PROBLEM SET #2 SOLUTIONS

#1 (a) $d(P,Q)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

(b) YES. $\|\vec{r}_1-\vec{r}_2\|=\|\langle x_1-x_2,\, y_1-y_2,\, z_1-z_2\rangle\|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

(c) $1=\|\vec{r}-\vec{r}_0\|=$ distance between (x,y,z) and $(1,1,1)$.

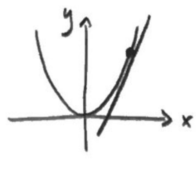
That is, the set of all points distance 1 from $(1,1,1)$.

A sphere of radius 1 centered at $(1,1,1)$.

$$(x-1)^2+(y-1)^2+(z-1)^2=1$$

#2 $\vec{v}=7\frac{\vec{u}}{\|\vec{u}\|}=7\frac{\langle 5,-1,2\rangle}{\sqrt{25+1+4}}=\frac{7}{\sqrt{30}}\langle 5,-1,2\rangle=\left\langle \frac{35}{\sqrt{30}},\frac{-7}{\sqrt{30}},\frac{14}{\sqrt{30}}\right\rangle$

#3 $y=x^2$



(a) $m=2x\big|_{x=3}=6 \implies \langle 1,6\rangle$

(b) Negative reciprocal is $-\frac{1}{6} \implies \langle -6,1\rangle$

#4 (a) $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$, so $\vec{a}\cdot(\vec{a}\times\vec{b})=0$.

(b) $\vec{a}\times\vec{b}$ is parallel to $\vec{a}\times\vec{b}$ (i.e., itself), so $(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{b})=\vec{0}$.

#5 $\vec{u}\cdot\vec{v}=-2-8-50=-60\neq 0 \implies$ Not perpendicular.

$-2\vec{u}=\vec{v} \implies$ Parallel!

#6 (a) $\vec{e}_1 \cdot \vec{e}_2 = -12 + 12 = 0$ ✓

$\vec{e}_1 \cdot \vec{e}_3 = 0$ ✓

$\vec{e}_2 \cdot \vec{e}_3 = 0$ ✓

(b) $$\text{proj}_{\vec{e}_1} \vec{v} = \left(\frac{\vec{e}_1 \cdot \vec{v}}{\|\vec{e}_1\|} \right) \frac{\vec{e}_1}{\|\vec{e}_1\|} = \frac{-10}{\sqrt{25}} \frac{\langle 3,4,0 \rangle}{\sqrt{25}} = -\frac{10}{25} \langle 3,4,0 \rangle = \langle \frac{-6}{5}, \frac{-8}{5}, 0 \rangle$$

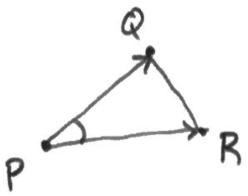
$$\text{proj}_{\vec{e}_2} \vec{v} = \left(\frac{\vec{e}_2 \cdot \vec{v}}{\|\vec{e}_2\|} \right) \frac{\vec{e}_2}{\|\vec{e}_2\|} = \frac{5}{\sqrt{25}} \frac{\langle -4,3,0 \rangle}{\sqrt{25}} = \frac{5}{25} \langle -4,3,0 \rangle = \langle -\frac{4}{5}, \frac{3}{5}, 0 \rangle$$

$$\text{proj}_{\vec{e}_3} \vec{v} = \left(\frac{\vec{e}_3 \cdot \vec{v}}{\|\vec{e}_3\|} \right) \frac{\vec{e}_3}{\|\vec{e}_3\|} = \frac{5}{\sqrt{25}} \frac{\langle 0,0,5 \rangle}{\sqrt{25}} = \frac{5}{25} \langle 0,0,5 \rangle = \langle 0,0,1 \rangle$$

(c) $$\text{proj}_{\vec{e}_1} \vec{v} + \text{proj}_{\vec{e}_2} \vec{v} + \text{proj}_{\vec{e}_3} \vec{v} = \langle -\frac{6}{5}, \frac{-8}{5}, 0 \rangle + \langle -\frac{4}{5}, \frac{3}{5}, 0 \rangle + \langle 0,0,1 \rangle$$

$$= \langle -2, -1, 1 \rangle = \vec{v}. \quad ✓$$

#7



$\angle QPR$

(a) $\overrightarrow{PQ} = \langle 0, 1, 2 \rangle$, $\overrightarrow{PR} = \langle -2, 1, -1 \rangle$

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \|\overrightarrow{PQ}\| \|\overrightarrow{PR}\| \cos(\theta)$$

$$-1 = \sqrt{5} \sqrt{6} \cos(\theta)$$

$$\theta = \arccos\left(-\frac{1}{\sqrt{30}} \right)$$

(b)

$$A = \underbrace{\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|}_{\text{Half the area of the parallelogram}} = \frac{1}{2}\sqrt{9+16+4} = \frac{\sqrt{29}}{2}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ -2 & 1 & -1 \end{vmatrix} = \langle -3, -4, 2 \rangle$$

#8

$$V = |\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)|$$

$$= \|\vec{v}_1\| \|\vec{v}_2 \times \vec{v}_3\| \left|\cos(\theta_{v_1, v_2 \times v_3})\right|$$

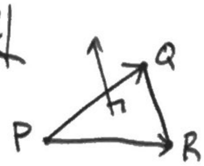
$$= \|\vec{v}_1\| \|\vec{v}_2\| \|\vec{v}_3\| \sin(\theta_{v_2, v_3}) \left|\cos(\theta_{v_1, v_2 \times v_3})\right|$$

$$= (3)(4)(5)\sin(30^\circ)\left|\cos(60^\circ)\right|$$

$$= (60)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= 15$$

#9



$$\vec{PQ} = \langle -5, -2, 4 \rangle$$
$$\vec{PR} = \langle 4, 1, -3 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & -2 & 4 \\ 4 & 1 & -3 \end{vmatrix} = \langle 2, 1, 3 \rangle$$

NEED: ① Point on plane $(3, 0, -1)$ ✓
② Normal vector to plane $\vec{PQ} \times \vec{PR}$ ✓

$$2(x-3) + 1(y-0) + 3(z+1) = 0$$

#10 NEED: ① Point on line $(2,1,0)$ ✓

② Parallel vector to line $\langle 1,3,0\rangle \times \langle -1,1,-1\rangle$ ✓

$$\langle 1,3,0\rangle \times \langle -1,1,-1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ -1 & 1 & -1 \end{vmatrix} = \langle -3, 1, 4\rangle$$

Parametric Equations
$$\begin{cases} x(t) = 2 - 3t \\ y(t) = 1 + t \\ z(t) = 0 + 4t \end{cases} \qquad t \in \mathbb{R}$$

Symmetric Equation
$$\frac{x-2}{-3} = \frac{y-1}{1} = \frac{z-0}{4}$$

Vector Equation
$$\vec{r}(t) = \langle 2-3t, 1+t, 0+4t\rangle, \quad t \in \mathbb{R}$$

#11
$$\int \left(\frac{1}{t^2+1}\,\hat{i} + t e^{t^2}\,\hat{j} + \sqrt{t}\,\hat{k} \right) dt = \left\langle \arctan(t) + A,\ \frac{1}{2} e^{t^2} + B,\ \frac{2}{3} t^{3/2} + C \right\rangle$$

ASIDE:

$$\int \frac{1}{t^2+1}\,dt = \arctan(t) + A$$

$$\int t e^{t^2}\,dt = \frac{1}{2}\int e^u\,du = \frac{1}{2} e^u + B = \frac{1}{2} e^{t^2} + B$$

Let $u = t^2$

$du = 2t\,dt$

$$\int \sqrt{t}\,dt = \int t^{1/2}\,dt = \frac{2}{3} t^{3/2} + C$$

#12. $\vec{r}(t) = \langle \ln(t), 2t, t^2 \rangle \qquad (0,2,1) \longleftrightarrow t_0 = 1$

Normal Plane needs $\vec{r}'(t_0)$ (or $\vec{T}(t_0)$)

$$\vec{r}'(t) = \left\langle \frac{1}{t}, 2, 2t \right\rangle$$

$$\vec{r}'(1) = \langle 1, 2, 2 \rangle$$

$$\boxed{1(x-0) + 2(y-2) + 2(z-1) = 0}$$

Osculating Plane needs $\vec{B}(t_0)$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle \frac{1}{t}, 2, 2t \rangle}{\sqrt{\frac{1}{t^2} + 4 + 4t^2}} = \frac{\langle \frac{1}{t}, 2, 2t \rangle}{\sqrt{\frac{1+4t^2+4t^4}{t^2}}} = \frac{\langle \frac{1}{t}, 2, 2t \rangle}{\sqrt{\frac{(2t^2+1)^2}{t^2}}}$$

$$= \frac{t}{2t^2+1} \left\langle \frac{1}{t}, 2, 2t \right\rangle = \left\langle \frac{1}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1} \right\rangle$$

$$\vec{T}'(t) = \left\langle \frac{-4t}{(2t^2+1)^2}, \frac{2(2t^2+1) - 2t(4t)}{(2t^2+1)^2}, \frac{4t(2t^2+1) - 2t^2(4t)}{(2t^2+1)^2} \right\rangle$$

$$\vec{T}'(1) = \left\langle -\frac{4}{9}, -\frac{2}{9}, \frac{4}{9} \right\rangle, \quad \vec{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\|\vec{T}'(1)\| = \sqrt{\frac{16+4+16}{81}} = \sqrt{\frac{36}{81}} = \frac{6}{9} = \frac{2}{3}$$

$$\vec{N}(1) = \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = \frac{3}{2}\left\langle -\frac{4}{9}, -\frac{2}{9}, \frac{4}{9} \right\rangle = \left\langle -\frac{2}{3}, \frac{-1}{3}, \frac{2}{3} \right\rangle$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \frac{1}{3 \cdot 3} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{vmatrix} = \frac{1}{9}\langle 6, -6, 3 \rangle = \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle$$

From the $\frac{1}{3}$ in $\vec{T}(1)$ and $\frac{1}{3}$ in $\vec{N}(1)$

$$\boxed{\frac{2}{3}(x-0) - \frac{2}{3}(y-2) + \frac{1}{3}(z-1) = 0}$$

#13 $\vec{v}(t) = \int \vec{a}(t)\,dt = \langle -\cos(t) + A,\ 2\sin(t) + B,\ 3t^2 + C \rangle$

$\langle 0, 0, -1 \rangle = \vec{v}(0) = \langle -1 + A,\ B,\ C \rangle$

$A = 1,\ B = 0,\ C = -1$

$\vec{v}(t) = \langle -\cos(t) + 1,\ 2\sin(t),\ 3t^2 - 1 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle -\sin(t) + t + A,\ -2\cos(t) + B,\ t^3 - t + C \rangle$

$\langle 0, 1, -4 \rangle = \vec{r}(0) = \langle A,\ -2 + B,\ C \rangle$

$A = 0,\ B = 3,\ C = -4$

$$\boxed{\vec{r}(t) = \langle -\sin(t) + t,\ -2\cos(t) + 3,\ t^3 - t - 4 \rangle}$$

#14 $\vec{r}(t) = \langle 2t, \sin(3t), -\cos(3t) \rangle$

$(0, 0, -1) \longleftrightarrow t = 0$

$(\pi, -1, 0) \longleftrightarrow t = \frac{\pi}{2}$

$\vec{r}\,'(t) = \langle 2, 3\cos(3t), 3\sin(3t) \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{4 + 9\cos^2(3t) + 9\sin^2(3t)} = \sqrt{4 + 9} = \sqrt{13}$

$$L = \int_{t=0}^{t=\pi/2} \sqrt{13}\,dt = \left[\sqrt{13}\,t\right]_0^{\pi/2} = \sqrt{13}\left(\frac{\pi}{2} - 0\right) = \boxed{\frac{\sqrt{13}\,\pi}{2}}$$

#15 $x^2+y^2=9$, $x+z=1$

(a) Choose $x = 3\cos(t)$, $y = 3\sin(t)$, So $z = 1-3\cos(t)$ $\Rightarrow$ $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 1-3\cos(t) \rangle$

(b) Need: ① Point $(3,0,-2)$ ✓

② Parallel vector to line, $\vec{r}\,'(t_0)$ ✓

$(3,0,-2) \longleftrightarrow t_0 = 0$

$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 3\sin(t) \rangle$

$\vec{r}\,'(0) = \langle 0, 3, 0 \rangle$

Parametric Equations $\begin{cases} x(t) = 3 \\ y(t) = 3t \\ z(t) = -2 \end{cases}$ $t \in \mathbb{R}$

#16 $\vec{r}(t) = \langle t, (t-1)^2, 0 \rangle$

$\vec{r}\,'(t) = \langle 1, 2(t-1), 0 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1+4(t-1)^2} = \sqrt{1+4t^2-8t+4} = \sqrt{4t^2-8t+5}$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{4t^2-8t+5}}, \frac{2t-2}{\sqrt{4t^2-8t+5}}, 0 \right\rangle$$

$$\vec{T}\,'(t) = \left\langle -\frac{1}{2}(4t^2-8t+5)^{-3/2}(8t-8), \frac{2(4t^2-8t+5)^{1/2} - (2t-2)\frac{1}{2}(4t^2-8t+5)^{-1/2}(8t-8)}{4t^2-8t+5}, 0 \right\rangle$$

$$\vec{T}'(t) = \left\langle \frac{4-4t}{(4t^2-8t+5)^{3/2}}, \frac{2(4t^2-8t+5)-(t-1)(8)(t-1)}{(4t^2-8t+5)^{3/2}}, 0 \right\rangle$$

$$= \left\langle \frac{4-4t}{(4t^2-8t+5)^{3/2}}, \frac{\cancel{8t^2}\cancel{-16t}+10-8(\cancel{t^2}\cancel{-2t}+1)}{(4t^2-8t+5)^{3/2}}, 0 \right\rangle$$

$$= \left\langle \frac{4-4t}{(4t^2-8t+5)^{3/2}}, \frac{2}{(4t^2-8t+5)^{3/2}}, 0 \right\rangle$$

$$\|\vec{T}'(t)\| = \sqrt{\frac{(4-4t)^2+2^2+0^2}{(4t^2-8t+5)^3}} = \sqrt{\frac{16(1-2t+t^2)+4}{(4t^2-8t+5)^3}}$$

$$= \sqrt{\frac{20-32t+16t^2}{(4t^2-8t+5)^3}} = \sqrt{\frac{4(5-8t+4t^2)}{(4t^2-8t+5)^3}}$$

$$= \sqrt{\frac{4}{(4t^2-8t+5)^2}} = \frac{2}{4t^2-8t+5}$$

$$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \left(\frac{2}{4t^2-8t+5}\right)\left(\frac{1}{\sqrt{4t^2-8t+5}}\right) = \boxed{\frac{2}{(4t^2-8t+5)^{3/2}}}$$

(b) $K'(t) = -3(4t^2-8t+5)^{-5/2}(8t-8)$

$(1,0,0) \longleftrightarrow t_0 = 1$

$K'(0) > 0$

$K'(2) < 0$

$$\begin{array}{ccc} + & & - \\ \longleftarrow & \underset{1}{+} & \longrightarrow K'(t) \\ \nearrow & & \searrow \end{array}$$

$\boxed{\text{(1,0,0) is the relative maximum!}}$