PROBLEM SET #5 SOLUTIONS

#1 $\vec{F} = \langle ye^x, e^x + e^y \rangle$

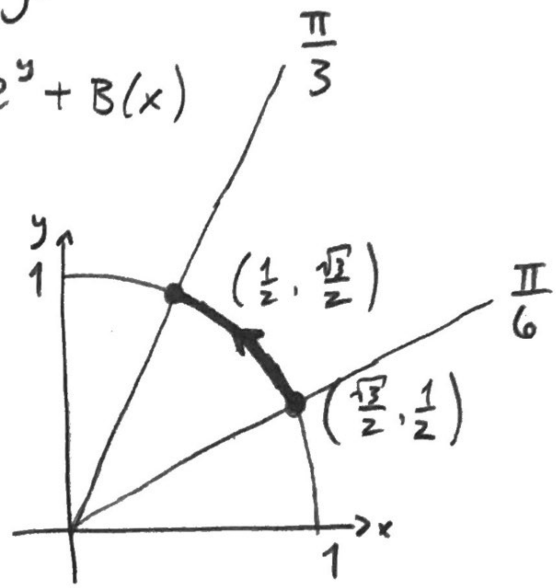
$$P_y = e^x \quad \checkmark$$
$$Q_x = e^x + 0 \quad \checkmark$$

Same!

$$f = \int P\,dx = \int ye^x\,dx = ye^x + A(y)$$

$$f = \int Q\,dy = \int e^x + e^y\,dy = ye^x + e^y + B(x)$$

$$\therefore\ f(x,y) = ye^x + e^y$$



#2 $\vec{F} = \langle y, x + y^2 \rangle$

(a) $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

$$\frac{\pi}{6} \le t \le \frac{\pi}{3}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\pi/6}^{\pi/3} \left\langle \sin(t), \cos(t) + \sin^2(t) \right\rangle \cdot \left\langle -\sin(t), \cos(t) \right\rangle dt$$

$$= \int_{\pi/6}^{\pi/3} -\sin^2(t) + \underbrace{\cos^2(t)}_{1-\sin^2(t)} + \sin^2(t)\cos(t)\,dt$$

$$= \int_{\pi/6}^{\pi/3} 1 - 2\sin^2(t) + \sin^2(t)\cos(t)\,dt$$

$$= \int_{\pi/6}^{\pi/3} 1 - \cancel{2} \cdot \frac{1-\cos(2t)}{\cancel{2}} + \underbrace{\sin^2(t)\cos(t)\,dt}_{u\text{-sub.}}$$

$$= \left[\cancel{t} - \cancel{t} + \frac{1}{2}\sin(2t) + \frac{\sin^3(t)}{3} \right]_{t=\pi/6}^{t=\pi/3}$$

$$= \left(\frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{\sin^3\left(\frac{\pi}{3}\right)}{3} \right) - \left(\frac{1}{2}\sin\left(\frac{\pi}{3}\right) + \frac{\sin^3\left(\frac{\pi}{6}\right)}{3} \right)$$

$$= \left(\cancel{\frac{\sqrt{3}}{4}} + \frac{\sqrt{3}}{8} \right) - \left(\cancel{\frac{\sqrt{3}}{4}} + \frac{1}{24} \right) = \frac{\sqrt{3}}{8} - \frac{1}{24} = \boxed{\frac{3\sqrt{3}-1}{24}}$$

(b) $P_y = 1$ ✓ $\quad f = \int P\,dx = \int y\,dx = xy + A(y)$

$Q_x = 1 + 0$ ✓ $\quad f = \int Q\,dy = \int x + y^2\,dy = xy + \frac{y^3}{3} + B(x)$

$$\therefore f(x,y) = xy + \frac{y^3}{3}$$

(c) $\int_C \nabla f \cdot d\vec{r} = f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) - f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ by FTLI

$$= \left(\cancel{\frac{\sqrt{3}}{4}} + \frac{\sqrt{3}}{8} \right) - \left(\cancel{\frac{\sqrt{3}}{4}} + \frac{1}{24} \right) = \boxed{\frac{3\sqrt{3}-1}{24}}$$

#3 $\vec{F} = \langle \sin(yz), \sin(zx), \sin(xy) \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(yz) & \sin(zx) & \sin(xy) \end{vmatrix}$$

$$= \left\langle x\cos(xy) - x\cos(zx), -\left(y\cos(xy) - y\cos(yz)\right), z\cos(zx) - z\cos(yz) \right\rangle$$

$$= \left\langle x\left(\cos(xy) - \cos(zx)\right), y\left(\cos(yz) - \cos(xy)\right), z\left(\cos(zx) - \cos(yz)\right) \right\rangle.$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(\sin(yz)\right) + \frac{\partial}{\partial y}\left(\sin(zx)\right) + \frac{\partial}{\partial z}\left(\sin(xy)\right) = 0 + 0 + 0 = 0.$$

#4

Figure 1

(a) $(\nabla \times \vec{F})\big|_P$ points in the $\hat{k}$ direction by the right-hand rule for curl.

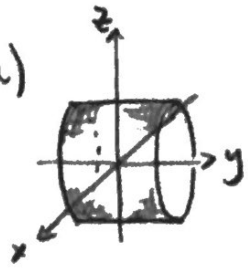
(b) $(\nabla \cdot \vec{F})\big|_P = 0$ because the same amount of "stuff" going <u>into</u> P equals the same amount of "stuff" going <u>out</u> of P.

Figure 2

(a) $(\nabla \times \vec{F})\big|_P = \vec{0}$ because there is clearly no rotation.

(b) $(\nabla \cdot \vec{F})\big|_P > 0$ because more "stuff" is going <u>out</u> of P than <u>into</u> P.

#5 (a)



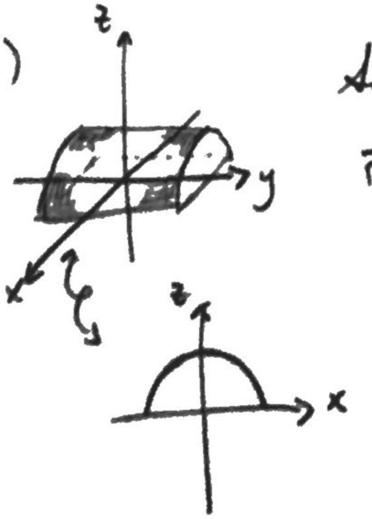
$$\begin{cases} x = 3\cos(u) \\ y = v \\ z = 3\sin(u) \end{cases}$$

$\vec{r} = \langle 3\cos(u), v, 3\sin(u) \rangle$

$0 \leq u \leq 2\pi$

$-4 \leq v \leq 4$

(b)



Same as (a), but different bounds!

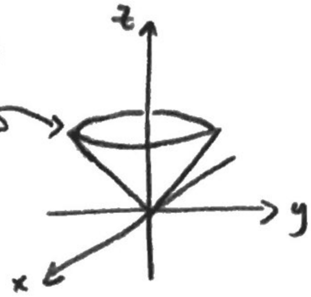
$\vec{r} = \langle 3\cos(u), v, 3\sin(u) \rangle$

$0 \leq u \leq \pi$

$-4 \leq v \leq 4$

(c)

at $z=1$, the radius is 1.



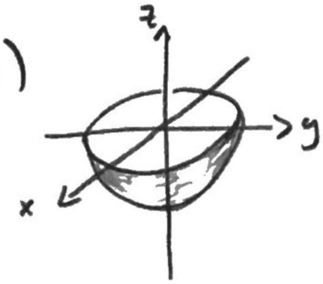
$$\begin{cases} x = u\cos(v) \\ y = u\sin(v) \\ z = +\sqrt{u^2} = u \end{cases}$$

$\vec{r} = \langle u\cos(v), u\sin(v), u \rangle$

$0 \le u \le 1$

$0 \le v \le 2\pi$

(d)



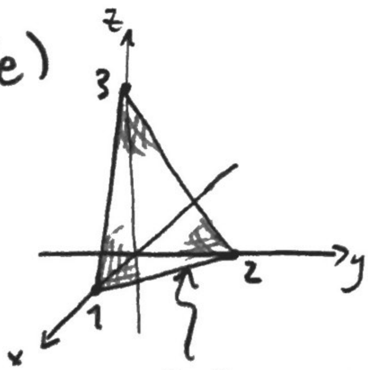
$$\begin{cases} x = \sqrt{5}\cos(u)\sin(v) \\ y = \sqrt{5}\sin(u)\sin(v) \\ z = \sqrt{5}\cos(v) \end{cases}$$

$\vec{r} = \langle \sqrt{5}\cos(u)\sin(v), \sqrt{5}\sin(u)\sin(v), \sqrt{5}\cos(v) \rangle$

$0 \le u \le 2\pi$

$\frac{\pi}{2} \le v \le \pi$

(e)



$z=0$, so $x + \frac{y}{2} = 1$

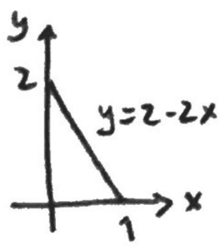
$y = 2-2x$

$$\begin{cases} x = u \\ y = v \\ z = 3\left(1 - u - \frac{v}{2}\right) \end{cases}$$

$\vec{r} = \langle u, v, 3\left(1-u-\frac{v}{2}\right) \rangle$

$0 \le v \le 2-2u$

$0 \le u \le 1$



$y = 2-2x$ <u>OR</u> $v = 2-2u$

#6 $\vec{r} = \langle u+v,\ u-v,\ 1+2u+v \rangle, \quad 0 \le u \le 2,\ 0 \le v \le 2$

(a) $SA = \iint_S dS = \int_0^2 \int_0^2 \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$

$$\vec{r}_u = \langle 1, 1, 2 \rangle, \quad \vec{r}_v = \langle 1, -1, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \langle 3, 1, -2 \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{14}$$

$$SA = \int_0^2 \int_0^2 \sqrt{14} \, du \, dv = \sqrt{14}(2)(2) = \boxed{4\sqrt{14}}$$

(b) $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle 3, 1, -2 \rangle$ at every point.

$$3(x-2) + 1(y-0) - 2(z-4) = 0.$$