HW 11

Q5 - Q6

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Question 5

(a) Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder).

proof: by induction on n.

Base case: n = 1.

$n = 1^3 + 2(1) = 3$, which is divisible by 3.

inductive hypothesis: for any positive integer $k \geq 1$, $k^3 + 2k$ is divisible by 3.

Induction step: we will show that $(k+1)^3 + 2(k+1)$ is divisible by 3.

$$\begin{aligned}
&(k+1)^3 + 2(k+1) \\
&= k^3 + 3k^2 + 3k + 1 + 2k + 1 \\
&= (k^3 + 2k) + (3k^2 + 3k + 3)
\end{aligned}$$

$k^3 + 2k$, by the inductive hypothesis, is divisible by 3, and $3k^2 + 3k + 3$ is also divisble by 3. Therefore, $(k+1)^3 + 2(k+1)$ is divisble by 3.

(b) Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

proof: by strong induction on n.

Base case: n = 2.

Since 2 is a prime number, it can be written as $2 \cdot 1$.

inductive hypothesis: for any positive integer $k \geq 2$, k is a product of primes.

Induction step: we will show that $k+1$ is also a product of primes.

if $k+1$ is a prime number, then it can be expressed as $(k+1) \cdot 1$. If $k+1$ is a composite number, it can be expressed as $k+1 = a \cdot b$ where a and b are greater than 2. The equation can rewritten in terms of a:

$$a = \frac{k+1}{b}$$

Since $b \geq 2$,

$$a = \frac{k+1}{b} \leq k+1$$

Same logic is applied to b, where

$$b = \frac{k+1}{a} \leq k+1$$

By the inductive hypothesis, a and b, which are greater than 2, can be expressed as products of prime numbers. Therefore, $a \cdot b = k + 1$ can be expressed as a product of prime numbers.

Question 6

Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(n+2)}{6}$$

Exercise 7.4.1

(a) Verify that P(3) is true.

$$\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = \frac{3(3+1)(3+2)}{6} = 14$$

(b) Express P(k).

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(k+2)}{6}$$

(c) Express P(k + 1).

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(k+3)}{6}$$

(d) In an inductive proof that for every positive integer n, what must be proven in the base case?

P(1) is true.

(e) In an inductive proof that for every positive integer n, what must be proven in the inductive step?

for all positive integers, P(k) implies P(k + 1)

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

P(k)

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(n+2)}{6}$$

Proof: by induction on n.
Base Case: n = 1.

$$\sum_{j=1}^{n} 1^2 = \frac{1(1+1)(1+2)}{6} = \frac{6}{6} = 1$$

Inductive hypothesis: for any positive integer,

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(k+2)}{6}$$

Induction step: we will show that,

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(k+3)}{6}$$

Proof:

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2$$

$$= \frac{k(k+1)(k+2)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(k+2) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(k+2)(k+3)}{6}$$

Exercise 7.4.3

(c) Prove that for $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Proof: by induction on n.
Base case: $n = 1$.

$$\frac{1}{1^2} = 2 - \frac{1}{1} = 1$$

Inductive Hypothesis: for any positive integer k,

$$\sum_{j=1}^{k} \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

Induction step: we will show that

$$\sum_{j=1}^{k+1} \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Proof:

$$\sum_{j=1}^{k+1} \frac{1}{(k+1)^2} = \sum_{j=1}^{k} \frac{1}{k^2} + \frac{1}{k+1}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}$$

$$\leq 2 - \frac{k}{k(k+1)}$$

$$\leq 2 - \frac{1}{k(k+1)}$$

Exercise 7.5.1

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Proof: by induction on n.

Base case: n = 1.

$3^{2 \cdot 1} - 1 = 8$

8 is divisible by 4.

Inductive Hypothesis: for any positive integer k, $3^{2k} - 1 = 4m$ where m is an integer.

Inductive step: we will show that $3^{2(k+1)} - 1$ is divisible by 4. By the inductive hypothesis,

$$3^{2k} = 4m + 1$$

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^2 \cdot 3^{2k} - 1 = 9 \cdot 3^{2k} - 1 = 9(4m+1) - 1 = 36m + 8$$

$36m + 8$ is divisible by 4, therefore, $3^{2(k+1)} - 1$ is divisible by 4.