PROBLEM SET #4 SOLUTIONS

#1 $f(x,y,z) = e^{x+y} + \sin(z+x^2)$

(a) $\nabla f = \langle e^{x+y} + 2x\cos(z+x^2),\ e^{x+y} + 0,\ 0 + \cos(z+x^2) \rangle$

(b) $$(D_{\vec{u}} f)(0,0,0) = \nabla f(0,0,0) \cdot \frac{\vec{u}}{\|\vec{u}\|}$$

$$= \langle 1,1,1 \rangle \cdot \frac{\langle 1,1,1 \rangle}{\sqrt{1^2+1^2+1^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(c) Want $\vec{F} \cdot \nabla f = 0$. Let $\vec{F}(x,y,z) = \langle f_1, f_2, f_3 \rangle$.

$$0 = \langle f_1, f_2, f_3 \rangle \cdot \langle e^{x+y} + 2x\cos(z+x^2),\ e^{x+y},\ \cos(z+x^2) \rangle$$

Let $f_1 = 1,\ f_2 = -1,\ f_3 = -2x$.

$$\vec{F}(x,y,z) = \langle 1, -1, -2x \rangle.$$

#2 $0 = \nabla F(x_0,y_0,z_0) \cdot (\vec{r} - \vec{r}_0)$

$$0 = F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$$

$$z - z_0 = \nabla f(x_0,y_0) \cdot \langle x-x_0, y-y_0 \rangle$$

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

#3 $f(x,y,z) = \dfrac{1}{1+2x^2+3y^2} = (1+2x^2+3y^2)^{-1}$

(a) $$\nabla f = \left\langle \frac{-4x}{(1+2x^2+3y^2)^2}, \frac{-6y}{(1+2x^2+3y^2)^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle \frac{-4}{36}, \frac{-6}{36} \right\rangle = \left\langle -\frac{1}{9}, -\frac{1}{6} \right\rangle$$

(b) $\left(D_{\vec{u}} f\right)(1,1) = \nabla f(1,1) \cdot \dfrac{\vec{u}}{\|\vec{u}\|}$

$$= \left\langle -\frac{1}{9}, -\frac{1}{6} \right\rangle \cdot \frac{\langle -3, -4 \rangle}{\sqrt{9+16}}$$

$$= \frac{1}{5}\left(\frac{1}{3} + \frac{2}{3}\right) = \frac{1}{5}$$

(c) $z - \frac{1}{6} = -\frac{1}{9}(x-1) - \frac{1}{6}(y-1)$.

#4 $z^2 - x^2 - y^2 = 0$ (implicit function)

(a) Cone.

(b) $\nabla F = \langle -2x, -2y, 2z \rangle$

$\nabla F(3,4,5) = \langle -6, -8, 10 \rangle$

$0 = -6(x-3) - 8(y-4) + 10(z-5)$

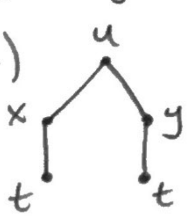
(c) Point: $(3,4,5)$

Vector: $\nabla F(3,4,5) = \langle -6, -8, -10 \rangle$.

$$\begin{cases} x(t) = 3 - 6t \\ y(t) = 4 - 8t \\ z(t) = 5 + 10t \end{cases} \quad t \in \mathbb{R}$$

#5 $u(x,y) = x^2\sqrt{y^2+3}$, $\quad x = t^2 - 1$, $\quad y = t - 1$

(a)



(b) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$

$$= \left(2x\sqrt{y^2+3}\right)(2t) + \left(x^2 \tfrac{1}{\cancel{2}}(y^2+3)^{-1/2}\cancel{2}y\right)(1)$$

at $t=2$, we have $x=3$, $y=1$

$$\left.\frac{du}{dt}\right|_{t=2} = (12)(4) + \left(\frac{9}{2}\right)(1) = \frac{105}{2}.$$

#6 $\displaystyle\int_C (x^2+y^2+z^2)\,ds$, $\quad \vec{r}\,'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$

$$\|\vec{r}\,'(t)\| = \sqrt{1+4\sin^2(2t)+4\cos^2(2t)} = \sqrt{5}$$

$$= \int_0^{2\pi} \left(t^2 + \underbrace{\cancel{\cos^2(2t)+\sin^2(2t)}}_{1}\right)\sqrt{5}\,dt$$

$$= \sqrt{5}\left[\frac{t^3}{3}+t\right]_0^{2\pi} = \sqrt{5}\left(\frac{8\pi^3}{3}+2\pi\right)$$

#7 $\vec{F}(x,y) = \langle y^2e^{xy}, (1+xy)e^{xy}\rangle$

(a) $\vec{r}(t) = \langle 0, 2t\rangle$, $\quad 0 \le t \le 1$

(b) $P_y = 2ye^{xy} + y^2xe^{xy}$ ✓

$Q_x = ye^{xy} + (1+xy)ye^{xy} = 2ye^{xy} + y^2xe^{xy}$ ✓

(c) $f(x,y) = \int P\,dx = \int y^2e^{xy}\,dx = \dfrac{y^2}{y}e^{xy} + A(y) = ye^{xy} + A(y)$

$f(x,y) = \int Q\,dy = \int (1+xy)e^{xy}\,dy = \dfrac{1+xy}{x}e^{xy} - \dfrac{x}{x^2}e^{xy} + B(x)$

$1+xy$	e^{xy}
x (+)	$\frac{1}{x}e^{xy}$
0 (−)	$\frac{1}{x^2}e^{xy}$

$$= \cancel{\frac{1}{x}e^{xy}} + ye^{xy} - \cancel{\frac{1}{x}e^{xy}} + B(x)$$

$$= ye^{xy} + B(x)$$

Letting $A(y)=0$, $B(x)=0$, we have $f(x,y)=ye^{xy}$.

(d) $$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \underbrace{\langle 4t^2, 1\rangle}_{\vec{F}(\vec{r}(t))}\cdot\underbrace{\langle 0,2\rangle}_{\vec{r}\,'(t)}\,dt$$

$$= \int_0^1 2\,dt = 2.$$

(e) $$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(0,2) - f(0,0)$$

$$= 2 - 0$$

$$= 2.$$

#8 $\vec{F}(x,y,z) = \langle xy, yz, zx\rangle$, $\vec{r}(t) = \langle t, t^2, t^3\rangle$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2\rangle$

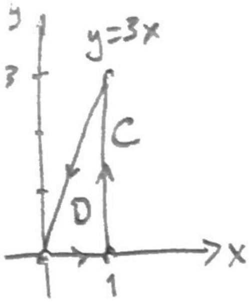
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle t^3, t^5, t^4\rangle\cdot\langle 1, 2t, 3t^2\rangle\,dt$$

$$= \int_0^1 \left(t^3 + 2t^6 + 3t^6\right)dt$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_0^1$$

$$= \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

#9



$$\oint_C \underbrace{xy^2}_{P(x,y)}\,dx + \underbrace{y\arctan(y)}_{Q(x,y)}\,dy$$

$$= \iint_D (Q_x - P_y)\,dA$$

$$= \int_0^3 \int_{y/3}^1 (0 - 2xy)\,dx\,dy \quad \begin{pmatrix}\text{Type II} \\ \text{region}\end{pmatrix}$$

$$= \int_0^3 \left[-x^2y\right]_{x=y/3}^{x=1} dy$$

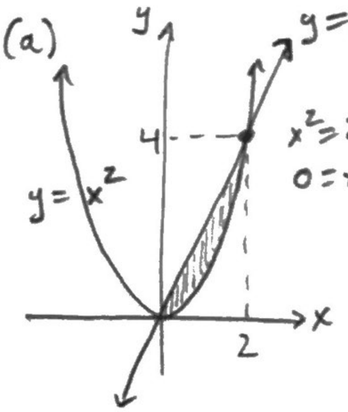
$$= \int_0^3 \left(-y + \frac{y^3}{9}\right) dy$$

$$= \left[-\frac{y^2}{2} + \frac{y^4}{36}\right]_{y=0}^{y=3}$$

$$= -\frac{9}{2} + \frac{81}{36}$$

$$= -\frac{9}{4}$$

#10 (a)



$x^2 = 2x$

$0 = x(x-2)$

(b) $\displaystyle\int_0^2 \int_{x^2}^{2x} xy\,dy\,dx$ (Type I)

$\displaystyle\int_0^4 \int_{y/2}^{\pm\sqrt{y}} xy\,dx\,dy$ (Type II)

(c) $\int_0^2 \int_{x^2}^{2x} xy \, dy \, dx = \int_0^2 \left[\frac{xy^2}{2}\right]_{y=x^2}^{y=2x} dx$

$= \int_0^2 \frac{1}{2}x\left(4x^2 - x^4\right) dx$

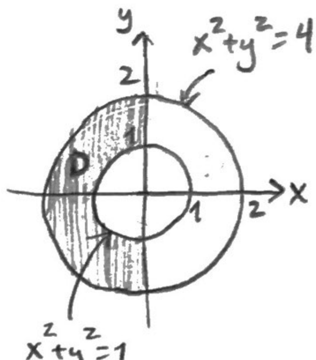
$= \int_0^1 \frac{1}{2}\left(4x^3 - x^5\right) dx$

$= \frac{1}{2}\left[x^4 - \frac{x^6}{6}\right]_{x=0}^{x=2}$

$= \frac{1}{2}\left(16 - \frac{64}{6}\right)$

$= \frac{8}{3}$

#11



$\iint_D xy^2 \, dA$

$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 r\cos(\theta)\, r^2\sin^2(\theta)\, r\,dr\,d\theta$

$= \int_{\pi/2}^{3\pi/2} \int_1^2 \cos(\theta)\sin^2(\theta)\, r^4 \,dr\,d\theta$

$= \int_{\pi/2}^{3\pi/2} \cos(\theta)\sin^2(\theta) \left[\frac{r^5}{5}\right]_1^2 d\theta$

u-sub
$u = \sin(\theta)$
$du = \cos(\theta)d\theta$

$= \frac{31}{5}\left[\frac{\sin^3(\theta)}{3}\right]_{\pi/2}^{3\pi/2}$

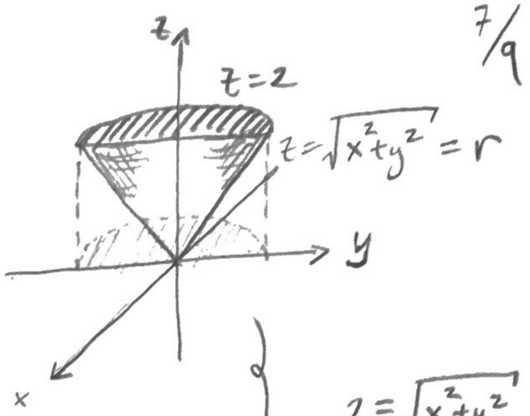
$= \frac{31}{15}(-1-1)$

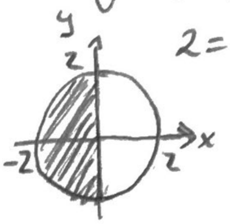
$= \frac{-62}{15}$

#12

$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$

cone





$2=\sqrt{x^2+y^2}$

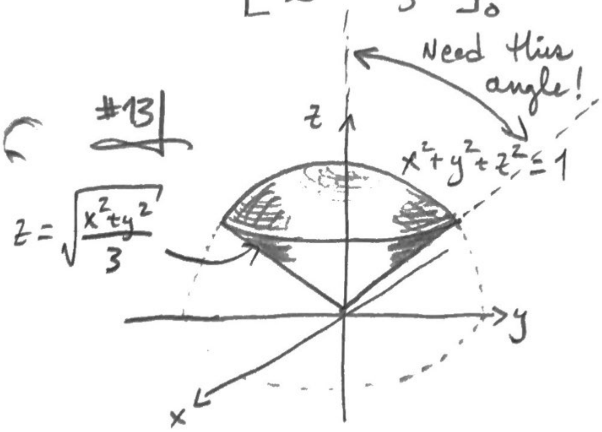
$2=r$

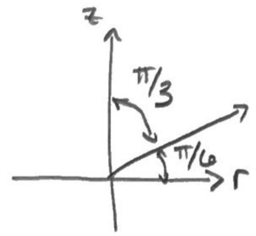
$$=\int_{\pi/2}^{3\pi/2}\int_{0}^{2}\int_{r}^{2} r^2\, r\,dz\,dr\,d\theta$$

$$=\int_{\pi/2}^{3\pi/2}\int_{0}^{2} r^3(2-r)\,dr\,d\theta$$

$$=(\pi)\left[\frac{r^4}{2}-\frac{r^5}{5}\right]_0^2=(\pi)\left(8-\frac{32}{5}\right)=\frac{8\pi}{5}$$

#13





$$z=\sqrt{\frac{x^2+y^2}{3}}=\frac{r}{\sqrt{3}}$$

$$\iiint_E \sin\left((x^2+y^2+z^2)^{3/2}\right)dV=\int_0^{\pi/3}\int_0^{2\pi}\int_0^{1}\sin(\rho^3)\,\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

$u=\rho^3$

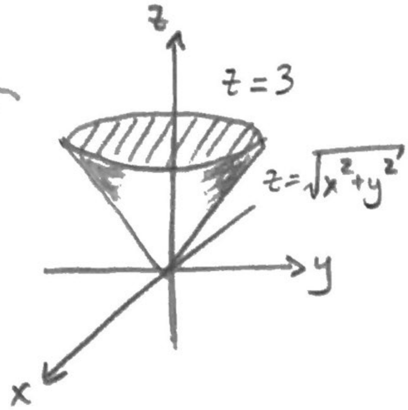
$du=3\rho^2 d\rho$

$$=\int_0^{\pi/3}\int_0^{2\pi}\left[\frac{-\cos(\rho^3)}{3}\right]_0^1\sin(\phi)\,d\theta\,d\phi$$

$$=\int_0^{\pi/3}(2\pi)\left(\frac{1-\cos(1)}{3}\right)\sin(\phi)\,d\phi$$

$$=\frac{2\pi(1-\cos(1))}{3}\left[-\cos(\phi)\right]_0^{\pi/3}=\frac{2\pi(1-\cos(1))}{3}\left(1-\frac{1}{2}\right)=\frac{\pi(1-\cos(1))}{3}$$

#11



$0 \le \rho \le (z=3)$

$\rho\cos(\phi) = 3$

$\rho = 3\sec(\phi)$

$0 \le \rho \le 3\sec(\phi)$

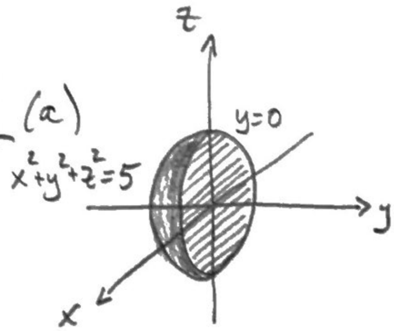
$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/2$

$$\iiint_E x\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{3\sec(\phi)} \rho\cos(\theta)\sin(\phi)\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^{3\sec(\phi)} \rho^3\cos(\theta)\sin^2(\phi)\,d\rho\,d\theta\,d\phi$$

#15 (a)



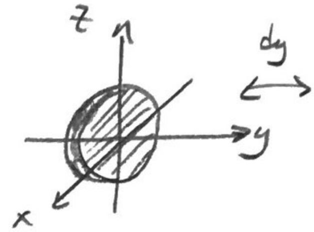
(b) $$\int_0^{\pi}\int_{\pi}^{2\pi}\int_0^{\sqrt{5}} 2\rho\sin(\theta)\sin(\phi)\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\sin^2(\phi)\,d\phi \int_{\pi}^{2\pi}\sin(\theta)\,d\theta \int_0^{\sqrt{5}} 2\rho^3\,d\rho$$

$$= \frac{1}{2}\left[\phi - \frac{1}{2}\sin(2\phi)\right]_0^{\pi}\left[-\cos(\theta)\right]_{\pi}^{2\pi}\left[\frac{2\rho^4}{4}\right]_0^{\sqrt{5}}$$

$$= \frac{1}{\cancel{2}}(\pi)(-\cancel{2})\left(\frac{25}{2}\right)$$

$$= \frac{-25\pi}{2}$$



(c) $\int_0^{2\pi}\int_0^{\sqrt{5}}\int_{-\sqrt{5-r^2}}^{0} 2y\, r\, dy\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^{\sqrt{5}} r\left[y^2\right]_{-\sqrt{5-r^2}}^{0} dr\, d\theta$

$= \int_0^{2\pi}\int_0^{\sqrt{5}} -r(5-r^2)\, dr\, d\theta$

$= -\int_0^{2\pi}\int_0^{\sqrt{5}} (5r-r^3)\, dr\, d\theta$

$= -(2\pi)\left[\frac{5r^2}{2}-\frac{r^4}{4}\right]_0^{\sqrt{5}}$

$= -(2\pi)\left(\frac{25}{2}-\frac{25}{4}\right)$

$= \frac{-25\pi}{2}$

(d) $\int_{-\sqrt{5}}^{\sqrt{5}}\int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}}\int_{-\sqrt{5-x^2-z^2}}^{0} 2y\, dy\, dz\, dx$

The double integral part (projection onto the xz-plane):