

New York University 수학학사, Columbia University 통계학 석사,
20년 경력의 미국 수학 전문가 이연욱 선생님의 미국수학 시리즈

이연욱(Brian Rhee) 선생님의

AP Calculus AB & BC

이연욱 지음 (Brian Rhee)

미국 토마스 제퍼슨 과학고 및 유수의 명문고
학생들에게 인기 만점인 이연욱 선생님의
AP Calculus 완결판

AP Calculus

핵심 내용,비법 & 노하우 공개

이 책을 제대로 공부한 학생
95% 이상이 AP Calculus
시험에서 5점 만점 기록 !!



MR. RHEE'S BRILLIANT MATH SERIES

AP CALCULUS

By Brian Rhee

Reflects Changes to the New
2017 AP Exam

45 Topic-Specific Lessons with
Key Summaries

Complete Review of both AP Calculus
AB and BC in this Comprehensive
Test Preparation Book

ABOUT AP Calculus AB and BC Exams

The AP Calculus AB and BC exams are intended to measure the extent to which a student has mastered the subject matters of the AP Calculus course. Although the AP Calculus courses focus on differential and integral calculus, students need strong foundations in Algebra, Geometry, and Trigonometry.

Each AP exam is 3 hours and 15 minutes long, and its format is as follows:

Section 1 Multiple choice — 45 Questions (1 hour and 45 minutes)

Part A: 30 questions for 60 minutes (calculator is not permitted)

Part B: 15 questions for 45 minutes (graphing calculator is required)

Section 2 Free Response — 6 Questions (1 hour and 30 minutes)

Part A: 2 questions for 30 minutes (graphing calculator is required)

Part B: 4 questions for 60 minutes (calculator is not permitted)

Contents

Common Topics For AP Calculus AB & BC	9
Lesson 1 The Limit of a Function	11
Lesson 2 Calculating Limits Using the Properties of Limits	17
Lesson 3 Limits at Infinity	23
Lesson 4 Continuity	30
Lesson 5 Average Rate of Change and Instantaneous Rate of Change	36
Lesson 6 Derivatives	43
Lesson 7 Differentiation Rules	48
Lesson 8 Differentiation Rules	54
Lesson 9 The Chain Rule	60
Lesson 10 Implicit Differentiation	66
Lesson 11 Derivatives of Inverse Trig Functions and Higher Derivatives	72
Lesson 12 Indeterminate Forms And L'Hospital's Rule	79
Lesson 13 Related Rates	86
Lesson 14 Linear Approximations And Differentials	92
Lesson 15 Maximum And Minimum Values	99
Lesson 16 The Mean Value Theorem And Rolle's Theorem	106
Lesson 17 Understanding A Curve From The First And Second Derivatives	111
Lesson 18 Optimization Problems	118
Lesson 19 Indefinite Integrals	123
Lesson 20 The Definite Integral	129

**MR. RHEE'S BRILLIANT
MATH SERIES**

TABLE of CONTENTS

Lesson 21	Numerical Approximations Of Integration	135
Lesson 22	The Fundamental Theorem Of Calculus	141
Lesson 23	The U-Substitution Rule	148
Lesson 24	Area Between Curves	155
Lesson 25	Average Value Of A Function and Arc Length	161
Lesson 26	Volumes Of Solids Of Revolution	166
Lesson 27	Volumes Of Solids Of Cross-Sections	174
Lesson 28	Differential Equations	181
AP Calculus BC Topics Only		189
Lesson 29	Logarithmic Differentiation	191
Lesson 30	Indeterminate Products and Indeterminate Powers	196
Lesson 31	Derivative And Arc Length Of Parametric Equations	203
Lesson 32	Volumes By Cylindrical Shells	209
Lesson 33	Integration By Parts	217
Lesson 34	Trigonometric Integrals	223
Lesson 35	Integration By Partial Fractions	229
Lesson 36	Improper Integrals	235
Lesson 37	Differential Equations	241
Lesson 38	Derivative, Arc Length, And Area With Polar Coordinates	249
Lesson 39	Sequences	257
Lesson 40	Convergence And Divergence Of Series, Part I	262
Lesson 41	Convergence And Divergence Of Series, Part II	271
Lesson 42	Strategy For Testing Series	278
Lesson 43	Power Series	284
Lesson 44	Representations Of Functions As Power Series	290
Lesson 45	Taylor And Maclaurin Series	296

AP CALCULUS

BC TOPICS

ONLY

LESSON 29

Logarithmic Differentiation

Logarithmic Differentiation

For a complicated functions involving product or quotient or functions for which differentiation rules do not apply such as x^x , use **Logarithmic differentiation**.

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation and simplify using the logarithmic properties.
2. Differentiate implicitly with respect to x . Recall that whenever differentiating y , multiply the result by $\frac{dy}{dx}$.
3. Solve for y' .

Tip

The logarithmic properties are as follows:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln x^n = n \ln x$
4. $\ln(xy) = \ln x + \ln y$
5. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

Example 1 Applying logarithmic differentiation

Differentiate $y = x^x$

Solution Take natural logarithms of both sides of the equation and simplify using the logarithmic properties.

$$\begin{aligned}y &= x^x \\ \ln y &= \ln x^x \\ \ln y &= x \ln x\end{aligned}$$

Differentiate y implicitly with respect to x and solve for $\frac{dy}{dx}$.

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \ln x + 1 \\ \frac{dy}{dx} &= y(\ln x + 1) \\ \frac{dy}{dx} &= x^x(\ln x + 1)\end{aligned}$$

Therefore, the derivative of $y = x^x$ is $\frac{dy}{dx} = x^x(\ln x + 1)$.

Tip

Recall that

- Power rule: $(x^n)' = nx^{n-1}$, where the base is variable and exponent is constant.
- Rule for differentiating exponential function: $(a^x)' = a^x \ln a$, where the base is constant and exponent is variable.

Thus, you cannot apply the power rule to differentiate the function $y = x^x$, where the base is variable and exponent is variable. Simply put, $y \neq x(x)^{x-1}$.

Example 2 Applying logarithmic differentiation

Differentiate $y = \frac{x\sqrt{x^2+1}}{(1-2x)^2}$.

Solution Take natural logarithms of both sides of the equation and simplify using the logarithmic properties.

$$\begin{aligned}\ln y &= \ln \frac{x\sqrt{x^2+1}}{(1-2x)^2} \\ \ln y &= \ln x + \frac{1}{2} \ln(x^2+1) - 2 \ln(1-2x)\end{aligned}$$

Differentiating y implicitly with respect to x gives

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot (2x) - 2 \cdot \frac{1}{1-2x} \cdot (-2) \\ \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} + \frac{x}{x^2+1} + \frac{4}{1-2x}\end{aligned}$$

Solving for $\frac{dy}{dx}$, we get

$$\begin{aligned}\frac{dy}{dx} &= y \left(\frac{1}{x} + \frac{x}{x^2+1} + \frac{4}{1-2x} \right) \\ \frac{dy}{dx} &= \frac{x\sqrt{x^2+1}}{(1-2x)^2} \left(\frac{1}{x} + \frac{x}{x^2+1} + \frac{4}{1-2x} \right)\end{aligned}$$

EXERCISES

For questions 1-6, use the logarithmic differentiation to find the derivative of the following functions.

1. $y = (3x - 1)^5(x^2 + 2x - 1)^6$

2. $y = \frac{e^{x^2}(x^4 + 1)}{\sqrt{x - 1}}$

3. $y = \sqrt[3]{\frac{x^3 - 1}{x^3 + 1}}$

4. $y = x^{\sqrt{x}}$

5. $y = x^{\sin x}$

6. $y = (\ln x)^x$

Answers

1. $\frac{dy}{dx} = (3x - 1)^5(x^2 + 2x - 1)^6 \left(\frac{15}{3x - 1} + \frac{12(x + 1)}{x^2 + 2x - 1} \right)$
2. $\frac{dy}{dx} = \frac{e^{x^2}(x^4 + 1)}{\sqrt{x - 1}} \left(2x + \frac{4x^3}{x^4 + 1} - \frac{1}{2(x - 1)} \right)$
3. $\frac{dy}{dx} = \sqrt[3]{\frac{x^3 - 1}{x^3 + 1}} \left(\frac{2x^2}{x^3 - 1} \cdot \frac{1}{x^3 + 1} \right)$
4. $\frac{dy}{dx} = x^{\sqrt{x}} \left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} \right)$
5. $\frac{dy}{dx} = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right)$
6. $\frac{dy}{dx} = (\ln x)^x \left(\ln(\ln x) + \frac{1}{\ln x} \right)$

LESSON 30

Indeterminate Products and Indeterminate Powers

L'Hospital's Rule

Suppose you have indeterminate forms of

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Tip

1. L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives.
2. L'Hospital's Rule is also valid for one-sided limits: that is, $x \rightarrow a$ can be replaced by any of the followings: $x \rightarrow a^+$, $x \rightarrow a^-$.

Indeterminate Products

Suppose $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$. Then $\lim_{x \rightarrow a} f(x)g(x) = 0 \cdot \infty$. This kind of limit is called an **Indeterminate form of type $0 \cdot \infty$** . The L'Hospital's Rule does not work on products, it only works on quotients.

Steps in finding the limit of an indeterminate form of type $0 \cdot \infty$

1. Rewrite the product fg as a quotient shown below.

$$fg = \frac{f}{\frac{1}{g}} \quad \text{or} \quad fg = \frac{g}{\frac{1}{f}}$$

Then

$$\lim_{x \rightarrow a} fg = \lim_{x \rightarrow a} \frac{f}{\frac{1}{g}} = \frac{0}{0} \quad \text{or} \quad \lim_{x \rightarrow a} fg = \lim_{x \rightarrow a} \frac{g}{\frac{1}{f}} = \frac{\infty}{\infty}$$

the indeterminate form of type $0 \cdot \infty$ changes to an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

2. Use L'Hospital's Rule.

Tip

When you rewrite the product fg as a quotient either $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$, select one that leads you to the simpler limit after you apply L'Hospital's Rule. If you get a more complicated expression than the one you started with, select the other quotient.

Example 1 Finding the limit of an indeterminate product

Evaluate $\lim_{x \rightarrow 0^+} x \ln x$.

Solution Plugging-in $x = 0^+$ into $x \ln x$, we get an indeterminate form of $0 \cdot \infty$. Rewrite the product $x \ln x$ as $\frac{\ln x}{1/x}$ so that the indeterminate form of $0 \cdot \infty$ becomes an indeterminate form of $\frac{\infty}{\infty}$. Use L'Hospital's Rule to evaluate the limit.

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} && \text{Use L'Hospital's Rule} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} (-x) \\ &= 0\end{aligned}$$

Tip

In case you rewrite the product $x \ln x$ as a quotient, $\frac{x}{1/\ln x}$, and apply L'Hospital's Rule,

$$\begin{aligned}\lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{x}{\frac{1}{\ln x}} && \text{Use L'Hospital's Rule} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{-(\ln x)^{-2} \cdot \frac{1}{x}} \\ &= \lim_{x \rightarrow 0^+} -x(\ln x)^2\end{aligned}$$

you get a more complicated expression than the one started with. So, select other quotient.

Indeterminate Differences

Suppose $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$. Then $\lim_{x \rightarrow a} [f(x) - g(x)] = \infty - \infty$. This kind of limit is called an **Indeterminate form of type $\infty - \infty$** . L'Hospital's Rule does not work on differences, it only works on quotients. Thus, rewrite the difference as a quotient and use L'Hospital's Rule.

Example 2 Finding the limit of an indeterminate difference

Evaluate $\lim_{x \rightarrow \infty} (xe^{\frac{1}{x}} - x)$.

Solution

Plugging-in $x = \infty$ into $xe^{\frac{1}{x}} - x$, we get an indeterminate form of $\infty - \infty$. Rewrite the difference as a quotient such that $xe^{\frac{1}{x}} - x = x(e^{\frac{1}{x}} - 1) = \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}}$.

$$\lim_{x \rightarrow \infty} (xe^{\frac{1}{x}} - x) = \lim_{x \rightarrow \infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \frac{0}{0}$$

Thus, the indeterminate form of $\infty - \infty$ becomes an indeterminate form of $\frac{0}{0}$. Use L'Hospital's Rule to evaluate the limit.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} &= \lim_{x \rightarrow \infty} \frac{e^{\frac{1}{x}} \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} e^{\frac{1}{x}} \\ &= 1 \end{aligned}$$

Indeterminate Powers

If $\lim_{x \rightarrow a} [f(x)]^{g(x)}$ has the following indeterminate power forms,

$$0^0 \quad \infty^0 \quad 1^\infty$$

let $y = \lim_{x \rightarrow a} [f(x)]^{g(x)}$ and takes the natural logarithm on both sides of the equation so that the indeterminate power form becomes an indeterminate product form.

$$y = \lim_{x \rightarrow a} [f(x)]^{g(x)} \implies \ln y = \lim_{x \rightarrow a} g(x) \ln f(x) = 0 \cdot \infty$$

Then, rewrite the product as a quotient and use L'Hospital's Rule.

Tip

Notice that 0^∞ , ∞^∞ , 1^0 are **NOT** indeterminate power forms.

Example 3 Finding the limit of an indeterminate power

Evaluate $\lim_{x \rightarrow 0^+} x^x$.

Solution Plugging-in $x = 0^+$ into x^x , we get an indeterminate power form of 0^0 . Let $y = \lim_{x \rightarrow 0^+} x^x$ and take the natural logarithm on both sides of the equation.

$$y = \lim_{x \rightarrow 0^+} x^x$$
$$\ln y = \lim_{x \rightarrow 0^+} x \ln x$$

Since $\lim_{x \rightarrow 0^+} x \ln x = 0$ shown in the example 1,

$$\ln y = 0$$
$$y = e^0 = 1$$

Therefore, $y = \lim_{x \rightarrow 0^+} x^x = 1$.

EXERCISES

1. Evaluate $\lim_{x \rightarrow \infty} e^{-x} \ln x$.

2. Evaluate $\lim_{x \rightarrow \infty} x^2 e^{-x}$.

3. Evaluate $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

4. Evaluate $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$.

5. Evaluate $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 1})$.

6. Evaluate $\lim_{x \rightarrow 0^+} x^{\sin x}$.

Answers

1. 0

4. e

2. 0

5. 0

3. 1

6. 1