Multivariable Calculus Problem Set 2

Topics: vectors, dot and cross products, projections, lines and planes, vector-valued functions, space curves, tangent lines, applications of space curves

Exercise 1.
(a) Write down the formula that computes the distance d between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.
(b) Let $\mathbf{r}_1$ and $\mathbf{r}_2$ denote the position vectors corresponding to points P and Q, respectively. Does $\|\mathbf{r}_1 - \mathbf{r_2}\|$ equal the formula from part (a)?
(c) Let $\mathbf{r} = \langle x, y, z\rangle$ be an arbitrary position vector, and also let $\mathbf{r}_0 = \langle 1, 1, 1\rangle$. Based on your answers in parts (a) and (b), describe in words the set of all points (x, y, z) such that $\|\mathbf{r} - \mathbf{r}_0\| = 1$.

Exercise 2. Consider the vector $\mathbf{u} = \langle 5, -1, 2\rangle$. Compute a vector $\mathbf{v}$ that points in the same direction as $\mathbf{u}$ and has length 7.

Exercise 3. Consider the curve $y = x^2$.
(a) Find a vector that is parallel to the tangent line to the curve at the point $(3, 9)$.
(b) Find a vector that is perpendicular to the tangent line to the curve at the point $(3, 9)$.

Exercise 4. Let $\mathbf{a}$ and $\mathbf{b}$ be any two nonzero vectors.
(a) What is $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$?
(b) What is $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$?

Exercise 5. Are the vectors $\mathbf{u} = \langle -1, 2, -5\rangle$ and $\mathbf{v} = \langle 2, -4, 10\rangle$ perpendicular, parallel, or neither?

Exercise 6. Let $\mathbf{v} = \langle -2, -1, 1\rangle$, $\mathbf{e}_1 = \langle 3, 4, 0\rangle$, $\mathbf{e}_2 = \langle -4, 3, 0\rangle$, and $\mathbf{e}_3 = \langle 0, 0, 5\rangle$.
(a) Show that $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are all orthogonal to each other.
(b) Compute the vector projections $\mathbf{proj}_{\mathbf{e}_1}\mathbf{v}$, $\mathbf{proj}_{\mathbf{e}_2}\mathbf{v}$ and $\mathbf{proj}_{\mathbf{e}_3}\mathbf{v}$.
(c) Using the vector projections computed in part (b), verify that $\mathbf{v} = \mathbf{proj}_{\mathbf{e}_1}\mathbf{v} + \mathbf{proj}_{\mathbf{e}_2}\mathbf{v} + \mathbf{proj}_{\mathbf{e}_3}\mathbf{v}$.

Exercise 7. Consider the triangle formed by the points $P(1, 1, 1)$, $Q(1, 2, 3)$, and $R(-1, 2, 0)$.
(a) Compute the angle $\measuredangle QPR$.
(b) Compute the area of the triangle ΔPQR.

Exercise 8. There are three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. The magnitudes/lengths of the vectors are $\|\mathbf{v}_1\| = 3$, $\|\mathbf{v}_2\| = 4$ and $\|\mathbf{v}_3\| = 5$. The vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ form a 30 degree angle. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2 \times \mathbf{v}_3$ form a 60 degree angle. Find the volume of the parallelepiped formed by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Exercise 9. Find an equation for the plane that passes through the points $P(3, 0, -1)$, $Q(-2, -2, 3)$, and $R(7, 1, -4)$.

Exercise 10. Find the parametric, symmetric, and vector equations for the line that passes through the point $(2, 1, 0)$ and is perpendicular to both $\langle 1, 3, 0\rangle$ and $\langle -1, 1, -1\rangle$.

Exercise 11. Evaluate $\displaystyle\int \left(\frac{1}{t^2+1}\hat{\mathbf{i}} + te^{t^2}\hat{\mathbf{j}} + \sqrt{t}\hat{\mathbf{k}}\right) dt.$

Exercise 12. Compute the normal and osculating planes for the curve $\mathbf{r}(t) = \langle \ln(t), 2t, t^2 \rangle$ at the point $(0, 2, 1)$.

Exercise 13. Consider the acceleration vector $\mathbf{a}(t) = \sin(t)\hat{\mathbf{i}} + 2\cos(t)\hat{\mathbf{j}} + 6t\hat{\mathbf{k}}$, initial position $\mathbf{r}(0) = \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$, and initial velocity $\mathbf{v}(0) = -\hat{\mathbf{k}}$. Compute the position vector $\mathbf{r}(t)$.

Exercise 14. Consider the curve described by the position vector $\mathbf{r}(t) = \langle 2t, \sin(3t), -\cos(3t) \rangle$. Compute the arc length of the curve from point $(0, 0, -1)$ to point $(\pi, -1, 0)$.

Exercise 15. Consider the cylinder $x^2 + y^2 = 9$ and the plane $x + z = 1$.
(a) Parameterize the curve of intersection between the given cylinder and plane.
(b) Find parametric equations for the tangent line to the curve of intersection at the point $(3, 0, -2)$.

Exercise 16. Consider the curve $\mathbf{r}(t) = \langle t, (t-1)^2, 0 \rangle$.
(a) Compute the curvature $\kappa(t)$.
(b) Using the first derivative test, show that $(1, 0, 0)$ is the point on the curve with the greatest curvature.