/g^1| < 1$ which is equivalent to $|\gamma| < 2\sqrt{2}$. Moreover, it is easily seen that $\gamma < c\Delta t/\Delta x$. Hence, a sufficient condition for stability of the second step of the hybrid scheme is

$$\frac{c\Delta t}{\Delta x} < 2\sqrt{2}.$$

This is more relaxed than the CFL condition for the advection equation (A.3). Therefore, if the CFL condition is satisfied in the advection equation, the fractional step is always stable with the suggested numerical scheme.

Appendix B. Energy estimates and dissipation

Appendix B.1. Velocity field

To derive the energy estimates for the Boussinesq-type equations, we define the depth-averaged velocity as,

$$\bar{u} = \frac{1}{H}\int_{-h}^{\epsilon\eta} u dz.$$

Then the velocity u can be expressed as $u = \bar{u} + \mu^2 u_1$ where

$$\int_{-h}^{\epsilon\eta} u_1 dz = 0. \tag{B.1}$$

Then the kinematic boundary condition at the bottom and zero divergence implies

$$w = -h_x u - \bar{u}_x(z+h) + O(\mu^2).$$

Appendix B.2. Energy integrals

The potential energy density per horizontal area is

$$V = \int_{-h}^{\epsilon\eta} gz dz = \frac{1}{2}\epsilon^2 g\eta^2 - \frac{1}{2}gh^2,$$

where the last term $\frac{1}{2}gh^2$ is the equilibrium energy. The kinematic energy density has two contributions,

$$T = T_u + T_w; \quad T_u = \frac{1}{2}\epsilon^2\int_{-h}^{\epsilon\eta} u^2 dz, \quad T_w = \frac{1}{2}\epsilon^2\mu^2\int_{-h}^{\epsilon\eta} w^2 dz.$$

For the horizontal part, T_u is

$$T_u = \frac{1}{2}\epsilon^2\int_{-h}^{\epsilon\eta} u^2 dz = \frac{1}{2}\epsilon^2\int_{-h}^{\epsilon\eta} \bar{u}^2 + 2\mu^2\bar{u}u_2 + \mu^4 u_1^2 dz = \frac{1}{2}\epsilon^2 H\bar{u}^2 + O(\epsilon^2\mu^4),$$

since $\bar{u}$ is independent of z and by (B.1). Assuming $\frac{1}{H}\int_{-h}^{\epsilon\eta} u^2 dz = \bar{u}^2$, the vertical part is

$$\begin{aligned} T_u &= \frac{1}{2}\epsilon^2\mu^2\int_{-h}^{\epsilon\eta} h_x^2 u^2 + 2h_x u\bar{u}_x(z+h) + \bar{u}_x^2(z+h)^2 dz + O(\epsilon^2\mu^4) \\ &= \frac{1}{2}\epsilon^2\mu^2 H\left(h_x^2\bar{u}^2 + Hh_x\bar{u}\bar{u}_x + \frac{1}{3}H^2\bar{u}_x^2\right) + O(\epsilon^2\mu^4). \end{aligned}$$

Thus the energy of a wave can be approximated as

$$E = \epsilon^2 \left(E_0 + \mu^2 E_1 + O(\mu^4) \right)$$

where

$$\begin{aligned} E_0 &= \frac{1}{2} \left(g\eta^2 + H\bar{u}^2 \right), \\ E_1 &= \frac{1}{6} H^3 \bar{u}_x^2 + \frac{1}{2} H^2 h_x \bar{u} \bar{u}_x + \frac{1}{2} H h_x^2 \bar{u}^2. \end{aligned}$$