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이연욱(Brian Rhee) 선생님의

AP Calculus AB & BC

이연욱 지음 (Brian Rhee)

미국 토마스 제퍼슨 과학고 및 유수의 명문고
학생들에게 인기 만점인 이연욱 선생님의
AP Calculus 완결판

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ABOUT AP Calculus AB and BC Exams

The AP Calculus AB and BC exams are intended to measure the extent to which a student has mastered the subject matters of the AP Calculus course. Although the AP Calculus courses focus on differential and integral calculus, students need strong foundations in Algebra, Geometry, and Trigonometry.

Each AP exam is 3 hours and 15 minutes long, and its format is as follows:

Section 1 — Multiple choice — 45 Questions (1 hour and 45 minutes)

- Part A: 30 questions for 60 minutes (calculator is not permitted)
- Part B: 15 questions for 45 minutes (graphing calculator is required)

Section 2 — Free Response — 6 Questions (1 hour and 30 minutes)

- Part A: 2 questions for 30 minutes (graphing calculator is required)
- Part B: 4 questions for 60 minutes (calculator is not permitted)

Contents

AP CACULUS

BC TOPICS

ONLY

LESSON 29

Logarithmic Differentiation

Logarithmic Differentiation

For a complicated functions involving product or quotient or functions for which differentiation rules do not apply such as x^x, use **Logarithmic differentiation**.

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation and simplify using the logarithmic properties.
2. Differentiate implicitly with respect to x. Recall that whenever differentiating y, multiply the result by $\frac{dy}{dx}$.
3. Solve for y'.

Tip The logarithmic properties are as follows:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln x^n = n \ln x$
4. $\ln(xy) = \ln x + \ln y$
5. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$

Example 1 Applying logarithmic differentiation

Differentiate $y = x^x$

Solution Take natural logarithms of both sides of the equation and simplify using the logarithmic properties.

$$\begin{aligned} y &= x^x \\ \ln y &= \ln x^x \\ \ln y &= x \ln x \end{aligned}$$

Differentiate y implicitly with respect to x and solve for $\frac{dy}{dx}$.

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$
$$\frac{dy}{dx} = y(\ln x + 1)$$
$$\frac{dy}{dx} = x^x(\ln x + 1)$$

Therefore, the derivative of $y = x^x$ is $\dfrac{dy}{dx} = x^x(\ln x + 1)$.

Tip Recall that

- Power rule: $(x^n)' = nx^{n-1}$, where the base is variable and exponent is constant.
- Rule for differentiating exponential function: $(a^x)' = a^x \ln a$, where the base is constant and exponent is variable.

Thus, you cannot apply the power rule to differentiate the function $y = x^x$, where the base is variable and exponent is variable. Simply put, $y \neq x(x)^{x-1}$.

Example 2 Applying logarithmic differentiation

Differentiate $y = \dfrac{x\sqrt{x^2+1}}{(1-2x)^2}$.

Solution Take natural logarithms of both sides of the equation and simplify using the logarithmic properties.

$$\ln y = \ln \frac{x\sqrt{x^2+1}}{(1-2x)^2}$$
$$\ln y = \ln x + \frac{1}{2}\ln(x^2+1) - 2\ln(1-2x)$$

Differentiating y implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot(2x) - 2\cdot\frac{1}{1-2x}\cdot(-2)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{x}{x^2+1} + \frac{4}{1-2x}$$

Solving for $\dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{x}{x^2+1} + \frac{4}{1-2x}\right)$$
$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(1-2x)^2}\left(\frac{1}{x} + \frac{x}{x^2+1} + \frac{4}{1-2x}\right)$$

EXERCISES

For questions 1-6, use the logarithmic differentiation to find the derivative of the following functions.

1. $y = (3x-1)^5(x^2+2x-1)^6$

2. $y = \dfrac{e^{x^2}(x^4+1)}{\sqrt{x-1}}$

3. $y = \sqrt[3]{\dfrac{x^3-1}{x^3+1}}$

4. $y = x^{\sqrt{x}}$

5. $y = x^{\sin x}$

6. $y = (\ln x)^x$

Answers

1. $\dfrac{dy}{dx} = (3x-1)^5(x^2+2x-1)^6\left(\dfrac{15}{3x-1} + \dfrac{12(x+1)}{x^2+2x-1}\right)$

2. $\dfrac{dy}{dx} = \dfrac{e^{x^2}(x^4+1)}{\sqrt{x-1}}\left(2x + \dfrac{4x^3}{x^4+1} - \dfrac{1}{2(x-1)}\right)$

3. $\dfrac{dy}{dx} = \sqrt[3]{\dfrac{x^3-1}{x^3+1}}\left(\dfrac{2x^2}{x^3-1} \cdot \dfrac{1}{x^3+1}\right)$

4. $\dfrac{dy}{dx} = x^{\sqrt{x}}\left(\dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}\right)$

5. $\dfrac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

6. $\dfrac{dy}{dx} = (\ln x)^x\left(\ln(\ln x) + \dfrac{1}{\ln x}\right)$

LESSON 30

Indeterminate Products and Indeterminate Powers

L'Hospital's Rule

Suppose you have indeterminate forms of

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{or} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Tip

1. L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives.
2. L'Hospital's Rule is also valid for one-sided limits: that is, $x \to a$ can be replaced by any of the followings: $x \to a^+$, $x \to a^-$.

Indeterminate Products

Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$. Then $\lim_{x \to a} f(x)g(x) = 0 \cdot \infty$. This kind of limit is called an **Indeterminate form of type $0 \cdot \infty$.** The L'Hospital's Rule does not work on products, it only works on quotients.

Steps in finding the limit of an indeterminate form of type $0 \cdot \infty$

1. Rewrite the product fg as a quotient shown below.

$$fg = \frac{f}{\frac{1}{g}} \qquad \text{or} \qquad fg = \frac{g}{\frac{1}{f}}$$

   Then

$$\lim_{x \to a} fg = \lim_{x \to a} \frac{f}{\frac{1}{g}} = \frac{0}{0} \qquad \text{or} \qquad \lim_{x \to a} fg = \lim_{x \to a} \frac{g}{\frac{1}{f}} = \frac{\infty}{\infty}$$

   the indeterminate form of type $0 \cdot \infty$ changes to an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

2. Use L'Hospital's Rule.

Tip

When you rewrite the product fg as a quotient either $fg = \dfrac{f}{1/g}$ or $fg = \dfrac{g}{1/f}$, select one that leads you to the simpler limit after you apply L'Hospital's Rule. If you get a more complicated expression than the one you started with, select the other quotient.

Example 1 Finding the limit of an indeterminate product

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution Plugging-in $x = 0^+$ into $x \ln x$, we get an indeterminate form of $0 \cdot \infty$. Rewrite the product $x \ln x$ as $\dfrac{\ln x}{1/x}$ so that the indeterminate form of $0 \cdot \infty$ becomes an indeterminate form of $\frac{\infty}{\infty}$. Use L'Hospital's Rule to evaluate the limit.

$$\begin{aligned} \lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} && \text{Use L'Hospital's Rule} \\ &= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \to 0^+} (-x) \\ &= 0 \end{aligned}$$

Tip In case you rewrite the product $x \ln x$ as a quotient, $\dfrac{x}{1/\ln x}$, and apply L'Hospital's Rule,

$$\begin{aligned} \lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{x}{\frac{1}{\ln x}} && \text{Use L'Hospital's Rule} \\ &= \lim_{x \to 0^+} \frac{1}{-(\ln x)^{-2} \cdot \frac{1}{x}} \\ &= \lim_{x \to 0^+} -x(\ln x)^2 \end{aligned}$$

you get a more complicated expression than the one started with. So, select other quotient.

Indeterminate Differences

Suppose $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$. Then $\lim_{x \to a} [f(x) - g(x)] = \infty - \infty$. This kind of limit is called an **Indeterminate form of type $\infty - \infty$.** L'Hospital's Rule does not work on differences, it only works on quotients. Thus, rewrite the difference as a quotient and use L'Hospital's Rule.

Example 2 Finding the limit of an indeterminate difference

Evaluate $\lim\limits_{x\to\infty}(xe^{\frac{1}{x}} - x)$.

Solution
Plugging-in $x = \infty$ into $xe^{\frac{1}{x}} - x$, we get an indeterminate form of $\infty - \infty$. Rewrite the difference as a quotient such that $xe^{\frac{1}{x}} - x = x(e^{\frac{1}{x}} - 1) = \dfrac{e^{\frac{1}{x}} - 1}{\frac{1}{x}}$.

$$\lim_{x\to\infty}(xe^{\frac{1}{x}} - x) = \lim_{x\to\infty}\frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \frac{0}{0}$$

Thus, the indeterminate form of $\infty - \infty$ becomes an indeterminate form of $\frac{0}{0}$. Use L'Hospital's Rule to evaluate the limit.

$$\begin{aligned}\lim_{x\to\infty}\frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} e^{\frac{1}{x}} \\ &= 1\end{aligned}$$

Indeterminate Powers

If $\lim\limits_{x\to a}[f(x)]^{g(x)}$ has the following indeterminate power forms,

$$0^0 \qquad \infty^0 \qquad 1^\infty$$

let $y = \lim\limits_{x\to a}[f(x)]^{g(x)}$ and takes the natural logarithm on both sides of the equation so that the indeterminate power form becomes an indeterminate product form.

$$y = \lim_{x\to a}[f(x)]^{g(x)} \qquad \Longrightarrow \qquad \ln y = \lim_{x\to a} g(x)\ln f(x) = 0\cdot\infty$$

Then, rewrite the product as a quotient and use L'Hospital's Rule.

Tip Notice that 0^∞, ∞^∞, 1^0 are **NOT** indeterminate power forms.

Example 3 Finding the limit of an indeterminate power

Evaluate $\lim_{x \to 0^+} x^x$.

Solution Plugging-in $x = 0^+$ into x^x, we get an indeterminate power form of 0^0. Let $y = \lim_{x \to 0^+} x^x$ and take the natural logarithm on both sides of the equation.

$$y = \lim_{x \to 0^+} x^x$$
$$\ln y = \lim_{x \to 0^+} x \ln x$$

Since $\lim_{x \to 0^+} x \ln x = 0$ shown in the example 1,

$$\ln y = 0$$
$$y = e^0 = 1$$

Therefore, $y = \lim_{x \to 0^+} x^x = 1$.

EXERCISES

1. Evaluate $\lim_{x\to\infty} e^{-x}\ln x$.

2. Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

3. Evaluate $\lim_{x\to\infty} x^{\frac{1}{x}}$.

4. Evaluate $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

5. Evaluate $\lim\limits_{x\to\infty}(x-\sqrt{x^2-1})$.

6. Evaluate $\lim\limits_{x\to 0^+} x^{\sin x}$.

Answers

1. 0	2. 0	3. 1
4. e	5. 0	6. 1