Multivariable Calculus Problem Set 5

Topics: line integrals, conservative vector fields, the fundamental theorem for line integrals, curl and divergence, parameterizing surfaces, surface integrals

Exercise 1. Show that $\mathbf{F}(x, y) = \langle ye^x, e^x + e^y \rangle$ is a conservative vector field. Then, find a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

Exercise 2. Let C be the curve along the circle $x^2 + y^2 = 1$ from the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and let $\mathbf{F}(x, y) = \langle y, x + y^2 \rangle$.

(a) Without using the fundamental theorem for line integrals, compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

(b) Show that $\mathbf{F}(\mathbf{x}, \mathbf{y})$ is a conservative vector field. Then, find a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

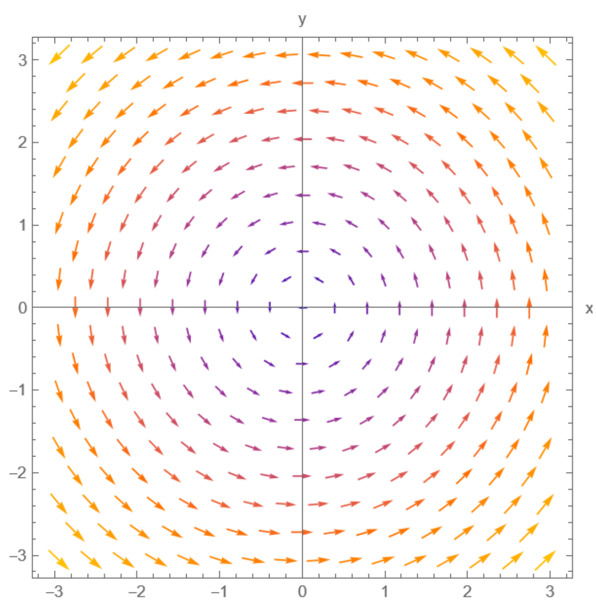
(c) Compute $\int_C \nabla f \cdot d\mathbf{r}$ using the fundamental theorem for line integrals. (You should get the same answer as you did in part (a)).

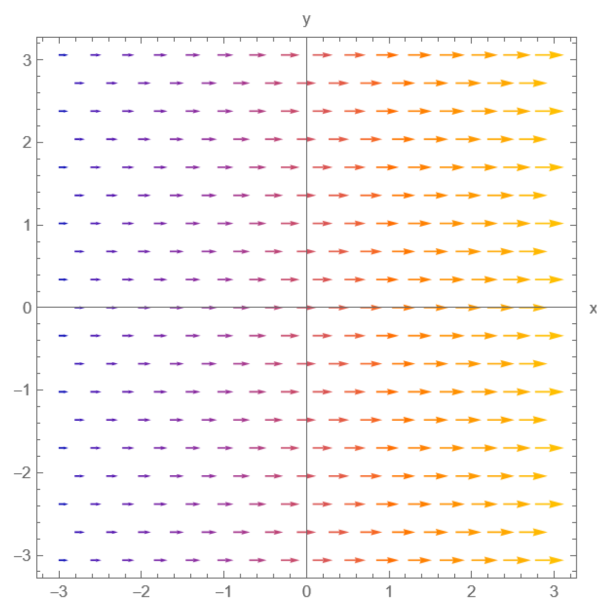
Exercise 3. Let $\mathbf{F}(x, y, z) = \sin(yz)\hat{\mathbf{i}} + \sin(zx)\hat{\mathbf{j}} + \sin(xy)\hat{\mathbf{k}}$. Compute the curl and divergence of this vector field.

Exercise 4. The images shown below are of vector fields that are constant in z, that is, the vector field is the same at every z value. For each vector field shown, determine the following at any point P where $x = 2$ and $y = 1$:

(a) Is the curl of the vector field at the point P the zero vector? If not, is it pointing in the $\hat{\mathbf{k}}$ direction or the $-\hat{\mathbf{k}}$ direction?

(b) Is the divergence of the vector field at the point P positive, negative or zero?





Exercise 5. Parameterize the following surfaces:
(a) The part of the cylinder $x^2 + z^2 = 9$ between $y = -4$ and $y = 4$.
(b) The part of the cylinder $x^2 + z^2 = 9$ above the xy-plane, and between $y = -4$ and $y = 4$.
(c) The part of the cone $z = \sqrt[+]{x^2 + y^2}$ below $z = 1$.
(d) The part of the sphere $x^2 + y^2 + z^2 = 5$ where $z \leq 0$.
(e) The part of the plane $x + \frac{y}{2} + \frac{z}{3} = 1$ that resides in the first octant.

Exercise 6. Let S be the surface described by the parameterization $\mathbf{r}(u, v) = \langle u+v, u-v, 1+2u+v \rangle$ with $0 \leq u \leq 2$ and $0 \leq v \leq 2$.
(a) Compute the surface area of S.
(b) Find an equation for the tangent plane of S at the point $(2, 0, 4)$.