Pseudo-Spectral Methods for 2D Incompressible Navier Stokes Equation

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Introduction

- Implementation of a pseudo-spectral method for solving the 2D incompressible forced Naiver-Stokes equations.
- Turbulence simulations at increasing Reynolds number to observe and analyze the transition from laminar to turbulent flows.
- Comparisons of the actual Reynolds number, derived using different mathematical norms.

Navier-Stokes Equations

The incompressible Navier-Stokes equations:

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \tag{1}$$

$$\nabla \cdot u = 0 \tag{2}$$

Vorticity Equation and Stream Function

By applying curl to the Navier-Stokes equations:

$$\partial_t w + (u \cdot \nabla) w - (w \cdot \nabla) u = \nu \Delta w \tag{3}$$

In 2D, this simplifies to:

$$\partial_t w = \nu \Delta w - (u \cdot \nabla) w \tag{4}$$

Solving for the stream function ψ:

$$w = -\Delta \psi \tag{5}$$

$$u = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \tag{6}$$

Discretization and Implementation

- Space discretization using an equidistant square grid.
- Fourier transform of the vorticity equation:

$$\partial_t \hat{w} = -\nu(k_x^2 + k_y^2)\hat{w} - \hat{u} \cdot \nabla \hat{w} \tag{7}$$

- Crank-Nicolson method for time stepping:

$$\hat{w}^{n+1} = \frac{\left(\frac{1}{\Delta t} + \frac{1}{2}\nu(k_x^2 + k_y^2)\right) \hat{w}^n - \hat{u}^n \cdot \nabla \hat{w}^n}{\frac{1}{\Delta t} - \frac{1}{2}\nu(k_x^2 + k_y^2)} \tag{8}$$

Stability Conditions for time stepping

- CFL Stability Condition (adaptive time steps):

$$\Delta x = \frac{\text{length of domain}}{\#\text{ of grid points}}$$

$$\|u(t)\|_\infty \Delta t \leq \Delta x$$

$$\Delta t \leq \frac{\Delta x}{\|u(t)\|_\infty} \tag{9}$$

Algorithm: Solving Vorticity Equation

Algorithm 1 Fourier-Spectral Method for Vorticity Equation

Initialize vorticity $\hat{\omega} = \mathrm{fft2}(\mathit{vorticity})$
while $t_0 < T$ **do**
  Compute stream function $\hat{\psi} = -\hat{\omega}/(k_x^2 + k_y^2)$
  Compute $u = \mathrm{real}(\mathrm{ifft2}(k_y \cdot \hat{\psi}))$ and $v = \mathrm{real}(\mathrm{ifft2}(-k_x \cdot \hat{\psi}))$
  Compute $w_x = \mathrm{real}(\mathrm{ifft2}(k_x \cdot \hat{\omega}))$ and $w_y = \mathrm{real}(\mathrm{ifft2}(k_y \cdot \hat{\omega}))$
  Compute $\widehat{\mathbf{u} \cdot \nabla\omega} = \mathrm{fft2}(u \cdot w_x + v \cdot w_y)$
  Compute adaptive time step $dt = \Delta x/\|\mathbf{u}\|_\infty$
  Update vorticity using Crank–Nicolson method
  Update vorticity field $\omega = \mathrm{real}(\mathrm{ifft2}(\hat{\omega}_{next}))$
  Update $\hat{\omega} = \hat{\omega}_{next}$
  Increment time $t_0 = t_0 + dt$
end while

Reynolds Number

In this simulation, we compare three different Reynolds numbers, which are calculated as follows:

- **Reynolds number based on the vorticity gradient**:

$$\text{Re}_{\text{prob}} = \frac{\|\mathbf{u} \cdot \nabla\omega\|}{\|\nu\Delta\omega\|}$$

  where $\mathbf{u}$ is the velocity field, ω is the vorticity, and ν is the viscosity.

- **Maximum Reynolds number**:

$$\text{Re}_{\text{max}} = \frac{\|\mathbf{u}\|_\infty \cdot \sigma}{\nu}$$

  where $\|\mathbf{u}\|_\infty$ is the maximum velocity magnitude, σ is the standard deviation of the Gaussians, and ν is the viscosity.

- **Mean Reynolds number**:

$$\text{Re}_{\text{mean}} = \frac{\langle\|\mathbf{u}\|\rangle * \sigma}{\nu}$$

  where $\langle\|\mathbf{u}\|\rangle$ is the mean velocity magnitude and ν is the viscosity.

Reynolds Number Based on Vorticity Gradient

The Reynolds number based on the vorticity gradient, $\mathrm{Re}_{\mathrm{prob}}$, is given by:

$$\mathrm{Re}_{\mathrm{prob}} = \frac{\|\mathbf{u} \cdot \nabla\omega\|}{\|\nu\Delta\omega\|}$$

- $\mathbf{u}$ is the velocity field.
- ω is the vorticity.
- ν is the viscosity.
- This Reynolds number measures the relative importance of the advective transport of vorticity compared to its diffusive dissipation.
- Higher values of $\mathrm{Re}_{\mathrm{prob}}$ indicate that advective effects are dominant over viscous effects.

Maximum Reynolds Number

The maximum Reynolds number, $\mathrm{Re}_{\mathrm{max}}$, is given by:

$$\mathrm{Re}_{\mathrm{max}} = \frac{\|\mathbf{u}\|_\infty \sigma}{\nu}$$

- $\|\mathbf{u}\|_\infty$ is the maximum velocity magnitude.
- σ is the standard deviation of the Gaussians.
- ν is the viscosity.
- This Reynolds number evaluates the flow based on the peak velocity observed in the system.
- It highlights the influence of the highest velocity points on the flow characteristics.

Mean Reynolds Number

The mean Reynolds number, $\mathrm{Re}_{\mathrm{mean}}$, is given by:

$$\mathrm{Re}_{\mathrm{mean}} = \frac{\langle \|\mathbf{u}\| \rangle \sigma}{\nu}$$

- $\langle \|\mathbf{u}\| \rangle$ is the mean velocity magnitude.
- σ is the standard deviation of the Gaussians.
- ν is the viscosity.
- This Reynolds number considers the average behavior of the velocity field.
- It provides an overall assessment of the flow's characteristics, considering peak and average velocities.

Comparison of Reynolds Numbers

By comparing the three different Reynolds numbers, we gain insights into different aspects of the flow:

- $\mathrm{Re}_{\mathrm{prob}}$ provides a measure of the relative importance of advective and diffusive effects based on the vorticity gradient.
- $\mathrm{Re}_{\mathrm{max}}$ focuses on the influence of the maximum velocity in the system, highlighting areas with peak flow velocities.
- $\mathrm{Re}_{\mathrm{mean}}$ offers an average view of the flow characteristics, considering the overall behavior of the velocity field.

Comparison and Insights:

- High $\mathrm{Re}_{\mathrm{prob}}$ indicates strong advective transport, while low values suggest diffusion is more significant.
- High $\mathrm{Re}_{\mathrm{max}}$ emphasizes the impact of extreme velocities, which can indicate potential areas of instability or turbulence.
- $\mathrm{Re}_{\mathrm{mean}}$ helps to understand the general flow dynamics, balancing the contributions of both high and low-velocity regions.

Parameter Setting for Simulation

- Domain Range

$$X = [0, 2\pi], Y = [0, 2\pi]$$

- # of grid points, N

$$N = 2^9$$

- simulation time

$$t_0 = 0, T = 15$$

- viscosity

$$\nu = 0.01, 0.001$$

Initial Condition for Vorticity

We define the initial vorticity function $\omega(x, y)$ using multiple Gaussian distributions as follows:
Let $n = 15$ be the number of Gaussians in each row and column, L be the length of the domain, and $\sigma = 0.5$ be the standard deviation for the Gaussians. The spacing between the centers of the Gaussians is given by:

$$\text{spacing} = \frac{L}{n+1}$$

The positions of the Gaussian centers are:

$$x_i = i \cdot \text{spacing} \quad \text{and} \quad y_j = j \cdot \text{spacing}$$

where $i, j \in \{1, 2, \dots, n\}$.
The vorticity $\omega(x, y)$ is then defined as:

$$\omega_0(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_{ij} \exp\left(-\frac{(x - x_i)^2 + (y - y_j)^2}{\sigma^2}\right)$$

where ϵ_{ij} is a random variable taking values ± 1.

Initial Vorticity

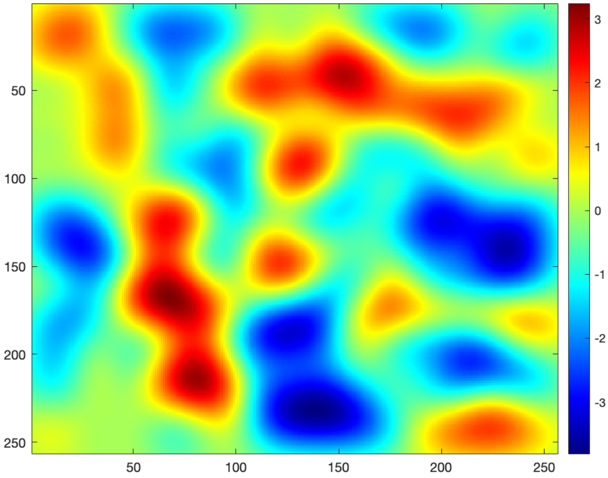

Figure: Initial Vorticity

Observations: Simulation with $\nu = 0.01$

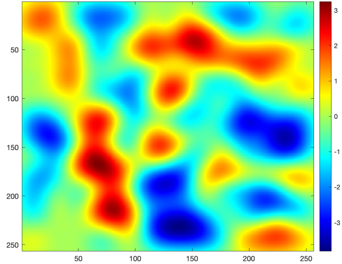

Figure: Initial Vorticity

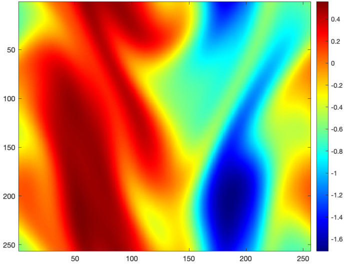

Figure: Vorticity at $T = 15$

Observations: Simulation with $\nu = 0.001$

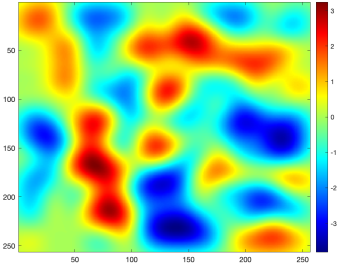

Figure: Initial Vorticity

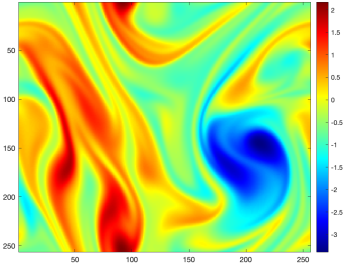

Figure: Vorticity at $T = 15$

Observations: Reynolds Number with $\nu = 0.001$

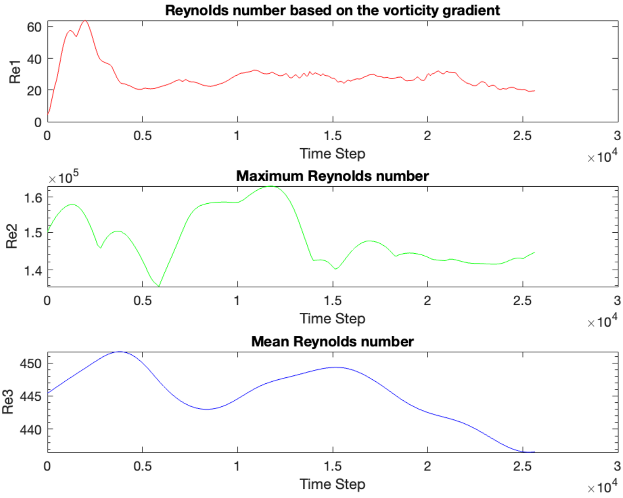

Figure: Reynolds Number with $\nu = 0.001$

Results: Vorticity at $T = 15$ with Different Viscosity

- The initial vorticity distribution evolves differently depending on the viscosity (ν).
- With $\nu = 0.01$:
  - The vorticity field shows more diffusive behavior.
  - The intensity of vortices is lower compared to lower viscosity.
  - Flow structures are smoother and less defined.
- With $\nu = 0.001$:
  - The vorticity field exhibits more complex and turbulent behavior.
  - The intensity of vortices is higher.
  - Flow structures are sharper and more distinct.

Results: Comparison of Reynolds Numbers with $\nu = 0.001$

- **Reynolds Number based on the vorticity gradient ($\mathrm{Re}_{\mathrm{prob}}$)**:
  - Shows significant fluctuations, indicating periods of strong advective effects.
  - High peaks correspond to increased vorticity transport, potentially indicating turbulence or vortex formation.
  - Intermittent spikes in the pattern suggest that the flow experiences periods of instability, where the flow behavior becomes more chaotic and less predictable.
- **Maximum Reynolds Number ($\mathrm{Re}_{\mathrm{max}}$)**:
  - Shows relatively smoother fluctuations compared to $\mathrm{Re}_{\mathrm{prob}}$.
  - High values indicate regions with high velocity, which may correlate with peak flow events or localized turbulence.
  - The overall pattern tends to rise and fall, reflecting changes in the maximum flow velocities.
- **Mean Reynolds Number ($\mathrm{Re}_{\mathrm{mean}}$)**:
  - Exhibits less fluctuation compared to $\mathrm{Re}_{\mathrm{prob}}$ and $\mathrm{Re}_{\mathrm{max}}$.
  - The overall pattern is more stable, indicating the general flow behavior without being influenced by extreme values.

Results: Significance of $\mathrm{Re}_{\mathrm{prob}}$

- **Significance of $\mathrm{Re}_{\mathrm{prob}}$**:
  - Indicates the balance between advective and diffusive effects in the flow.
  - High $\mathrm{Re}_{\mathrm{prob}}$ suggests stronger advective transport, leading to more turbulent behavior.
- **Sudden Changes in $\mathrm{Re}_{\mathrm{prob}}$**:
  - Sudden increases may indicate transitions to turbulence or the formation of new vortices.
  - Decreases may signal a stabilization of the flow or enhanced diffusion.
- **Implications**:
  - Monitoring $\mathrm{Re}_{\mathrm{prob}}$ helps predict flow regime changes.
  - Crucial for understanding and controlling turbulence in practical applications.

Results: Increase in $\mathrm{Re}_{\text{prob}}$ during Simulation with $\nu = 0.001$

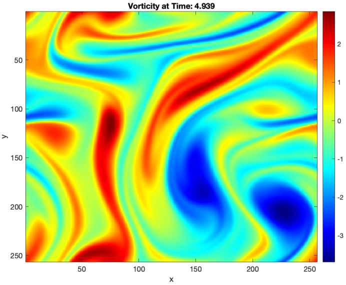

Figure: Vorticity at Time: 4.939

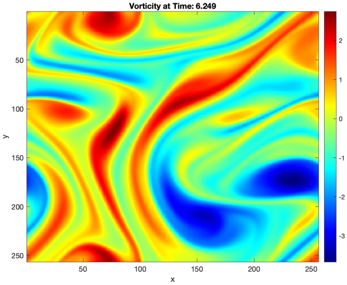

Figure: Vorticity at Time: 6.249

Observation: Increase in $\mathrm{Re}_{\mathrm{prob}}$

The plots show the vorticity field at two different times during the simulation where $\mathrm{Re}_{\mathrm{prob}}$ increased.

- **At Time 4.939**:
  - At this time, the vorticity field shows emerging regions of high vorticity.
  - These regions indicate increased advective transport, leading to a spike in $\mathrm{Re}_{\mathrm{prob}}$.
  - The interactions between these regions contribute to the increased complexity of the flow.
- **At Time 6.249**:
  - At this later time, the flow becomes more chaotic, with more defined and interacting vortices.
  - The increase in $\mathrm{Re}_{\mathrm{prob}}$ at this point suggests that advective effects are dominating, leading to a more turbulent flow.
  - Enhanced interactions between vortices lead to higher $\mathrm{Re}_{\mathrm{prob}}$ values, indicating stronger advective transport.

Conclusion

- **Fourier-Spectral Method**
  - Successfully implemented for solving 2D incompressible Navier-Stokes equations.
  - Utilized the Crank-Nicolson method for time-stepping.
- **Simulation Observations**
  - Vorticity field evolution is highly dependent on viscosity (ν).
  - Higher viscosity ($\nu = 0.01$) results in more diffusive behavior with smoother and less intense vortices.
  - Lower viscosity ($\nu = 0.001$) leads to complex, turbulent behavior with sharper and more defined vortices.
- **Reynolds Numbers Analysis**
  - Re_{prob}: Indicates the relative importance of advective vs. diffusive effects, with high values suggesting strong advective transport.
  - Re_{max}: Reflects influence of highest velocity points, showing regions with potential instability or turbulence.
  - Re_{mean}: Provides an average measure of flow characteristics, showing overall flow behavior.

Conclusion

- **Key Findings**
  - Increase in $\mathrm{Re}_{\mathrm{prob}}$ indicates transitions to turbulence, with more interactions between vortices.
  - Monitoring $\mathrm{Re}_{\mathrm{prob}}$ helps predict flow regime changes and is crucial for understanding and controlling turbulence.
- **Future Work**
  - Further analysis with different initial conditions and higher Reynolds numbers.
  - Application of the method to more complex geometries and boundary conditions.

References

- Lauber, M. (2020, November 19). Marin Lauber. Marin Lauber. https://marinlauber.github.io/2D-Turbulence/. Available at: `https://marinlauber.github.io/2D-Turbulence/`