Multivariable Calculus Problem Set 4

Topics: partial derivatives, chain rule, the gradient vector, directional derivatives, tangent planes, normal lines, double integrals, triple integrals, line integrals, conservative vector fields, the fundamental theorem for line integrals

Exercise 1. Let $f(x, y, z) = e^{x+y} + \sin(z + x^2)$.
(a) Compute the gradient of f. That is, compute ∇f.
(b) Compute the directional derivative $D_{\mathbf{u}} f$ at the point $(0, 0, 0)$, where $\mathbf{u} = \langle 1, 1, 1 \rangle$.
(c) Construct a nonzero vector function $\mathbf{F}(x, y, z)$ that is perpendicular to ∇f.

Exercise 2. What is an equation for the tangent plane of the surface $F(x, y, z) = k$ at the point $P_0(x_0, y_0, z_0)$, that is for *implicit functions*? What about $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$, that is for *explicit functions*?

Exercise 3. Let $f(x, y) = \dfrac{1}{1 + 2x^2 + 3y^2}$.
(a) Compute the gradient of f at the point where $(x, y) = (1, 1)$, given by $\nabla f(1, 1)$.
(b) Compute the rate of change of f at the point where $(x, y) = (1, 1)$ in the direction of the vector $\mathbf{u} = \langle -3, -4 \rangle$.
(c) Find an equation for the tangent plane of $f(x, y)$ at the point $P_0(1, 1, 1/6)$.

Exercise 4. Consider the surface $z^2 - x^2 - y^2 = 0$.
(a) What type of surface is this?
(b) Find an equation for the tangent plane of this surface at the point $(3, 4, 5)$.
(c) Find the parametric equations for the normal line of this surface at the point $(3, 4, 5)$.

Exercise 5. Let $u(x, y) = x^2\sqrt{y^2 + 3}$, $x = t^2 - 1$, and $y = t - 1$.
(a) Draw the tree diagram for this problem.
(b) Using your tree diagram, compute $\dfrac{du}{dt}$ at the point when $t = 2$.

Exercise 6. Compute the line integral $\int_C (x^2 + y^2 + z^2)ds$, where C is the curve described by $\mathbf{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle$, $0 \leq t \leq 2\pi$.

Exercise 7. Consider the vector field $\mathbf{F}(x, y) = \langle y^2 e^{xy}, (1 + xy)e^{xy} \rangle$. Let C be the line segment from $(0, 0)$ to $(0, 2)$.
(a) Parameterize the line segment C.
(b) Show that $\mathbf{F}$ is a conservative vector field.
(c) Compute a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.
(d) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ *without* using the fundamental theorem for line integrals.
(e) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the fundamental theorem for line integrals.

Problem 8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle$ and C is the twisted cubic given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.

Exercise 9. Use Green's theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C xy^2 dx + y \arctan(y) dy,$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$. Set up and solve as an iterated integral using a Type II region.

Problem 10. Let D be the region bounded by the curves $y = 2x$ and $y = x^2$.
(a) Draw the region D. Be sure to label all curves and points of intersection.
(b) Set up the double integral $\iint_D xy dA$ as BOTH an iterated Type I integral and an iterated Type II integral.
(c) Compute $\iint_D xy dA$ by using the iterated Type I integral from part (b).

Problem 11. Let D be the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, such that $x \leq 0$. Compute the double integral $\iint_D xy^2 dA$ using polar coordinates.

Problem 12. Evaluate the integral

$$\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2) dz dy dx$$

by converting to cylindrical coordinates.

Problem 13. Let E be the solid region within the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{\dfrac{x^2 + y^2}{3}}$. Compute the integral

$$\iiint_E \sin\left((x^2 + y^2 + z^2)^{3/2}\right) dV$$

by converting to spherical coordinates.

Exercise 14. Set up (but DO NOT evaluate) the integral $\iiint_E x dV$ in spherical coordinates, where E is the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 3$.

Exercise 15. Let E be the solid region inside the sphere $x^2 + y^2 + z^2 = 5$ with $y \leq 0$.
(a) Sketch the region E. Be sure to include labels!
(b) Compute the integral $\displaystyle\iiint_E 2y dV$ using **spherical coordinates**.
(c) Compute the integral $\displaystyle\iiint_E 2y dV$ using **cylindrical coordinates**.
(d) Set up (but DO NOT evaluate) the integral $\displaystyle\iiint_E 2y dV$ using **Cartesian coordinates**. Use the differential $dV = dy dz dx$.

POSSIBLY USEFUL FORMULAE

The tangent plane for an explicit function $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The tangent plane for an implicit function $F(x, y, z) = k$ at the point $P_0(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The relationships between Cartesian coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ) are

$$\begin{gathered} x = \rho \cos(\theta) \sin(\phi) \\ y = \rho \sin(\theta) \sin(\phi) \\ z = \rho \cos(\phi) \\ x^2 + y^2 + z^2 = \rho^2 \\ dV = \rho^2 \sin(\phi) d\rho d\theta d\phi \end{gathered}$$

The line integrals for scalar and vector fields over a curve C are respectively given by

$$\int_C f ds = \int_{t=a}^{t=b} f(\mathbf{r}(t)) \left\| \mathbf{r}'(t) \right\| dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

The Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simply closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$