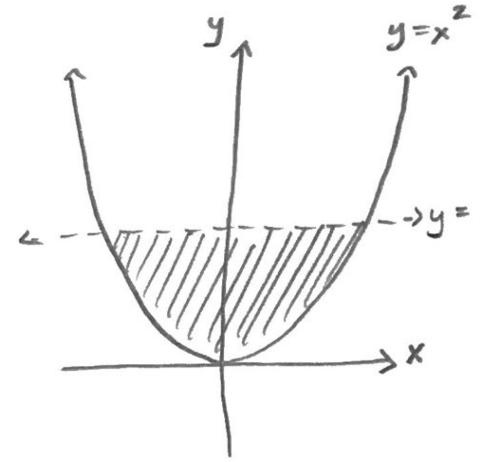
PROBLEM SET #3 SOLUTIONS

#1 $h(x,y) = e^{\sqrt{y-x^2}} + \dfrac{1}{\sqrt{1-y}}$

$y - x^2 \geq 0$ $\qquad$ $1 - y > 0$

$y \geq x^2$ $\qquad$ $1 > y$



#2 (a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3y - x^3}{x^2y^2 - x^2} = \lim_{(x,y)\to(1,1)} \frac{\cancel{x^3}(y-1)}{\cancel{x^2}(y^2-1)}$

$\displaystyle = \lim_{(x,y)\to(1,1)} \frac{\cancel{y-1}}{x(y+1)\cancel{(y-1)}} = \frac{1}{(1)(2)} = \boxed{\frac{1}{2}}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$

PATH #1: $y = x$

$\displaystyle\lim_{x\to 0} \frac{2x^2}{x^2+3x^2} = \frac{2}{4} = \frac{1}{2}$

PATH #2: $y = 2x$

$\displaystyle\lim_{x\to 0} \frac{4x^2}{x^2+3(4x^2)} = \frac{4}{13}$

$\times$ $\boxed{\text{DNE}}$

(c) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2+y^2} = \lim_{(x,y)\to(1,0)} \frac{y(x-1)}{(x-1)^2+y^2}$

This is just like 1b, but with $(x-1)$ instead of x.

PATH #1: $x - 1 = y$ (goes through $(1,0)$ ✓)

$\displaystyle\lim_{y\to 0} \frac{y^2}{y^2+y^2} = \frac{1}{2}$

PATH #2: $x-1=2y$ (goes through $(1,0)$ ✓)

$$\lim_{y\to 0} \frac{2y^2}{4y^2+y^2} = \frac{2}{5} \quad \boxed{DNE}$$

#3 $\displaystyle\lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right)$.

$$-|x||y|\overbrace{\left|\sin\left(\frac{1}{x^2+y^2}\right)\right|}^{\le 1} \le xy\sin\left(\frac{1}{x^2+y^2}\right) \le |x||y|\overbrace{\left|\sin\left(\frac{1}{x^2+y^2}\right)\right|}^{\le 1}$$

$$-|x||y| \le \qquad\qquad \le |x||y|$$

$$\underbrace{\lim_{(x,y)\to(0,0)} -|x||y|}_{=0} \le \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) \le \underbrace{\lim_{(x,y)\to(0,0)} |x||y|}_{=0}$$

$$\therefore \lim_{(x,y)\to(0,0)} xy\sin\left(\frac{1}{x^2+y^2}\right) = \boxed{0}.$$

#4 $f(x,y,z) = x^2y^3z^4 + \sin(x^2+5y)$

$$\frac{\partial f}{\partial x} = 2xy^3z^4 + \cos(x^2+5y)\cdot 2x$$

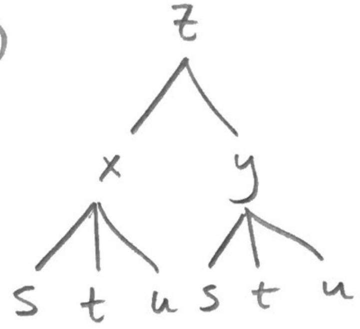
$$\frac{\partial f}{\partial y} = x^2 3y^2 z^4 + \cos(x^2+5y)\cdot 5$$

$$\frac{\partial f}{\partial z} = x^2y^3 4z^3 + 0$$

$$\frac{\partial^2 f}{\partial z\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 2xy^3 4z^3 + 0 = 8xy^3z^3$$

#5 $z = x^4 + x^2 y, \quad x = s + 2t - u, \quad y = stu^2$

(a)



(b) $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

$= (4x^3 + 2xy)(2) + (0 + x^2)(su^2)$

when $(s,t,u) = (4,2,1)$,

we have $(x,y) = (7,8)$.

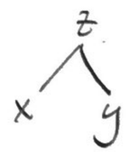
So, $\left.\dfrac{\partial z}{\partial t}\right|_{(s,t,u)=(4,2,1)} = (4\cdot 7^3 + 2\cdot 7\cdot 8)(2) + (7^2)(4\cdot 1^2)$

$= (1372 + 112)(2) + (49)(4)$

$= (1484)(2) + 196$

$= \boxed{3164}$

#6 $\dfrac{\partial}{\partial x}\left(\tan(x^2 z^2) = \cosh(y^2 x) + 2\ln z\right)$



$\sec^2(x^2z^2)\cdot\left(2xz^2 + x^2 2z\dfrac{\partial z}{\partial x}\right) = \sinh(y^2x)\cdot y^2 + \dfrac{2}{z}\dfrac{\partial z}{\partial x}$

$2xz^2\sec^2(x^2z^2) + \underbrace{2x^2z\sec^2(x^2z^2)\dfrac{\partial z}{\partial x}} = \sinh(y^2x)\cdot y^2 + \underbrace{\dfrac{2}{z}\dfrac{\partial z}{\partial x}}$

$$\boxed{\dfrac{\partial z}{\partial x} = \dfrac{2xz^2\sec^2(x^2z^2) - y^2\sinh(y^2x)}{\dfrac{2}{z} - 2x^2z\sec^2(x^2z^2)}}$$

#7 Want $h'(t=4)$. $\quad \dfrac{dh}{dt} = ?$

$$\begin{tikzcd}[column sep=small] & h \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[d, no head] & & y \arrow[d, no head] \\ t & & t \end{tikzcd}$$

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{\partial h}{\partial x}\right)\left(\frac{1}{2}(4t+a)^{-1/2}\cdot 4\right) + \left(\frac{\partial h}{\partial y}\right)\left(\cos\left(\frac{t\pi}{8}\right)\cdot\frac{\pi}{8}\right)$$

at $t=4$, $\quad (x,y)=(5,1)$.

So, $$\left.\frac{dh}{dt}\right|_{t=4} = h_x(5,1)\cdot\frac{2}{\sqrt{25}} + h_y(5,1)\cdot \cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\cdot\frac{\pi}{8}$$

$$= (0.5)\left(\frac{2}{5}\right) = \boxed{\frac{1}{5} \text{ meters per minute}}$$