# The Quantum Combination Lock

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Our quantum combination lock is a finite, structured space $\mathcal{L}$ modeled on the Cartesian product $\mathbb{Z}_5^3$, where all coordinates are taken modulo 5. Each element $s \in \mathcal{L}$ is called a *setting*, and the zero element is defined as $(0,0,0) \in \mathcal{L}$. For any two settings $s_1, s_2 \in \mathcal{L}$, we define

$$s_1 + s_2 := \left((s_1^1 + s_2^1) \bmod 5,\ (s_1^2 + s_2^2) \bmod 5,\ (s_1^3 + s_2^3) \bmod 5\right),$$

and $-s$ by

$$-s := (5 - s^1,\ 5 - s^2,\ 5 - s^3) \bmod 5.$$

Multiplication of a scalar $a \in \mathbb{Z}_5$, and a setting $s \in \mathcal{L}$, is defined by

$$a \cdot s := \left((as^1) \bmod 5,\ (as^2) \bmod 5,\ (as^3) \bmod 5\right),$$

which endows $\mathcal{L}$ with the structure of a vector space over $\mathbb{Z}_5$. We next define the norm of a setting $s = (x, y, z) \in \mathcal{L}$ as:

$$\|s\| := \min(x, 5 - x) + \min(y, 5 - y) + \min(z, 5 - z),$$

and a metric between two settings $s_1, s_2 \in \mathcal{L}$ by

$$d(s_1, s_2) := \|s_1 - s_2\|,$$

where subtraction is taken in the group $\mathbb{Z}_5^3$. We define an inner product:

$$\langle s_1, s_2 \rangle := (s_1^1 s_2^1 + s_1^2 s_2^2 + s_1^3 s_2^3) \bmod 5,$$

which is bilinear and symmetric but does not induce a norm over $\mathbb{R}_{\geq 0}$ (and thus the space is thus not a Hilbert space in the classical sense).

If we let $c = (0,0,0)$ be the canonical code configuration, we can define a potential or deviation energy of a setting $s \in \mathcal{L}$

$$\texttt{weak\_field}(s) := d(c, s) = \|s\|,$$

and the energy stratum at level $\ell \in \{0, 1, \ldots, 6\}$ is the set:

$$\texttt{stratum}(\ell) := \{s \in \mathcal{L} \mid \|s\| = \ell\},$$

where the degeneracy (cardinality of each shell) is given by the sequence {1, 6, 18, 32, 36, 24, 8} which corresponds to the self-convolution of the coordinate-wise Manhattan profile on $\mathbb{Z}_5$. We can then define the neighbor set of a setting $s = (x, y, z) \in \mathcal{L}$ as:

$$\texttt{neighbors}(s) := \{(x \pm 1, y, z),\ (x, y \pm 1, z),\ (x, y, z \pm 1) \mod 5\}$$

This defines the 3-dimensional toroidal Cayley graph of $\mathbb{Z}_5^3$ with generator set:

$$S = \{(\pm 1, 0, 0),\ (0, \pm 1, 0),\ (0, 0, \pm 1)\}$$

The dual vector $s^* \in \mathcal{L}^*$ is defined trivially as

$$s^* := s$$

and is interpreted as a bra $\langle s|$ in the sense of Dirac notation.