HW 8

Q7 - Q10

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Question 7

Exercise 6.1.5

(b) What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank.

$|E| = \binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}$

$|S| = (52; 5)$

$p(E) = \frac{|E|}{|S|} = \frac{\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5}} = \frac{13 \cdot 4 \cdot 6 \cdot 11 \cdot 4 \cdot 4}{52 \cdot 51 \cdot 49 \cdot 5 \cdot 4} = \frac{88}{4165}$

(c) What is the probability that all 5 cards have the same suit?

$|E| = \binom{4}{1} \cdot \binom{13}{5}$

$|S| = \binom{52}{5}$

$p(E) = \frac{|E|}{|S|} = \frac{\binom{4}{1} \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{4 \cdot 13 \cdot 11 \cdot 9}{52 \cdot 51 \cdot 49 \cdot 5 \cdot 4} = \frac{33}{16660}$

(d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

$|E| = \binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{1}^3$

$|S| = \binom{52}{5}$

$p(E) = \frac{|E|}{|S|} = \frac{\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{1}^3}{\binom{52}{5}} = \frac{13 \cdot 2 \cdot 3 \cdot 2 \cdot 11 \cdot 10 \cdot 4^3}{52 \cdot 51 \cdot 49 \cdot 5 \cdot 4} = \frac{352}{833}$

Exercise 6.2.4

A 5-card hand is dealt from a perfectly shuffled deck of playing cards.

(a) The hand has at least one club.

let E be the event of getting at least one club in a 5-card hand. Then $\overline{E}$ would be the event of getting no club in a 5-card hand.

$|\overline{E}| = \binom{39}{5}$

$p(E) = 1 - p(\overline{E}) = 1 - \frac{\binom{39}{5}}{\binom{52}{5}} = 1 - \frac{39 \cdot 37 \cdot 19 \cdot 7 \cdot 3}{52 \cdot 51 \cdot 49 \cdot 5 \cdot 4} = 1 - \frac{27417}{123760} = 0.77847$

(b) The hand has at least two cards with the same rank.

let E be the event of getting at least two cards with the same rank in a 5-card hand. Then $\overline{E}$ would be the event of getting no two cards with the same rank in a 5-card hand.

$|\overline{E}| = \binom{13}{5} \cdot \binom{4}{1}^5$

$p(\overline{E}) = 1 - \frac{\binom{13}{5} \cdot \binom{4}{1}^5}{\binom{52}{5}} = 1 - \frac{11 \cdot 3 \cdot 4^3}{17 \cdot 49 \cdot 5} = 0.49292$

(c) The hand has exactly one club or exactly one spade.

Let A be the event of getting exactly one club in a 5-card hand and B be the event of getting exactly one spade.

$|A| = \binom{13}{1} \cdot \binom{39}{4}$

$|B| = \binom{13}{1} \cdot \binom{39}{4}$

Since the events are not mutually exclusive, the probability of having exactly one club or exactly one spade is:

$p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{\binom{13}{1} \cdot \binom{39}{4}}{\binom{52}{5}} + \frac{\binom{13}{1} \cdot \binom{39}{4}}{\binom{52}{5}} - \frac{\binom{13}{1} \cdot \binom{13}{1} \cdot \binom{26}{3}}{\binom{52}{5}} = 0.65377$

(d) The hand has at least one club or at least one spade.

Let E be the event of getting at least one club or at least one spade. Then $\overline{E}$ would be the event of getting no club and no spade.

$|\overline{E}| = \binom{26}{5}$

$|S| = \binom{52}{5}$

$p(E) = 1 - p(\overline{E}) = 1 - \frac{|E|}{|S|} = 1 - \frac{\binom{26}{5}}{\binom{52}{5}} = 1 - \frac{23 \cdot 11}{51 \cdot 49 \cdot 4} = 0.97469$

Question 8

Exercise 6.3.2

The letters {a, b, c, d, e, f, g} are put in a random order. Each permutation is equally likely. Define the following events:

A: The letter b falls in the middle (with three before it and three after it)

B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "agbdcef" would be an outcome in this event.

C: The letters "def" occur together in that order (e.g. "gdefbca")

(a) Calculate the probability of each individual event. That is, calculate p(A), p(B), and p(C).

The letter "b" is predetermined, so there are 6 positions left to consider. Therefore, $p(A) = \frac{6!}{7!} = \frac{1}{7}$. Half of the outcomes have the letter "c" comes before the letter "b" and the other half of the outcomes have the letter "c" after the letter "b". Therefore, $p(B) = \frac{1}{2}$. The string "def" is predetermined, so there are 4 positions left to consider. As the string would be treated as a one group, $p(C) = \frac{5!}{7!} = \frac{1}{42}$

(b) What is p(A|C)?

There are 2 cases of $A \cap C$: 1) the string "defb" followed by 3 blank positions for the letters "a", "c", and "g", and 2) 3 blank positions followed by the string "bdef". So $|A \cap C| = 3! \cdot 2$. The number of outcomes of the event C is $|C| = 5!$. Therefore, $p(A|C) = \frac{|A \cap C|}{|C|} = \frac{3! \cdot 2}{5!} = \frac{1}{10}$

(c) What is p(B|C)?

There are 5! number of outcomes of event C. Given the event B, the number of outcomes of event C would be $\frac{5!}{2}$. Therefore, $p(B|C) = \frac{|B \cap C|}{|C|} = \frac{\frac{5!}{2}}{5!} = \frac{5!}{2 \cdot 5!} = \frac{1}{2}$.

(d) What is p(A|B)?

There are 3 cases of outcomes of event B while the letter "b" is in the middle: 1) the letter "b" is immediately followed by the letter "c", 2) there is a letter between the letters "b" and "c", and 3) there are two letters between the the letters "b" and "c". Each cases are 5!, so $|A \cap B| = 3 \cdot 5!$. Given that $|B| = \frac{7!}{2}$, $p(A|B) = \frac{|A \cap B|}{|B|} = \frac{3 \cdot 5!}{\frac{7!}{2}} = \frac{2 \cdot 3 \cdot 5!}{7!} = \frac{1}{7}$.

(e) Which pairs of events among A, B, and C are independent?

$p(A) = \frac{1}{7} = p(A|B)$, so A and B are independent. $p(B) = \frac{1}{2} = p(B|C)$, so B and C are independent. However, $p(C) = \frac{1}{42} \neq \frac{1}{10}$, so A and C are not independent.

Exercise 6.3.6

$$p(H) = \tfrac{1}{3}$$
$$p(T) = \tfrac{2}{3}$$

(b) The first 5 flips come up heads. The last 5 flips come up tails.

$$\left(\tfrac{1}{3}\right)^5\left(\tfrac{2}{3}\right)^5$$

(c) The first flip comes up heads. The rest of the flips come up tails.

$$\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^9$$

Exercise 6.4.2

(a) Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

Let F be the event of choosing the fair die and $\overline{F}$ be the event of choosing the biased die.

$$p(F) = p(\overline{F}) = \tfrac{1}{2}$$

Let E be the event of getting the values of 4, 3, 6, 6, 5, 5 by rolling a die. Then,

$$p(E|F) = \left(\tfrac{1}{6}\right)^6$$
$$p(E|\overline{F}) = (0.15)^4(0.25)^2.$$

In order to test the probability of choosing a fair die given the event E takes place, $p(F|E)$ needs to be calculated:

$$p(F|E) = \frac{p(E|F)\cdot p(F)}{p(E|F)\cdot p(F) + p\left(E|\overline{F}\cdot p(\overline{F})\right)} = \frac{\left(\frac{1}{6}\right)^6\cdot\left(\frac{1}{2}\right)}{\left(\frac{1}{6}\right)^6\cdot\left(\frac{1}{2}\right) + (0.15)^4\cdot(0.25)^2\cdot\left(\frac{1}{2}\right)} = 0.404$$

Question 9

Exercise 6.5.2

A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

(a) What is the range of A?

$$\{0,1,2,3,4\}$$

(b) Give the distribution over the random variable A.

$$\left\{\left(0, \frac{\binom{48}{5}\binom{4}{0}}{\binom{52}{5}}\right), \left(1, \frac{\binom{48}{4}\binom{4}{1}}{\binom{52}{5}}\right), \left(2, \frac{\binom{48}{3}\binom{4}{2}}{\binom{52}{5}}\right), \left(3, \frac{\binom{48}{2}\binom{4}{3}}{\binom{52}{5}}\right), \left(4, \frac{\binom{48}{1}\binom{4}{4}}{\binom{52}{5}}\right)\right\}$$

Exercise 6.6.1

(a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?

The probability of choosing 2 girls is: $p(gg) = \frac{\binom{7}{2}\binom{3}{0}}{\binom{10}{2}}$

The probability of choosing 1 girl and 1 boy is : $p(gb) = \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}}$

The probabiity of choosing 2 boys is: $p(bb) = \frac{\binom{7}{0}\binom{3}{2}}{\binom{10}{2}}$

$E[G] = 0 \cdot p(bb) + 1 \cdot p(gb) + 2 \cdot p(gg) = 0 \cdot \frac{\binom{7}{0}\binom{3}{2}}{\binom{10}{2}} + 1 \cdot \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} + 2 \cdot \frac{\binom{7}{2}\binom{3}{0}}{\binom{10}{2}} = \frac{7}{5}$

Exercise 6.6.4

(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then X = 25. What is E[X]?

$$E[X] = \left(\tfrac{1}{6}\right)(1)^2 + \left(\tfrac{1}{6}\right)(2)^2 + \left(\tfrac{1}{6}\right)(3)^2 + \left(\tfrac{1}{6}\right)(4)^2 + \left(\tfrac{1}{6}\right)(5)^2 + \left(\tfrac{1}{6}\right)(6)^2 = \tfrac{91}{6}$$

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and Y = 4. What is E[Y]?

$$E[Y] = 0^2 \cdot \tfrac{1}{8} + 1^2 \cdot \tfrac{3}{8} + 2^2 \cdot \tfrac{3}{8} + 3^2 \cdot \tfrac{1}{8} = 3$$

Exercise 6.7.4

(a) **A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?**

Let a random variable C_j be 1 if a child gets his or her own coat and 0 otherwise. The random variable C is the sum of the C_j. Since the probability of getting his or her own coat is $\frac{1}{10}$, $p(C_1) = \frac{1}{10}$. So, $E[C_1] = 1 \cdot \frac{1}{10} = \frac{1}{10}$. Each C_j has the same distribution, so $E[C] = 10 \cdot E[C_1]$. Therefore, $E[C] = 10 \cdot E[C_1] = 10 \cdot \frac{1}{10} = 1$.

Question 10

Exercise 6.8.1

The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

(a) What is the probability that out of 100 circuit boards made exactly 2 have defects?

$$b(2; 100, 0.01) = \binom{100}{2}(0.01)^2(0.99)^{98}$$

(b) What is the probability that out of 100 circuit boards made at least 2 have defects?

$$\begin{aligned}1 - [b(1; 100, 0.01) + b(0; 100, 0.01)] &= 1 - \left[\binom{100}{1}(0.01)(0.99)^{99} + \binom{100}{0}(0.99)^{100}\right] \\ &= 1 - \left[100(0.01)(0.99)^{99} + (0.99)^{100}\right] = 1 - \left[(0.99)^{99} + (0.99)^{100}\right] \\ &= 1 - (0.99)^{99}(1 + 0.99) \\ &= 1 - 1.99(0.99)^{99}\end{aligned}$$

(c) What is the expected number of circuit boards with defects out of the 100 made?

Let X be the random variable representing the number of defect out of 100 circuit boards. Then,

$$E[X] = 100[(0.01 \cdot 1) + (0.99 \cdot 0)] = 1$$

(d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?

Let D be the random variable representing having a defect batch out of 50 batches. A batch can either have a defect in each circuit board or none at all, if there are 2 defect circuit boards, one batch is defected. Therefore the probability of having at least 2 defected circuit boards, or one defected batch, is

$$p(D) = 1 - b(0; 50, 0.01) = 1 - \binom{50}{0}(0.01)^0(0.99)^{50} = 1 - (0.99)^{50}$$

The expected number of circuit boards with defects out of the 100 made is:

$$E[D] = (50)(2)[0.01 \cdot 1 + 0.99 \cdot 0] = 1$$

When compared to the previous answers, the probabilities are different but the expected numbers are same. This is because the counting each circuit board to test whether it has a defect is not mutually independent.

Exercise 6.8.3

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

(b) What is the probability that you reach an incorrect conclusion if the coin is biased?

Given that the coin is biased, the incorrect conclusion is that the number of heads is more than or equal to 4. To calculate the probability of this conclusion, the probability of getting at most 3 heads is subtracted from 1:

$$1 - [b(0; 10, 0.3) + b(1; 10, 0.3) + b(2; 10, 0.3) + b(3; 10, 0.3)]$$

$$= 1 - \left[\binom{10}{0}(0.3)^0(0.7)^{10} + \binom{10}{1}(0.3)^1(0.7)^9 + \binom{10}{2}(0.3)^2(0.7)^8 + \binom{10}{3}(0.3)^3(0.7)^7\right]$$