HW 2

Q5 - Q9

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Question 5

Exercise 1.12.2

(b)

$$\begin{array}{l} p \to (q \land r) \\ \underline{\neg q \qquad\quad} \\ \therefore \neg p \end{array}$$

1	$p \to (q \land r)$	hypothesis
2	$p \to q$	simplification, 1
3	$\neg q$	hypothesis
4	$\neg p$	modus tollens, 2, 3

(e)

$$\begin{array}{l} p \land q \\ \neg p \lor r \\ \underline{\neg q \qquad} \\ \therefore r \end{array}$$

1	$\neg p \lor r$	hypothesis
2	$p \lor q$	hypothesis
3	$q \lor r$	resolution, 1, 2
4	$\neg q$	hypothesis
5	r	disjunction syllogism, 3, 4

Exercise 1.12.3

(c)

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

1	$p \lor q$	hypothesis
2	$\neg(\neg p) \lor q$	double negation, 1
3	$\neg p$	hypothesis
4	q	modus ponens, 2, 3

Exercise 1.12.5

j = I get a job.
c = I will buy a new car.
h = I will buy a new house.

(c) "I will buy a new car and a new house only if I get a job." = $(c \land h) \to j$
"I am not going to get a job." = $\neg j$
"I will not buy a new car." = $\neg c$

$$\begin{array}{l} (c \land h) \to j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

j	h	c	$(c \land h) \to j$	$\neg j$	$\neg c$
T	T	T	T	F	F
T	T	F	T	F	T
T	F	T	T	F	F
T	F	F	T	F	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	T	T	F
F	F	F	F	T	T

The argument is invalid. When j = h = F and c = T, the hypotheses are true and the conclusion is false.

(d) "I will buy a new car and a new house only if I get a job." $= (c \wedge h) \rightarrow j$
"I am not going to get a job." $= \neg j$
"I will buy a new house." $= h$
"I will not buy a new car." $= \neg c$

$$\begin{array}{l} (c \wedge h) \rightarrow j \\ \neg j \\ h \\ \hline \therefore \neg c \end{array}$$

j	h	c	$(c \wedge h) \rightarrow j$	$\neg j$	$\neg c$
T	T	T	T	F	F
T	T	F	T	F	T
T	F	T	T	F	F
T	F	F	T	F	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	T	T	F
F	F	F	F	T	T

The argument is valid. When h = T and c = j = F, the hypotheses are true and the conclusion is true.

Exercise 1.13.3

(b)

$$\begin{array}{l} \exists x(P(x) \vee Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

$P(x) = F$ over the domain of {a,b} because the conclusion must be false.

	P	Q
a	F	
b	F	

if $Q(a) = T$, then $Q(b) = F$ or vice versa because $\exists \neg Q(x)$ must be true.

	P	Q
a	F	T
b	F	F

Exercise 1.13.5

$S(x)$: x is the student in the class.
$M(x)$: x misses a class.
$D(x)$: x gets a detention.
$A(x)$: x gets an A.

(d) Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

∴ Penelope did not get a detention.

$\forall x(M(x) \rightarrow D(x))$
Penelope is a stduent in the class.
$\neg M$(Penelope)

$\neg D$(Penelope)

The argument is invalid. When $M(x) = F$ and $D(x) = T$, the hypotheses are true and the conclusion is false.

(e) Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

∴ Penelope did not get a detention.

$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$
Penelope is a student in the class
A(Penelope)

$\neg D$(Penelope)

The argument is valid.

1	$\forall((M(x) \vee D(x)) \rightarrow \neg A(x))$	hypothesis
2	Penelope is a student in the class.	hypothesis
3	$(M$(Penelope) $\vee D$(Penelope)) $\rightarrow \neg A$(Penelope)	Universal Instanitiation, 1, 2
4	A(Penelope)	hypothesis
5	$\neg(\neg A$(Penleope))	Double negation, 4
6	$\neg(M$(Penelope) $\vee D$(Penelope))	Modus tollens, 3, 5
7	$\neg M$(Penelope) $\wedge \neg D$(Penelope)	De Morgan's Law, 6
8	$\neg D$(Penelope) $\wedge \neg M$(Penelope)	Commutative Law, 7
9	$\neg D$(Penelope)	Simplification, 8

Question 6

Exercise 2.4.1

(d) The product of two odd integers is an odd integer.

Assume: x and y are odd integers.

Prove: xy is odd.

Proof: let $x = 2k + 1$ and $y = 2j + 1$ where k and j are integers.

$$\begin{aligned} xy &= (2k + 1)(2j + 1) \\ &= 4kj + 2j + 2k + 1 \\ &= 2(2kj + j + k) + 1 \end{aligned}$$

Since k and j are integers, $2kj + j + k$ is an integer.

$xy = 2p + 1$ where $p = 2kj + j + k$.

Therefore, xy is an od integer. ∎

Exercise 2.4.3

(b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Assume: x is a real number and $x \leq 3$

Prove: $12 - 7x + x^2 \geq 0$.

Proof: $12 - 7x + x^2$ is factored into two binomials:

$(x - 4)(x - 3) \geq 0$

If $x = 3$, $x - 4 = -1$ and $x - 3 = 0$, then $12 - 7x + x^2 = 0$.

If $x < 3$, $x - 4 < 0$ and $x - 3 < 0$, then $12 - 7x + x^2 > 0$.

Therefore, if $x \leq 3$ where x is a real number, $12 - 7x + x^2 \geq 0$. ∎

Question 7

Exercise 2.5.1

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Assume: n is an even integer.

Prove: $n^2 - 2n + 7$ is odd.

Proof: $n = 2k$ where k is an integer.

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 4k^2 - 4k + 6 + 1 \\ &= 2(2k^2 - 2k + 3) + 1 \end{aligned}$$

Since k is an integer, $2k^2 - 2k + 3$ is an integer.

$n^2 - 2n + 7 = 2p + 1$ where $p = 2k^2 - 2k + 3$.

Therefore, $n^2 - 2n + 7$ is odd. $\blacksquare$

Exercise 2.5.4

(a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Assume: $x > y$ when x and y are real numbers.

Prove: $x^3 + xy^2 > x^2y + y^3$.

Proof: Since $x > y$, $x \neq y \neq 0$.

$x^3 + xy^2 > x^2y + y^3$.

$x^3 - xy^2 + xy^2 - y^3 > 0$

$x^2(x - y) + y^2(x - y) > 0$

$(x^2 + y^2)(x - y) > 0$

Since x and y are real numbers, $x^2 > 0$, $y^2 > 0$, and $x^2 + y^2 > 0$.

Since $x > y$, $x - y > 0$.

Therefore, $(x^2 + y^2)(x - y)$ must be greater 0, hence, $x^3 + xy^2 > x^2y + y^3$. $\blacksquare$

(b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Assume: $x \leq 10$ and $y \leq 10$ when x and y are real numbers.

Prove: $x + y \leq 20$.

Proof: when $x = y = 10$, then $x + y = 20$.

when $x < 10$ and $y < 10$, then $x + y < 20$.

Therefore, the sum of x and y cannot exceed 20. $\blacksquare$

Exercise 2.5.5

(c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Method: Contrapositive

Assume: $\frac{1}{x}$ is rational where $x \neq 0$.

Prove: x is rational.

Proof: $\frac{1}{x} = \frac{a}{b}$ where a and b are rational numbers and $a \neq b \neq 0$.

Since x and $\frac{1}{x}$ are reciprocal to each other, $x = \frac{b}{a}$.

Since a and b are rational numbers, $\frac{b}{a}$ is rational.

Therefore, x is rational. $\blacksquare$

Question 8

Exercise 2.6.6

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

Assume: when x, y, and z are real numbers, the average of those numbers is less than all three numbers.

Proof:

$$\frac{x+y+z}{3} < x \quad \text{and} \quad \frac{x+y+z}{3} < y \quad \text{and} \quad \frac{x+y+z}{3} < z$$

$$x+y+z < 3x \quad \text{and} \quad x+y+z < 3y \quad \text{and} \quad x+y+z < 3z$$

When the three inequalities are added together, it gives:

$$3x + 3y + 3z < 3x + 3y + 3z$$

which sums up as $0 < 0$.

the inequality above is a false statement. Therefore, the average of the three numbers is not less than all three numbers. ■

(d) There is no smallest integer.

Assume: there is a smallest integer.

Proof: let x be the smallest integer.

Since x is an integer, $x - 1$ is also an integer, hence, $x - 1 < x$.

This contradicts the assumption that x is the smallest integer.

Therefore, there is no smallest integer. ■

Question 9

Exercise 2.7.2

(b) **If integers x and y have the same parity, then $x + y$ is even. The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.**

Case 1: x and y are both odd.

$x = 2k + 1$ where k is a real number.

$y = 2j + 1$ where j is a real number.

$x + y = 2k + 2j + 2 = 2(k + j + 1)$

Since k and j are real numbers, $k + j + 1$ is a real number.

Therefore, when x and y are odd, $x + y$ is even.

Case 2: x and y are both even.

$x = 2k$ where k is a real number.

$y = 2j$ where j is a real number.

$x + y = 2k + 2j = 2(k + j)$

Since k and j are real numbers, $k + j$ is a real number.

Therefore, when x and y are even, $x + y$ is also even. ■