# PROBLEM SET #6 SOLUTIONS

Integration review:

**#1**

(a) $$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

(b) $$\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

$$\begin{array}{c|c} x^2 & e^x \\ {}^{+} & \\ 2x & e^x \\ {}^{-} & \\ 2 & e^x \\ {}^{+} & \\ 0 & e^x \end{array}$$

(c) $$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)du$$

$u = x^2+1 \longleftrightarrow x^2 = u-1$

$\frac{du}{2} = \frac{\cancel{2}x\,dx}{\cancel{2}}$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} + C$$

(d) $$\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + C$$

$$\begin{array}{c|c} x & \sin(x) \\ {}^{+} & \\ 1 & -\cos(x) \\ {}^{-} & \\ 0 & -\sin(x) \end{array}$$

#2

(a) $$\int_0^{\pi/2} \frac{1}{2}\cos^2(\theta)\,d\theta = \int_0^{\pi/2} \frac{1}{2}\cdot\frac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(1+\cos(2\theta)\right)d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{\pi}{2}+0\right) - \frac{1}{4}\left(0+0\right) = \frac{\pi}{8}$$

(b) $$\int_0^{\pi/3} \sin^3(\theta)\,d\theta = \int_0^{\pi/3}\left(1-\cos^2(\theta)\right)\sin(\theta)\,d\theta$$

$$= \int_0^{\pi/3}\sin(\theta)\,d\theta - \int_0^{\pi/3} \underbrace{\cos^2(\theta)\sin(\theta)\,d\theta}_{u\text{-sub.}}$$

$$= \left[-\cos(\theta)\right]_0^{\pi/3} + \left[\frac{\cos^3(\theta)}{3}\right]_0^{\pi/3}$$

$$= \left(-\frac{1}{2}+1\right) + \left(\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{24} - \frac{1}{3} = \frac{5}{24}$$

(c) $$\int_0^{\pi/4} \sin^4(\phi)\cos(\phi)\,d\phi = \int_0^{\sqrt{2}/2} u^4\,du = \left[\frac{u^5}{5}\right]_0^{\sqrt{2}/2} = \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5 = \frac{\sqrt{2}}{40}$$

$u = \sin(\phi)$

$du = \cos(\phi)\,d\phi$

(d) $$\int_0^{\pi/3} \sin^2(y)\,dy = \frac{1}{2}\int_0^{\pi/3}\left(1-\cos(2y)\right)dy = \frac{1}{2}\left[y - \frac{1}{2}\sin(2y)\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - \frac{1}{2}\left(0-0\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

3/14

# Matching

Match each entry in the left column with the possible correct option(s) in the right column. Some of the options in the right column could be matched with more than one entry in the left column.

#1 F The cross product of two nonzero parallel vectors, **a** and **b**.

#2 I The magnitude of the cross product of two nonzero perpendicular vectors, **a** and **b**.

#3 I, J The dot product of two nonzero parallel vectors, **a** and **b**.

#4 L The dot product of two nonzero perpendicular vectors, **a** and **b**.

#5 N The cross product of two perpendicular unit vectors.

#6 L The dot product of two perpendicular unit vectors.

#7 L $\nabla \cdot (\nabla \times \mathbf{F})$

#8 H $\nabla(\nabla \times \mathbf{F})$

#9 F $\nabla \times (\nabla f)$

#10 L $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a})$

#11 F The curl of a conservative vector field.

#12 D, G $D_{\nabla f} f$

#13 A, E $D_{-\nabla f} f$

#14 K, Q The set of all points $(x, y, z) \in \mathbb{R}^3$ such that $\|\mathbf{r} - \mathbf{r}_0\| = 1$, where $\mathbf{r}_0 = \langle 1, 1, 1 \rangle$.

(A) Minimum rate of change

(B) $\|\mathbf{a}\| + \|\mathbf{b}\|$

(C) 1

(D) $\|\nabla f\|$

(E) $-\|\nabla f\|$

(F) $\mathbf{0}$

(G) Maximum rate of change

(H) Nonsense

(I) $\|\mathbf{a}\|\|\mathbf{b}\|$

(J) $-\|\mathbf{a}\|\|\mathbf{b}\|$

(K) The circle of radius 1 centered at $(1, 1, 1)$

(L) 0

(M) $(x-1) + (y-1) + (z-1) = 0$

(N) Unit vector

(O) $x - 1 = y - 1 = z - 1$

(P) $\langle x-1, y-1, z-1 \rangle$

(Q) $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$

Matching justifications:

**#1** Cross product of parallel vectors is $\vec{0}$.

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\underbrace{\sin(\theta)}_{=0} = 0$$

**#2** $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\underbrace{\cancel{\sin\left(\frac{\pi}{2}\right)}}_{1} = \|\vec{a}\|\|\vec{b}\|$

**#3** $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\underbrace{\cos(\theta)}_{\substack{=1 \text{ if } \theta=0 \\ =-1 \text{ if } \theta=\pi}} = \|\vec{a}\|\|\vec{b}\|$ OR $-\|\vec{a}\|\|\vec{b}\|$

**#4** Dot product of perpendicular vectors is 0.

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\overbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}^{0} = 0$$

**#5** $\|\vec{a}\times\vec{b}\| = \underbrace{\cancel{\|\vec{a}\|}}_{1}\underbrace{\cancel{\|\vec{b}\|}}_{1}\underbrace{\cancel{\sin\left(\frac{\pi}{2}\right)}}_{1} = 1$

So, $\vec{a}\times\vec{b}$ is also a unit vector.

**#6** Same as #4

**#7** $\nabla\cdot(\nabla\times\vec{F}) = 0$.

proof. $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \langle h_y - g_z, -(h_x - f_z), g_x - f_y \rangle$

So, $\nabla\cdot(\nabla\times\vec{F}) = (h_y - g_z)_x - (h_x - f_z)_y + (g_x - f_y)_z = 0$.

**#8** Nonsense! Can't take the gradient of a vector!

**#9** $\nabla \times (\nabla f) = \vec{0}$

proof. $$\nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle 0-0, -(0-0), 0-0 \rangle = \vec{0}.$$

**#10** $\vec{b} \times \vec{a}$ is perpendicular to $\vec{b}$ and $\vec{a}$.

So, $\vec{a} \cdot (\vec{b} \times \vec{a}) = 0$.

**#11** $\nabla \times \vec{F} = \vec{0}$ if $\vec{F}$ is conservative.

**#12** $$D_{\nabla f} f = \nabla f \cdot \frac{\nabla f}{\|\nabla f\|} = \frac{\|\nabla f\|^2}{\|\nabla f\|} = \|\nabla f\|$$

**#13** $$D_{-\nabla f} f = \nabla f \cdot \frac{-\nabla f}{\|-\nabla f\|} = \frac{-\|\nabla f\|^2}{\|\nabla f\|} = -\|\nabla f\|$$

**#14** all points distance 1 from $(1,1,1)$.
(That's just the sphere of radius 1 centered at $(1,1,1)$).

# Practice problems:

**#1** $\vec{a} = \langle -1, 1, 0 \rangle$ , $\vec{b} = \langle 2, -1, 2 \rangle$ , $\vec{c} = \langle 1, 1, 0 \rangle$

(a) $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 2 & -1 & 2 \end{vmatrix} = \langle 2, 2, -1 \rangle$$

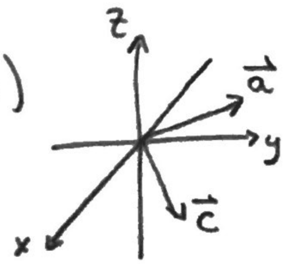
$$\sqrt{9} = \sqrt{2}\sqrt{9} \sin(\theta)$$

$$\theta = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$$

(b) $$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} = \frac{-3}{\sqrt{2}}$$

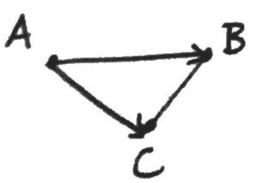
$$\text{proj}_{\vec{a}} \vec{b} = \left(\text{comp}_{\vec{a}} \vec{b}\right) \frac{\vec{a}}{\|\vec{a}\|} = \left(-\frac{3}{\sqrt{2}}\right) \frac{\langle -1, 1, 0 \rangle}{\sqrt{2}} = \left\langle \frac{3}{2}, \frac{-3}{2}, 0 \right\rangle$$

(c)




$\vec{a} \times \vec{c}$ is in the <u>negative</u> z-direction.

**#2**




$\overrightarrow{AB} = \langle 1, -1, 0 \rangle$ , $\overrightarrow{AC} = \langle 1, 0, -1 \rangle$

<u>NEED</u>: ① Point ✓ $A(0,1,1)$

② Normal vector ✓ $$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$$\boxed{1(x-0) + 1(y-1) + 1(z-1) = 0}$$

**#3** $7\dfrac{\vec{u}}{\|\vec{u}\|} = 7\dfrac{\langle 2,3,-4\rangle}{\sqrt{4+9+16}} = \dfrac{7}{\sqrt{29}}\langle 2,3,-4\rangle$

$= \boxed{\left\langle \dfrac{14}{\sqrt{29}}, \dfrac{21}{\sqrt{29}}, \dfrac{-28}{\sqrt{29}} \right\rangle}$

**#4** $\vec{v}(t) = \int \vec{a}(t)\,dt = \left\langle t^2+A, -\cos(t)+B, \frac{1}{2}\sin(2t)+C \right\rangle$

$\langle 1,0,0\rangle = \vec{v}(0) = \langle A, -1+B, C\rangle$

$A=1, \ B=1, \ C=0$

$\vec{v}(t) = \left\langle t^2+1, -\cos(t)+1, \frac{1}{2}\sin(2t)\right\rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \frac{t^3}{3}+t+D, -\sin(t)+t+E, -\frac{1}{4}\cos(2t)+F \right\rangle$

$\langle 0,1,0\rangle = \vec{r}(0) = \left\langle D, E, -\frac{1}{4}+F\right\rangle$

$D=0, \ E=1, \ F=\frac{1}{4}$

$\boxed{\vec{r}(t) = \left\langle \frac{t^3}{3}+t, -\sin(t)+t+1, -\frac{1}{4}\cos(2t)+\frac{1}{4} \right\rangle}$

**#5** $f(x,y,z) = e^{x+y} + \sin(z+x^2)$

(a) $\nabla f = \left\langle e^{x+y} + 2x\cos(z+x^2), e^{x+y}, \cos(z+x^2)\right\rangle$

(b) $D_{\vec{u}} f\Big|_{(0,0,0)} = \nabla f\Big|_{(0,0,0)} \cdot \dfrac{\vec{u}}{\|\vec{u}\|} = \langle 1,1,1\rangle \cdot \dfrac{\langle 1,1,1\rangle}{\sqrt{3}}$

$= \dfrac{3}{\sqrt{3}} = \boxed{\sqrt{3}}$

#6 $z^2 - x^2 - y^2 = 0$

(a) Cone. $z = \pm\sqrt{x^2+y^2}$

(b) NEED:

① Point ✓ $(3,4,5)$

② Normal vector $\nabla F\big|_{(3,4,5)}$

$$F(x,y,z) = z^2 - x^2 - y^2$$

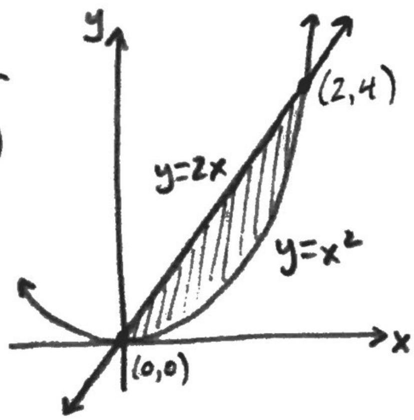
$$\nabla F = \langle -2x, -2y, 2z \rangle$$

$$\nabla F\big|_{(3,4,5)} = \langle -6, -8, 10 \rangle$$

$$\boxed{-6(x-3) - 8(y-4) + 10(z-5) = 0}$$

#7

(a)




$2x = x^2$

$x=0, x=2$

$y=0, y=4$

(b) Type 1 $\displaystyle\int_0^2 \int_{x^2}^{2x} xy \, dy \, dx$

Type 2 $\displaystyle\int_0^4 \int_{y/2}^{\pm\sqrt{y}} xy \, dx \, dy$

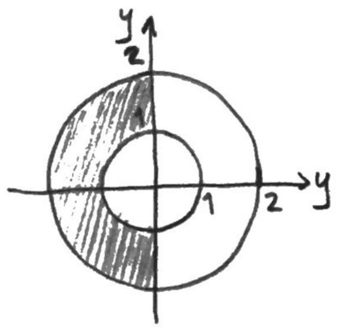
(c) $$\int_0^2 \int_{x^2}^{2x} xy \, dy \, dx = \int_0^2 x \left[\frac{y^2}{2}\right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 \frac{x}{2}\left(4x^2 - x^4\right) dx$$

$$= \frac{1}{2}\int_0^2 \left(4x^3 - x^5\right) dx$$

$$= \frac{1}{2}\left[x^4 - \frac{x^6}{6}\right]_{x=0}^{x=2} = \frac{1}{2}\left(16 - \frac{64}{6}\right) = \boxed{\frac{8}{3}}$$

#8




$$\iint_D xy^2 \, dA = \int_{\pi/2}^{3\pi/2} \int_1^2 r^4 \cos(\theta)\sin^2(\theta) \, dr \, d\theta$$

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$dA = r \, dr \, d\theta$

$$= \int_{\pi/2}^{3\pi/2} \underbrace{\cos(\theta)\sin^2(\theta)}\, d\theta \int_1^2 r^4 \, dr = \left[\frac{\sin^3(\theta)}{3}\right]_{\pi/2}^{3\pi/2} \left[\frac{r^5}{5}\right]_1^2$$

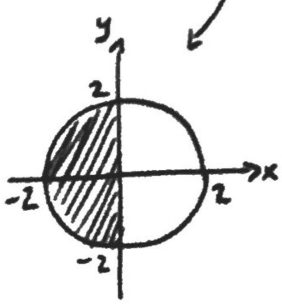
$u = \sin(\theta)$

$du = \cos(\theta) d\theta$

$$= \left(-\frac{1}{3} - \frac{1}{3}\right)\left(\frac{32-1}{5}\right) = \boxed{\frac{-62}{15}}$$

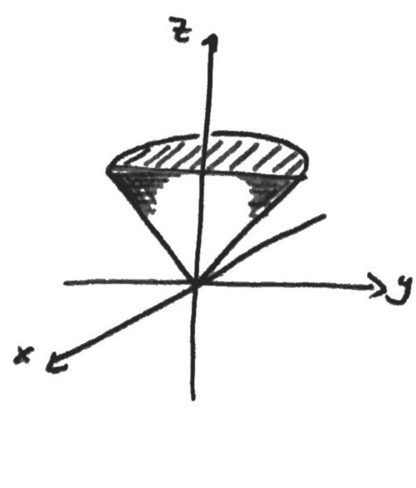
#9 $$\underbrace{\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2) \, dz \, dy \, dx$$

$\rightarrow$ cone $\leq z \leq 2$




$0 \leq r \leq 2$

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

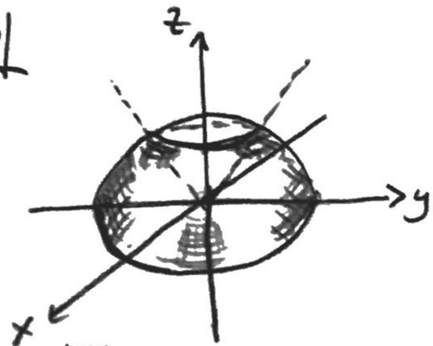


$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 \int_r^2 r^2 \, r \, dz \, dr \, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 r^3 \left[ z \right]_{z=r}^{z=2} dr \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 \left( 2r^3 - r^4 \right) dr \, d\theta = \pi \left[ \frac{r^4}{2} - \frac{r^5}{5} \right]_{r=0}^{r=2} = \pi \left( 8 - \frac{32}{5} \right)$$

$$= \boxed{\frac{8\pi}{5}}$$

#10




$$\iiint_E \sin\left( (x^2+y^2+z^2)^{3/2} \right) dV$$

$$= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^1 \sin(\rho^3) \, \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi$$

$$= \int_{\pi/4}^{\pi/2} \sin(\phi) \, d\phi \int_0^{2\pi} d\theta \int_0^1 \underbrace{\sin(\rho^3) \rho^2}_{\text{u-sub.}} d\rho$$

$u = \rho^3$

$du = 3\rho^2 d\rho$

$$= \left[ -\cos(\phi) \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[ \theta \right]_0^{2\pi} \left[ -\frac{\cos(\rho^3)}{3} \right]_0^1$$

$$= \left( -0 + \frac{\sqrt{2}}{2} \right) \left( 2\pi - 0 \right) \left( \frac{-\cos(1) + 1}{3} \right)$$

$$= \boxed{\frac{(1 - \cos(1)) \sqrt{2} \, \pi}{3}}$$

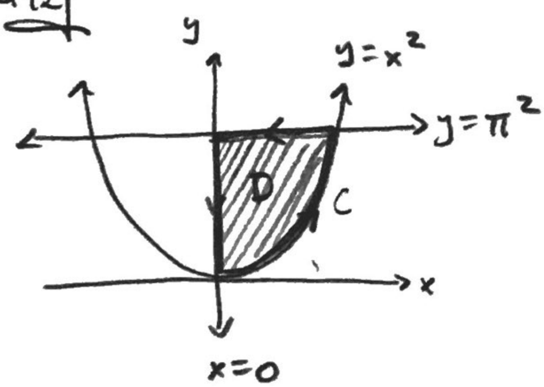
**#11** $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

$$= \int_0^1 \langle t^3, t^5, t^4\rangle \cdot \langle 1, 2t, 3t^2\rangle\, dt$$

$$= \int_0^1 \left(t^3 + 2t^6 + 3t^6\right) dt$$

$$= \int_0^1 \left(t^3 + 5t^6\right) dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

**#12**




$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(Q_x - P_y\right) dA$$

$$= \int_0^{\pi}\int_{x^2}^{\pi^2} \left(0 + \sin(x)\right) dy\,dx$$

$$= \int_0^{\pi} \sin(x)\left(\pi^2 - x^2\right) dx$$

$$= \int_0^{\pi} \pi^2 \sin(x)\,dx - \int_0^{\pi} x^2 \sin(x)\,dx$$

$$= \Big[-\pi^2\cos(x)\Big]_0^{\pi} - \Big[-x^2\cos(x) + 2x\cancel{\sin(x)}^{\,0} + 2\cos(x)\Big]_0^{\pi}$$

$$= \left(\cancel{\pi^2} + \pi^2\right) - \left(\cancel{\pi^2} - 2 - 2\right)$$

$$= \boxed{\pi^2 + 4}$$

→ Positive orientation.

→ Closed curve.

→ Simply connected region.

→ Continuous partial derivatives.

| | | |
|---|---|---|
| $x^2$ | | $\sin(x)$ |
| | $+$ | |
| $2x$ | | $-\cos(x)$ |
| | $-$ | |
| $2$ | | $-\sin(x)$ |
| | $+$ | |
| $0$ | | $\cos(x)$ |

#13

(a) $\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$

(b)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, 1+1 \rangle = \langle 0, 0, 2 \rangle$$

(c) Counter-clockwise.

(d) $P_y = e^x$ ✓ Yes! Conservative!

$Q_x = e^x$ ✓

$$f(x,y) = \int y e^x \, dx = y e^x + A(y)$$

$$f(x,y) = \int (e^x + e^y) \, dy = y e^x + e^y + B(x)$$

$$\therefore \boxed{f(x,y) = y e^x + e^y}$$

#14 $f(x,y) = x^2 + xy + y^2 + y$

① Find the critical points.

$0 = f_x = 2x + y \longrightarrow y = -2x$

$0 = f_y = x + 2y + 1$

$0 = x - 4x + 1$

$3x = 1$

$x = \frac{1}{3}$

$y = -\frac{2}{3}$

$\left(\frac{1}{3}, -\frac{2}{3}\right)$

② $D = f_{xx} f_{yy} - f_{xy}^2$

$= (2)(2) - (1)^2$

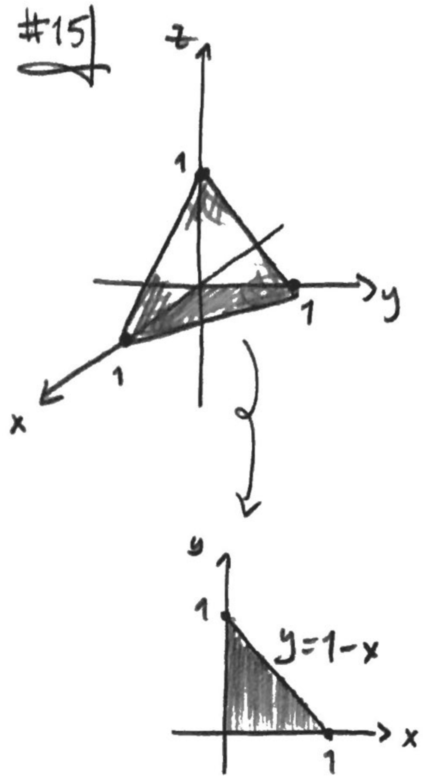
$= 3 > 0$

$D\left(\frac{1}{3}, -\frac{2}{3}\right) > 0$

$f_{xx}\left(\frac{1}{3}, -\frac{2}{3}\right) = 2 > 0$

$$\boxed{\text{Local minimum. } f\left(\tfrac{1}{3}, -\tfrac{2}{3}\right) = -\tfrac{1}{3}}$$

#15




(a) $x = u$

$y = v$

$z = 1 - u - v$

$$\begin{cases} \vec{r}(u,v) = \langle u, v, 1-u-v \rangle \\ 0 \le u \le 1 \\ 0 \le v \le 1-u \end{cases}$$

(b) $\vec{r}_u \times \vec{r}_v = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$

(c) $\displaystyle\iint_S x\, dS = \int_0^1 \int_0^{1-u} u \, \|\langle 1,1,1 \rangle\| \, dv\, du$

$\displaystyle = \sqrt{3} \int_0^1 u(1-u)\, du = \sqrt{3} \int_0^1 (u - u^2)\, du = \sqrt{3}\left(\frac{1}{2} - \frac{1}{3}\right)$

$= \boxed{\dfrac{\sqrt{3}}{6}}$

#16

(a) ① E is a simple and solid region.

② S is a closed surface with positive orientation.

③ The components of $\vec{F}$ have continuous partial derivatives on an open set containing E.

(b) All the assumptions for the divergence theorem hold true.

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_E \nabla\cdot\vec{F}\,dV = \iiint_E 3\,dV$$

$$= \int_0^1 \int_0^1 \int_0^{e^{-x-y}} 3\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^1 3e^{-x-y}\,dy\,dx$$

$$= 3\int_0^1 e^{-x}\,dx \int_0^1 e^{-y}\,dy$$

$$= 3\left[-e^{-x}\right]_0^1 \left[-e^{-y}\right]_0^1$$

$$= 3\left(1-e^{-1}\right)\left(1-e^{-1}\right)$$

$$= \boxed{3\left(1-e^{-1}\right)^2}$$