A Guide to Complex Numbers

Your Name

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1 Introduction

Complex numbers extend the real numbers by introducing the imaginary unit i such that $i^2 = -1$. A complex number can be written as $z = x + iy$ where $x, y \in \mathbb{R}$.

2 Basic Properties

- **Real part:** $\Re(z) = x$
- **Imaginary part:** $\Im(z) = y$
- **Complex conjugate:** $\overline{z} = x - iy$
- **Modulus:** $|z| = \sqrt{x^2 + y^2}$
- **Argument:** $\arg(z) = \arctan\left(\frac{y}{x}\right)$ (the angle with the real axis)

3 Arithmetic with Complex Numbers

Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

4 Polar and Exponential Forms

A complex number can also be represented in polar (trigonometric) and exponential forms:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $r = |z|$, $\theta = \arg(z)$.

5 The Complex Logarithm $\ln(z)$

The logarithm of a complex number $z = re^{i\theta}$ is defined as:

$$\ln(z) = \ln|z| + i\arg(z)$$

- $\ln|z|$ is the natural logarithm of the modulus.
- $\arg(z)$ is the argument (angle), and since it is multi-valued (differing by $2\pi n$), $\ln(z)$ is also multi-valued:

$$\ln(z) = \ln r + i(\theta + 2\pi n), \quad n \in \mathbb{Z}$$

- The **principal value** often takes $\theta \in (-\pi, \pi]$.

5.1 Examples

$$\ln(1) = 0$$
$$\ln(-1) = i\pi$$
$$\ln(i) = i\left(\frac{\pi}{2}\right)$$

6 Applications of Complex Numbers

Complex numbers arise in many mathematical and physical contexts, such as:

- Solving polynomial equations
- Engineering (AC circuits, signal processing)
- Quantum mechanics

7 References