Multivariable Calculus Problem Set 6

Integration Practice

Some trig identities that you will need to know:

$$\sin^2(x) = \frac{1}{2}\big(1 - \cos(2x)\big) \qquad \cos^2(x) = \frac{1}{2}\big(1 + \cos(2x)\big)$$

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

Exercise 1. Evaluate the following integrals.

(a) $\displaystyle\int \frac{(\ln x)^2}{x} dx$ ($u-$substitution)

(b) $\displaystyle\int x^2 e^x dx$ (integration by parts)

(c) $\displaystyle\int x^3\sqrt{x^2+1}dx$ ($u-$substitution)

(d) $\displaystyle\int x\sin(x)dx$ (integration by parts)

Exercise 2. Evaluate the following integrals. I've given you the answers so that you can check your answers.

(a) $\displaystyle\int_0^{\pi/2} \frac{1}{2}\cos^2(\theta)d\theta$

Answer: $\pi/8$

(c) $\displaystyle\int_0^{\pi/3} \sin^3(\theta)d\theta$

Answer: $5/24$

(d) $\displaystyle\int_0^{\pi/4} \sin^4(\phi)\cos(\phi)d\phi$

Answer: $\sqrt{2}/40$

(f) $\displaystyle\int_0^{\pi/3} \sin^2(y)dy$

Answer: $\pi/6 - \sqrt{3}/8$

Matching

Match each entry in the left column with the possible correct option(s) in the right column. Some of the options in the right column could be matched with more than one entry in the left column.

______ The cross product of two nonzero parallel vectors, $\mathbf{a}$ and $\mathbf{b}$.

______ The magnitude of the cross product of two nonzero perpendicular vectors, $\mathbf{a}$ and $\mathbf{b}$.

______ The dot product of two nonzero parallel vectors, $\mathbf{a}$ and $\mathbf{b}$.

______ The dot product of two nonzero perpendicular vectors, $\mathbf{a}$ and $\mathbf{b}$.

______ The cross product of two perpendicular unit vectors.

______ The dot product of two perpendicular unit vectors.

______ $\nabla \cdot (\nabla \times \mathbf{F})$

______ $\nabla(\nabla \times \mathbf{F})$

______ $\nabla \times (\nabla f)$

______ $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a})$

______ The curl of a conservative vector field.

______ $D_{\nabla f} f$

______ $D_{-\nabla f} f$

______ The set of all points $(x, y, z) \in \mathbb{R}^3$ such that $\|\mathbf{r} - \mathbf{r}_0\| = 1$, where $\mathbf{r}_0 = \langle 1, 1, 1 \rangle$.

(A) Minimum rate of change

(B) $\|\mathbf{a}\| + \|\mathbf{b}\|$

(C) 1

(D) $\|\nabla f\|$

(E) $-\|\nabla f\|$

(F) $\mathbf{0}$

(G) Maximum rate of change

(H) Nonsense

(I) $\|\mathbf{a}\|\|\mathbf{b}\|$

(J) $-\|\mathbf{a}\|\|\mathbf{b}\|$

(K) The circle of radius 1 centered at $(1, 1, 1)$

(L) 0

(M) $(x-1) + (y-1) + (z-1) = 0$

(N) Unit vector

(O) $x - 1 = y - 1 = z - 1$

(P) $\langle x-1, y-1, z-1 \rangle$

(Q) $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$

Practice Problems

Problem 1. Let $\mathbf{a} = \langle -1, 1, 0\rangle$, $\mathbf{b} = \langle 2, -1, 2\rangle$, and $\mathbf{c} = \langle 1, 1, 0\rangle$.
(a) Compute the angle between $\mathbf{a}$ and $\mathbf{b}$ using the cross product.
(b) Compute the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$. That is, compute $\mathbf{comp_a b}$ and $\mathbf{proj_a b}$.
(c) Draw and label $\mathbf{a}$ and $\mathbf{c}$ on the same set of axes. Using the right-hand rule, is $\mathbf{a} \times \mathbf{c}$ pointing in the positive or negative $z-$direction?

Problem 2. What is an equation for the plane that passes through the points $A(0, 1, 1)$, $B(1, 0, 1)$, and $C(1, 1, 0)$?

Problem 3. Construct a vector that is parallel to $\mathbf{u} = \langle 2, 3, -4\rangle$ and is of length 7.

Problem 4. Find the position vector $\mathbf{r}(t)$ of a particle that has the given acceleration and the specified initial velocity and position.

$$\mathbf{a}(t) = 2t\ \hat{\mathbf{i}} + \sin(t)\ \hat{\mathbf{j}} + \cos(2t)\ \hat{\mathbf{k}}, \quad \mathbf{v}(0) = \hat{\mathbf{i}}, \quad \mathbf{r}(0) = \hat{\mathbf{j}}$$

Problem 5. Let $f(x, y, z) = e^{x+y} + \sin(z + x^2)$.
(a) Compute the gradient of f. That is, compute ∇f.
(b) Compute the directional derivative $D_{\mathbf{u}} f$ at the point $(0, 0, 0)$, where $\mathbf{u} = \langle 1, 1, 1\rangle$.

Problem 6. Consider the surface $z^2 - x^2 - y^2 = 0$.
(a) What type of surface is this?
(b) Find an equation for the tangent plane of this surface at the point $(3, 4, 5)$.

Problem 7. Let D be the region bounded by the curves $y = 2x$ and $y = x^2$.
(a) Draw the region D. Be sure to label all curves and points of intersection.
(b) Set up the double integral $\iint_D xy\,dA$ as BOTH an iterated Type I integral and an iterated Type II integral.
(c) Compute $\iint_D xy\,dA$ by using the iterated Type I integral from part (b).

Problem 8. Let D be the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, such that $x \leq 0$. Compute the double integral $\iint_D xy^2\,dA$ using polar coordinates.

Problem 9. Evaluate the integral $\displaystyle\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\,dz\,dy\,dx$ by converting to cylindrical coordinates.

Problem 10. Let E be the solid region within the sphere $x^2 + y^2 + z^2 = 1$, below the cone $z = \sqrt{x^2 + y^2}$, and above the $xy-$plane. Compute the integral $\displaystyle\iiint_E \sin\left((x^2 + y^2 + z^2)^{3/2}\right) dV$ by converting to spherical coordinates.

Problem 11. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, yz, zx\rangle$ and C is the twisted cubic given by $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, $0 \leq t \leq 1$.

Problem 12. Using Green's theorem, evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle -y \sin(x), y^2 \rangle$ and C is the (positively oriented) curve that bounds the region in the *first quadrant* between $y = x^2$, $y = \pi^2$, and $x = 0$.

Problem 13. Let $\mathbf{F}(x, y, z) = \langle -y, x, 0 \rangle$.
(a) Compute the divergence of $\mathbf{F}$. That is, compute $\nabla \cdot \mathbf{F}$.
(b) Compute the curl of $\mathbf{F}$. That is, compute $\nabla \times \mathbf{F}$.
(c) Based on your answer in part (b), is the vector field $\mathbf{F}$ rotating clockwise or counter-clockwise in the $xy-$plane?
(d) Is the vector field $\langle ye^x, e^x + e^y \rangle$ conservative? If so, then find a potential function for this vector field.

Problem 14. Find and identify any local maximums, local minimums, and/or saddle points of the function $f(x, y) = x^2 + xy + y^2 + y$. What is the function value at each point?

Problem 15. Let S be the portion of the plane $x + y + z = 1$ in the first octant.
(a) Parameterize the surface S.
(b) Compute $\mathbf{r}_u \times \mathbf{r}_v$ using your parameterization from part (a).
(c) Evaluate the surface integral $\iint_S x dS$.

Problem 16.
(a) Write the assumptions and equation of the divergence theorem.
(b) Let $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$. Let E be the solid region under the surface $z = e^{-x-y}$ and above the unit square in the $xy-$plane (that is, $0 \leq x \leq 1$, $0 \leq y \leq 1$). Let S be the boundary surface of E. Using the divergence theorem, evaluate the integral $\oiint_S \mathbf{F} \cdot d\mathbf{S}$.

**Hint:* $e^{-x-y} = e^{-x}e^{-y}$.