STAT401_HW2

2019120256
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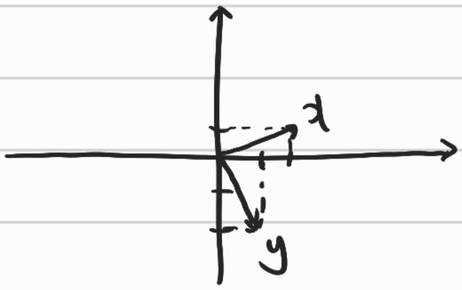
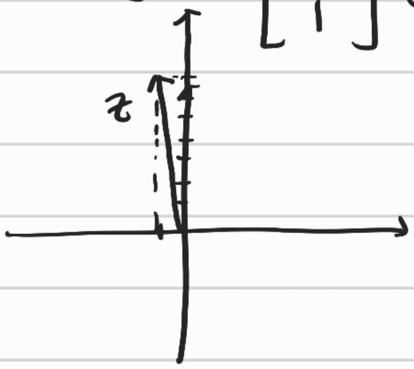
#Q1. Let $x' = (2,1)$, $y' = (1,-2)$

(a)



(b) $z = x - 3y = \begin{bmatrix}2\\1\end{bmatrix} - 3\cdot\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}-1\\7\end{bmatrix}$



(c) $\cos\theta = \dfrac{x'\cdot y}{L_x \cdot L_y} = \dfrac{2\cdot 1 - 1\cdot 2}{\sqrt{5}\cdot\sqrt{5}} = 0.$

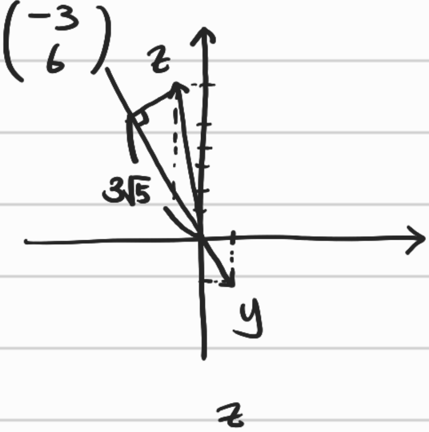
$\therefore\ \theta = 90^\circ$

(d) ∘ Projection of z on y

: $\left(\dfrac{z'y}{y'y}\right) y = -\dfrac{15}{5}\cdot\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix}$

∘ Length of the Projection

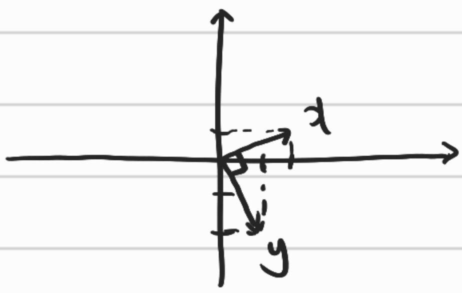
: $\sqrt{(-3)^2+6^2} = \underline{\underline{3\sqrt{5}}}$



(e) • Projection of x on y.

$$;\ \left(\frac{x'y}{y'y}\right)y = 0\cdot\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

• Length of Projection: 0.



Since $x \perp y$, the Length of the projection $= 0$.

#Q2. Let $x_1' = (3,0,0)$, $x_2' = (4,1,0)$, $x_3' = (5\ -6\ 2)$

(a) They are linear independent.

$$a\cdot\begin{bmatrix}3\\0\\0\end{bmatrix} + b\cdot\begin{bmatrix}4\\1\\0\end{bmatrix} + c\cdot\begin{bmatrix}5\\-6\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

1) $3a+4b+5c=0$

2) $b-6c=0$

3) $c=0$

This equation holds true only when $a=b=c=0$.
Therefore, these vectors are linearly independent.

(b)

Let $u_1 = x_1 = \begin{bmatrix}3\\0\\0\end{bmatrix}$

$$u_2 = \begin{bmatrix}4\\1\\0\end{bmatrix} - \frac{(4\cdot3)+(1\cdot0)+(0\cdot0)}{9}\begin{bmatrix}3\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$u_3 = \begin{bmatrix}5\\-6\\2\end{bmatrix} - \frac{5}{3}\begin{bmatrix}3\\0\\0\end{bmatrix} + 6\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

The unit length vectors are

$z_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $z_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $z_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

Q3.

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 4 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$$

(a) Find a vector x such that $Ax = 0$.

Row Reduced Form:

$$U = \begin{pmatrix} 3 & -1 & 2 \\ 4 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 4 & -2 & 3 \\ 3 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$Ax = 0 \Longleftrightarrow Ux = 0.$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$$

This equation holds true when $\begin{cases} x_2 = -x_1 \\ x_3 = -2x_1 \end{cases}$

Therefore $x = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$ could be one answer.

(Unit Length Vector)

(b) By Matrix U, one can show that rank of $A = 2$.

(or Use the fact that row② = row① + row③)

(c) Since rank of A is not 3,

A is a singular matrix.

#Q4. $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$

(a) Eigenvalues, Eigenvectors of A.

$|A - \lambda I| = 0.$

$\left| \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \begin{pmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{pmatrix} \right| = (9-\lambda)(6-\lambda) - 4 = 0.$

$\Leftrightarrow \lambda^2 - 15\lambda + 50 = (\lambda - 10)(\lambda - 5) = 0. \Rightarrow \lambda = 5, 10.$

Let $\lambda_1 = 10$

$A x_1 = \lambda_1 x_1$

$\begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 10 \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$

$\begin{cases} 9x_{11} - 2x_{12} = 10x_{11} \\ -2x_{11} + 6x_{12} = 10x_{12} \end{cases}$

$\rightarrow x_1 = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$.

Let $\lambda_2 = 5$

$A x_2 = \lambda_2 x_2$

$\begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = 5 \cdot \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$

$\begin{cases} 9x_{21} - 2x_{22} = 5x_{21} \\ -2x_{21} + 6x_{22} = 5x_{22}. \end{cases}$

$\Rightarrow x_2 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

Therefore, Eigenvalues and eigenvectors of A:

$\begin{cases} (\lambda_1, x_1) = \left(10, \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}\right) \\ (\lambda_2, x_2) = \left(5, \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\right) \end{cases}$

(b) $x_1' x_2 = \left(\frac{2}{\sqrt{5}}\right) \cdot \left(\frac{1}{\sqrt{5}}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = 0$

Therefore, the eigenvectors of A are perpendicular.

(c) Spectral Decomposition:

$A = \lambda_1 x_1 x_1' + \lambda_2 x_2 x_2'$

$= 10 \cdot \begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$

(d) Eigenvalues, Eigenvectors of A^{-3}

$$A^{-3} = (A^{-1})(A^{-1})(A^{-1}) = P\Lambda^{-1}P' P\Lambda^{-1}P' P\Lambda^{-1}P'$$
$$= P\Lambda^{-1}\Lambda^{-1}\Lambda^{-1}P'$$
$$= \sum_{i=1}^{2} \frac{1}{\lambda_i^3} \cdot x_i \cdot x_i' \text{ where } \lambda_1 = 10 \text{ and } \lambda_2 = 5.$$

(e) $A^{1/2} = \sqrt{\lambda_1}\, x_1 x_1' + \sqrt{\lambda_2}\, x_2 x_2'$

$$= \sqrt{10}\begin{pmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{pmatrix} + \sqrt{5}\begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4\sqrt{10}+\sqrt{5}}{5} & \frac{-2\sqrt{10}+2\sqrt{5}}{5} \\ \frac{-2\sqrt{10}+2\sqrt{5}}{5} & \frac{\sqrt{10}+4\sqrt{5}}{5} \end{pmatrix}$$

#Q5. Find maximum value of $\frac{(x'd)^2}{x'Ax}$ for any non-zero vector $x' = (x_1, x_2)$

$$\max_{x \neq 0} \frac{(x'd)^2}{x'Ax} = d'A^{-1} \cdot d$$

$$= \frac{1}{6} \cdot (3\ \ -3)\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$
$$= \frac{1}{6} \cdot (0\ \ -9) \cdot \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \frac{9}{2}$$

#Q6. Find $x' = (x_1, x_2)$ such that $x'x = 1$.

$$4x_1^2 + 4x_2^2 - 6x_1x_2 = (x_1\ \ x_2)\begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(Note that $4x_1^2 + 4x_2^2 - 6x_1x_2 > 0$)

Let $A = \begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$

Then, $|A - \lambda I| = \left|\begin{pmatrix} 4-\lambda & -3 \\ -3 & 4-\lambda \end{pmatrix}\right| = (4-\lambda)^2 - 9 = 0$

$\Leftrightarrow \lambda^2 - 8\lambda + 7 = (\lambda - 7)(\lambda - 1) = 0.$

$\lambda = 7, 1.$

Maximum $\max_{x \neq 0} \frac{x'Ax}{x'x} = 7$

Minimum $\min_{x \neq 0} \frac{x'Ax}{x'x} = 1.$

Q7.

① $\frac{z'Bz}{z'z} = \frac{z'P\Lambda P'z}{z'z}$ (P: orthogonal Matrix)

$= \frac{y'\Lambda y}{y'y}$ (with $y = P'z$)

$= \frac{\sum_{i=1}^{k} \lambda_i y_i^2}{\sum_{i=1}^{k} y_i^2} \geq \frac{\sum_{i=1}^{k} \lambda_k y_i^2}{\sum_{i=1}^{k} y_i^2}$

because $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_k$ and
$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_k y_k^2$
$\geq \lambda_k y_1^2 + \lambda_k y_2^2 + \cdots + \lambda_k y_k^2$

$= \frac{\lambda_k \cdot \cancel{\sum_{i=1}^{k} y_i^2}}{\cancel{\sum_{i=1}^{k} y_i^2}}$

$\frac{z'Bz}{z'z} \geq \lambda_k$, therefore $\min \frac{z'Bz}{z'z} = \lambda_k$ for any $z \neq 0$.

② $y = P'x_k = (x_1\ x_2 \cdots x_k)'x_k = \begin{pmatrix} x_1'x_k \\ x_2'x_k \\ \vdots \\ x_k'x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

since
$x_k'x_k = 1$ and
$x_1'x_k = x_2'x_k$
$= x_3'x_k = \cdots = x_{k-1}'x_k = 0$

$y'\Lambda y = (1, 0, \cdots, 0) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \lambda_k \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = (0, \cdots, 0, \lambda_k) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \lambda_k$.

and $y'y = (0 \cdots 0\ 1) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = 1$.

For this choice of $z = x_k$, $\frac{z'Bz}{z'z} = \frac{y'\Lambda y}{y'y} = \frac{\lambda_k}{1} = \lambda_k$