MATH1013 - Calculus

Lecturer: Lilia Ferrario

John Dedman Building, Room G29 (Ground floor),
Ph: 61250346,
email:Lilia.Ferrario@anu.edu.au

These lecture slides have been developed by
Rob Taggart, Rishni Ratnam, Ravi Shroff & Lilia Ferrario

Antiderivatives

Thus far we have concentrated on finding the derivative f' of a given function f. For example, given a function f which describes the position of a particle at time t, or g which measures the volume of water in a tank at time t, we have investigated how to find functions f' and g' which measure the instantaneous rate of change of f and g.

Now we look at the reverse question. That is, if we can measure the velocity of a particle at each time t, can we find the position of the particle? If we can measure the rate at which water leaks from a tank, can we determine the amount of water that leaked out in a given time period? That is, given *the derivative* of a function, f', can we determine f?

Definition

A function F is said to be an *antiderivative* (or *primitive*) of f on an interval I if $F'(x) = f(x)$ for all x in I.

True or False? (Assume $I = \mathbb{R}$).

(i) $\sin x$ is an antiderivative of $\cos x$.

(ii) $3x^2 + 5$ is an antiderivative of $6x$.

(iii) $2x^3 - \tan x$ is the only antiderivative of $6x^2 - \sec^2 x$

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Antiderivatives

In the following table, C is an arbitrary constant.

Function	Antiderivative
$cf(x)$	$cF(x) + C$
$f(x) + g(x)$	$F(x) + G(x) + C$
x^n (where $n \neq -1$)	$\dfrac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Differential equations. A *differential equation* is an equation involving the derivatives of a function. We can solve some very basic differential equations by looking for antiderivatives. Although antiderivatives usually involve arbitrary constants, we may be able to uniquely determine these constants by some extra conditions.

Example (physics!). A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/sec and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

What is area?

We are now going to turn our focus to the other main branch of calculus, which deals with *integrals*. We will see how the notion of integral naturally arises when we consider the area under a curve, much as the notion of derivative naturally arose when we considered the tangent line to a curve. The concepts of differential and integral calculus are related by the so-called Fundamental Theorem of Calculus, which is one of humanity's great accomplishments.

What is area?

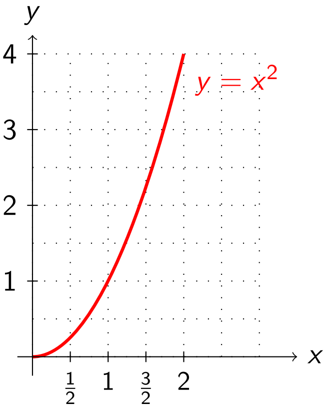
We are motivated by the following problem: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

What is area?

This question is easy to answer if the curve is a horizontal line, and only slightly more difficult if the curve is any non-vertical line. But what if the curve is not a line?

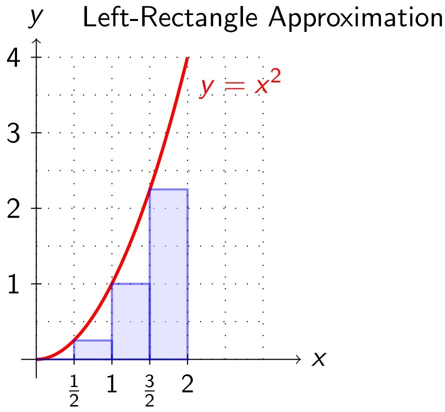
Example. Consider the curve $y = x^2$ between 0 and 2. Let's try to estimate the area under the curve:

Riemann Sums



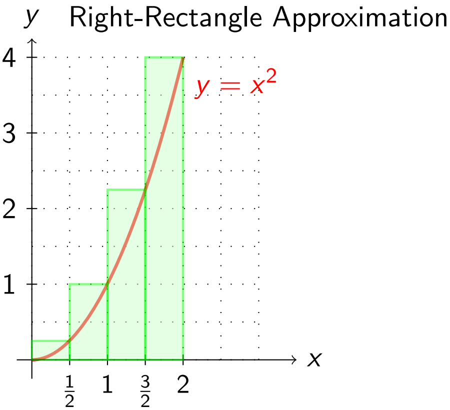
We want to find the area under the graph of $y = x^2$ in between $x = 0$ and $x = 2$.

Riemann Sums



Left Rectangle Approximation $=$
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

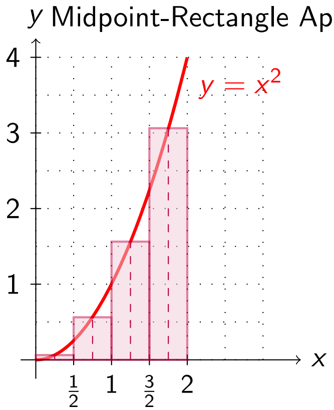
Riemann Sums



Left Rectangle Approximation $=$ $= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

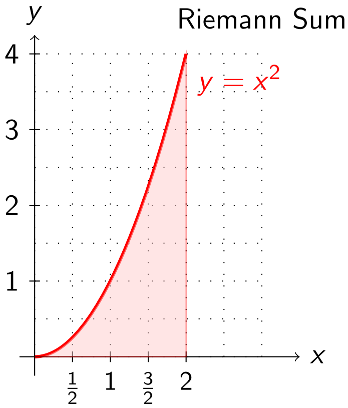
Right Rectangle Approximation $=$ $= \frac{1}{2} \cdot [(\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2 + 2^2] = 3.75.$

Riemann Sums



Midpoint Rectangle Approximation =
$= \frac{1}{2} \cdot [(\frac{1}{4})^2 + (\frac{3}{4})^2 + (\frac{5}{4})^2 + (\frac{7}{4})^2] = 2.625.$

Riemann Sums



Actual area is $2\frac{2}{3}$.

Area under a curve

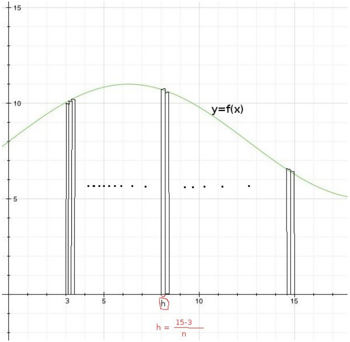


Figure: Computing area under $y = 3\sin(x/4) + 8$ in between $x = 3$ and $x = 15$

The above area is subdivided into n equal partitions (called *regular partitions*)and we draw left rectangles on each of them.
Let $x_0 = 3$, $x_1 = 3 + h$, $\ldots x_{n-1} = 15 - h$, $x_n = 15$.
Then, the approximated area is

$$h \cdot \left[f(x_0) + f(x_1) + \cdots + f(x_{n-1}) \right]$$

where $h = (15 - 3)/n$.

Taking the limit as $n \to \infty$, we compute the area.

We more generally define the **area** of the region S that lies under the graph of the continuous function f to be the limit of the sum of areas of approximating rectangles. That is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\triangle x + f(x_2)\triangle x + \ldots + f(x_n)\triangle x].$$

Note: It can be shown that this limit always exists as long as f is continuous. In fact, this limit may exist even if f is not continuous, but we will rarely consider this case.

Note: It can be shown that we get the same value if we use left endpoints rather than right endpoints. In fact, we can take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. These numbers $x_1^*, \ldots, x_n^*$ are called *sample points*.

Riemann Sums

A limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x = \lim_{n\to\infty} [f(x_1^*)\triangle x + \ldots + f(x_n^*)\triangle x]$$

arises when we compute the area under the curve $y = f(x)$. These limits occur often, even when $f < 0$.

Start with a function f defined on an interval $[a, b]$. Divide $[a, b]$ into n smaller subintervals by choosing points $x_1, \ldots, x_{n-1}$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

We call the resulting set of subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ a **partition** P of $[a, b]$.

Riemann Sums

We use the notation $\triangle x_i$ for the length of the ith subinterval $[x_{i-1}, x_i]$, so

$$\triangle x_i = x_i - x_{i-1}.$$

Then we choose **sample points** $x_1^*, \ldots, x_n^*$ in the subintervals so that x_i^* is in the ith subinterval $[x_{i-1}, x_i]$. The sample points could be left or right endpoints or any number in between. Here is an example of what a partition looks like:

A **Riemann sum** associated to a partition P and a function f is made by evaluating f at the sample points, multiplying by the lengths of the corresponding subintervals, and then adding:

$$\sum_{i=1}^{n} f(x_i^*)\triangle x_i = f(x_1^*)\triangle x_1 + \ldots + f(x_n^*)\triangle x_n.$$

Riemann Sums

Note: If $f(x_i^*)$ is negative, then $f(x_i^*)\triangle x_i$ is negative, and we must *subtract* the area of the corresponding rectangle.

If we take *all* possible partitions of $[a, b]$ and *all* possible choices of sample points, we can try to take the limit of *all* possible Riemann sums as n becomes large. However, since our subintervals have different lengths, we need to make sure that all these lengths $\triangle x_i$ approach 0. We do this by requiring that the largest subinterval length, $\max \triangle x_i$, approaches zero.

Definite integral

Definition

If f is a function defined on $[a, b]$, the **definite integral of** f **from** a **to** b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \triangle x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \triangle x_i$$

provided this limit exists. If it does exist, we say that f is *integrable* on $[a, b]$.

$y = x^2$

Definite integral

Remarks.

(i) The definition is saying that the definite integral is a **number** that can be approximated to within any degree of accuracy by a Riemann sum. The number does not depend on x. In fact, we can use any letter other than x without changing the value of the integral.

(ii) The symbol $\int$ is called an **integral sign**. In the above notation, $f(x)$ is called the **integrand**, a is the **lower limit of integration** and b is the **upper limit of integration**. Together they are **the limits of integration**. The symbol dx has no meaning by itself. The procedure of calculating an integral is called **integration**.

Theorem

If f is continuous on $[a, b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$. That is, the definite integral $\int_a^b f(x)\,dx$ exists.

Remarks.

(i) Although in the definition of partition we allowed subintervals to have different widths, for most intents and purposes it suffices to consider partitions where all subintervals have the same width. If our function f is integrable, this will not affect the value of the integral. Such partitions are called *regular partitions*.

(ii) If f takes on both positive and negative values, then the definite integral gives us the *net area* or *signed area* between the graph of f and the x-axis.

Examples.

Let's evaluate the following integrals by interpreting each in terms of areas:

1. $\int_0^1 \sqrt{1-x^2}\, dx$.
2. $\int_0^3 (x-1)\, dx$.

Note: The textbook goes into more detail on how to evaluate Riemann sums using the definition, and on the "Midpoint Rule". Both topics are recommended reading, although you will not be tested on the midpoint rule or on more complicated examples of Riemann sums.

Properties of the Definite Integral

If a, b, and c are real numbers and f, g, and $(f \pm g)$ are integrable on $[a, b]$:

1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$
2. $\displaystyle\int_a^a f(x)\,dx = 0.$
3. $\displaystyle\int_a^b c\,dx = c(b-a).$
4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$
5. $\displaystyle\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Properties of the Definite Integral: Example

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\, dx$.

Properties of the Definite Integral: Example

Suppose that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

Comparison properties of the integral

Suppose that $a \leq b$.

1. If $f(x) \geq 0$ for all x in $[a, b]$ then $\int_a^b f(x)\,dx \geq 0$.
2. If $f(x) \leq g(x)$ for all x in $[a, b]$ then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$
3. If $m \leq f(x) \leq M$ for all x in $[a, b]$ then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$
4. If $|f|$ is integrable on $[a, b]$ then
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Functions defined by an integral

Before we discuss the Fundamental Theorem of Calculus, we introduce the notion of a function *defined by an integral*.
Suppose we start with a continuous function f defined on the interval $[a, b]$, and we define a new function g on $[a, b]$ via the rule

$$g(x) = \int_a^x f(t)\, dt.$$

For example, $g(a) = \int_a^a f(t)\, dt = 0$, and $g(b) = \int_a^b f(t)\, dt$. If f is positive, then $g(x)$ can be interpreted as the area under the graph of f from a to x, the "area so far" function.

Theorem (Fundamental Theorem of Calculus)

Suppose a function f is continuous on $[a, b]$.

1. *If $g(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$, $g'(x) = f(x)$ for all $x \in (a, b)$.*
2. *If F is an antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Remark.

Note that part 1 says that if f is integrated, then differentiated, we get the original function f. Part 2 says that if we take a function F, differentiate it, then integrate the result, we get the original function F, but in the form $F(b) - F(a)$. Together, the two parts say that differentiation and integration are *inverse processes*.

Remarks on Part 1.

(a) Part 1 tells us that *every* function defined by an integral is differentiable, and $g'(x) = f(x)$.

(b) Also, *every* continuous function f on (a, b) has an antiderivative g, where g is defined as the integral of f from a to x.

Examples. Find the derivative of g, where g is given by

(i) $g(x) = \int_0^x \sqrt{1+t^2}\, dt$.

(ii) $g(x) = \int_5^x \sqrt{1+t^2}\, dt$.

(iii) $g(x) = \int_3^{17} \sqrt{1+t^2}\, dt$.

Remarks on Part 2.

(a) Note that the left hand side of the above equation is found by taking the limit of Riemann sums, which involves knowing the value of f at **all** points between a and b. The right hand side, however, only requires knowing the value of F at **two** points!

(b) A common misconception is that this formula is the *definition* of the definite integral. It is not, the definition of integral is the limit of Riemann sums.

(c) We use the notation $F(x)\Big|_a^b = F(b) - F(a)$. Another common notation is $F(x)\Big]_a^b$, which is used in the book.

(d) When we apply the FTC, part 2, we may use *any* antiderivative of f (recall that if $F' = f$, the most general form for an antiderivative of f is $F + C$, where C is an arbitrary constant).

Remarks on Part 2.

(e) The FTC, part 2 is quite plausible from a physical perspective. Recall that when we considered the "Distance Problem" several lectures ago, we determined that the area under the velocity curve was equal to the total distance travelled. That is, if velocity and position are given by $v(t)$ and $s(t)$ respectively at time t, then $s'(t) = v(t)$, so s is an antiderivative of v, and $\int_a^b v(t)\,dt = s(b) - s(a)$.

Before we do some examples, we introduce the notion of an **indefinite integral**. This is just a new notation and name for an antiderivative.

Definition

If $F'(x) = f(x)$, we write $\int f(x)\,dx = F(x) + C$ and call the symbol $\int f(x)\,dx$ the **indefinite integral** of f, and C is called the **constant of integration**.

Note that the definite integral is a *number*, whereas the indefinite integral is a *function*, or a *family of functions*. The Fundamental Theorem of Calculus, part 2 tells us that if f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$. Also note that an indefinite integral can represent either a particular antiderivative of f or an entire family of antiderivatives.

A table of indefinite integrals.

In the following table, c, C, and k are arbitrary constants.

f	$\int f(x)\,dx$
$cf(x)$	$c(\int f(x)\,dx) + C$
$f(x) \pm g(x)$	$(\int f(x)\,dx) \pm (\int g(x)\,dx) + C$
k	$kx + C$
x^n (where $n \neq -1$)	$\dfrac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Examples

Evaluate the following integrals:

(i) $\int_{\pi}^{2\pi} \cos\theta \, d\theta$

(ii) $\int_0^1 x(\sqrt[3]{x} + \sqrt[4]{x}) \, dx$

(iii) Find the indefinite integral: $\int (1-t)(2+t^2) \, dt$.

(iv) **True or False:**
$\int_0^{\pi} \sec^2 x \, dx = \tan x \Big|_0^{\pi} = 0.$

Applications

Suppose f is continuous on $[a, b]$ and F is an antiderivative of f, that is, $F' = f$. Then the FTC, part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

We know $F'(x)$ is the rate of change of $y = F(x)$ with respect to x, and $F(b) - F(a)$ is the *net* change in y when x changes from a to b. Thus the FTC part 2 says that the integral of a rate of change over an interval is the net change over that interval.
For instance,

1. If $V(t)$ is the volume of water in a tank at time t, then $V'(t)$ is the rate at which water flows into the tank at time t. Hence $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$ is the net change in the amount of water in the tank between times t_1 and t_2.

Applications

2. If the rate of growth of a population is $\frac{dP}{dt}$, then $\int_{t_1}^{t_2} \frac{dP}{dt}\, dt = P(t_2) - P(t_1)$, the net change in population during time t_1 to t_2.
3. If an object has position function s and velocity v, then $\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$, the net change of position (or *displacement*) of the particle during the period from t_1 to t_2.

Examples

Find the derivative of g, where g is given by

1. $g(x) = \int_1^x \sqrt{1+t^3}\, dt.$

2. $g(x) = \int_1^{x^4} \sec t\, dt.$

3. $g(x) = \int_2^{\frac{1}{x}} \sin^4 t\, dt.$

4. Evaluate the following integral:

$$\int_0^{\frac{\pi}{4}} \frac{1+\cos^2\theta}{\cos^2\theta}\, d\theta$$

Examples

1. If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?
2. If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)\,dt$ represent?
3. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\,dt$ represent?
4. If $v(t) = 3t - 5$ gives the velocity of a particle moving along a line, find the displacement and the total distance traveled by the particle during the time interval $0 \leq t \leq 3$.
5. Water flows from a storage tank at a rate of $r(t) = 200 - 4t$ l/min, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

The average value of a function

The average value of finitely many numbers $y_1, y_2, \ldots, y_n$ is given by

$$y_{\text{ave}} = \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

In general, how do we calculate the average value of a function $y = f(x)$, where $a \leq x \leq b$?

Definition

Suppose that f is continuous on $[a, b]$. Then the *average value* f_{ave} of f on $[a, b]$ is defined by the formula

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The average value is sometimes denoted by $\overline{f}$.

The average value of a function: Example

Find the average value of the function f over $[-2, 3]$, where

$$f(x) = 1 + x^2.$$

The Mean Value Theorem for Integrals

If $T(t)$ is the temperature (in $^\circ$C) at time t (in hours) and T_{ave} is the average temperature on the time interval $[0, 24]$. Is there a time t_0 in $[0, 24]$ when $T(t_0)$ is equal to T_{ave}? More generally, given a function f, is there a specific value c for which $f(c) = f_{\mathrm{ave}}$?

Yes! This is the *mean value theorem for integrals*.

Theorem (The Mean Value Theorem for integrals)

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

that is

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

The Mean Value Theorem for Integrals - Example

Find all numbers c that satisfy the conclusion of the MVT for integrals when $f(x) = 1 + x^2$ and $[a, b] = [-2, 3]$.

Integration by substitution

To apply the FTC, we must find antiderivatives. This may be difficult, like for

$$\int 2x\sqrt{1+x^2}\,dx.$$

However, there is a useful method called "integration by substitution" that we can apply. We illustrate it with the above example. If we let $u = 1 + x^2$, then $du = 2x\,dx$. Hence

$$\begin{aligned}\int 2x\sqrt{1+x^2}\,dx &= \int \sqrt{1+x^2}(2xdx) \\ &= \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(x^2+1)^{3/2} + C\end{aligned}$$

Integration by substitution

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

That is, *it is permissible to work with dx and du after integral signs as if they are differentials.*

Examples.

1. Evaluate $\int \sqrt{2x+1}\,dx$.
2. Evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.
3. Evaluate $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\,dx$.

Symmetries

Symmetries

Suppose f is integrable on $[-a, a]$, where $a > 0$. Then

1. If f is even $(f(-x) = f(x))$, then
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$
2. If f is odd $(f(-x) = f(x))$, then
$$\int_{-a}^{a} f(x)\,dx = 0.$$