# Multivariable Calculus Problem Set 4

**Topics:** partial derivatives, chain rule, the gradient vector, directional derivatives, tangent planes, normal lines, double integrals, triple integrals, line integrals, conservative vector fields, the fundamental theorem for line integrals

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**Exercise 1.** Let $f(x, y, z) = e^{x+y} + \sin(z + x^2)$.
(a) Compute the gradient of $f$. That is, compute $\nabla f$.
(b) Compute the directional derivative $D_{\mathbf{u}} f$ at the point $(0, 0, 0)$, where $\mathbf{u} = \langle 1, 1, 1 \rangle$.
(c) Construct a nonzero vector function $\mathbf{F}(x, y, z)$ that is perpendicular to $\nabla f$.

**Exercise 2.** What is an equation for the tangent plane of the surface $F(x, y, z) = k$ at the point $P_0(x_0, y_0, z_0)$, that is for *implicit functions*? What about $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$, that is for *explicit functions*?

**Exercise 3.** Let $f(x, y) = \dfrac{1}{1 + 2x^2 + 3y^2}$.
(a) Compute the gradient of $f$ at the point where $(x, y) = (1, 1)$, given by $\nabla f(1, 1)$.
(b) Compute the rate of change of $f$ at the point where $(x, y) = (1, 1)$ in the direction of the vector $\mathbf{u} = \langle -3, -4 \rangle$.
(c) Find an equation for the tangent plane of $f(x, y)$ at the point $P_0(1, 1, 1/6)$.

**Exercise 4.** Consider the surface $z^2 - x^2 - y^2 = 0$.
(a) What type of surface is this?
(b) Find an equation for the tangent plane of this surface at the point $(3, 4, 5)$.
(c) Find the parametric equations for the normal line of this surface at the point $(3, 4, 5)$.

**Exercise 5.** Let $u(x, y) = x^2\sqrt{y^2 + 3}$, $x = t^2 - 1$, and $y = t - 1$.
(a) Draw the tree diagram for this problem.
(b) Using your tree diagram, compute $\dfrac{du}{dt}$ at the point when $t = 2$.

**Exercise 6.** Compute the line integral $\int_C (x^2 + y^2 + z^2)ds$, where $C$ is the curve described by $\mathbf{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle$, $0 \leq t \leq 2\pi$.

**Exercise 7.** Consider the vector field $\mathbf{F}(x, y) = \langle y^2 e^{xy}, (1 + xy)e^{xy} \rangle$. Let $C$ be the line segment from $(0, 0)$ to $(0, 2)$.
(a) Parameterize the line segment $C$.
(b) Show that $\mathbf{F}$ is a conservative vector field.
(c) Compute a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.
(d) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ *without* using the fundamental theorem for line integrals.
(e) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the fundamental theorem for line integrals.

**Problem 8.** Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle$ and $C$ is the twisted cubic given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.

**Exercise 9.** Use Green's theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C xy^2dx + y\arctan(y)dy,$$

where $C$ is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$. Set up and solve as an iterated integral using a Type II region.

**Problem 10.** Let $D$ be the region bounded by the curves $y = 2x$ and $y = x^2$.
(a) Draw the region $D$. Be sure to label all curves and points of intersection.
(b) Set up the double integral $\iint_D xydA$ as BOTH an iterated Type I integral and an iterated Type II integral.
(c) Compute $\iint_D xydA$ by using the iterated Type I integral from part (b).

**Problem 11.** Let $D$ be the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, such that $x \leq 0$. Compute the double integral $\iint_D xy^2dA$ using polar coordinates.

**Problem 12.** Evaluate the integral

$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx$$

by converting to cylindrical coordinates.

**Problem 13.** Let $E$ be the solid region within the sphere $x^2 + y^2 + z^2 = 1$, below the cone $z = \sqrt{x^2 + y^2}$, and above the $xy-$plane. Compute the integral

$$\iiint_E \sin\left((x^2+y^2+z^2)^{3/2}\right)dV$$

by converting to spherical coordinates.

**Exercise 14.** Set up (but DO NOT evaluate) the integral $\iiint_E xdV$ in spherical coordinates, where $E$ is the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 3$.

**Exercise 15**. Let $E$ be the solid region inside the sphere $x^2 + y^2 + z^2 = 5$ with $y \leq 0$.
(a) Sketch the region $E$. Be sure to include labels!
(b) Compute the integral $\displaystyle\iiint_E 2ydV$ using **spherical coordinates**.
(c) Compute the integral $\displaystyle\iiint_E 2ydV$ using **cylindrical coordinates**.
(d) Set up (but DO NOT evaluate) the integral $\displaystyle\iiint_E 2ydV$ using **Cartesian coordinates**. Use the differential $dV = dydzdx$.

# POSSIBLY USEFUL FORMULAE

The tangent plane for an explicit function $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The tangent plane for an implicit function $F(x, y, z) = k$ at the point $P_0(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The relationships between Cartesian coordinates $(x, y, z)$ and spherical coordinates $(\rho, \theta, \phi)$ are

$$x = \rho \cos(\theta) \sin(\phi)$$
$$y = \rho \sin(\theta) \sin(\phi)$$
$$z = \rho \cos(\phi)$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$dV = \rho^2 \sin(\phi) d\rho d\theta d\phi$$

The line integrals for scalar and vector fields over a curve $C$ are respectively given by

$$\int_C f ds = \int_{t=a}^{t=b} f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

**The Fundamental Theorem for Line Integrals:** Let $C$ be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

**Green's Theorem:** Let $C$ be a positively oriented, piecewise-smooth, simply closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\oint_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$