HW 5

Q3 - Q5

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Question 3

Exercise 4.1.3

(b) $\boldsymbol{f(x) = \frac{1}{x^2-4}}$

$f(x)$ is not well-defined for $x = \pm 2$.

(c) $\boldsymbol{f(x) = \sqrt{x^2}}$

$f(x)$ is a well-defined function.
For every x in the domain, there is exactly one y of target for which $(x, y) \in f$.
The rangle of the function is $\mathbb{R}^+ \cup \{0\}$.

Exercise 4.1.5

(b) Let $\boldsymbol{A = \{2, 3, 4, 5\}}$ $\boldsymbol{f : A \to Z}$ such that $\boldsymbol{f(x) = x^2}$.

Range $= \{4, 9, 16, 25\}$

(d) $\boldsymbol{f : \{0, 1\}^5 \to Z}$. For $\boldsymbol{x \in \{0, 1\}^5}$, $\boldsymbol{f(x)}$ is the number of $\boldsymbol{1'}$s that occur in x.

The targets contain 5 digits of $0'$s and $1'$s. A target can have no $1'$s such as 00000 or have five $1'$s such as 11111. Therefore, Range $= \{0, 1, 2, 3, 4, 5\}$.

(h) Let $\boldsymbol{A = \{1, 2, 3\}}$. $\boldsymbol{f : A \times A \to \mathbb{Z} \times \mathbb{Z}}$, where $\boldsymbol{f(x, y) = (y, x)}$.

$A \times A = (x, y) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
Range $= (y, x) = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (3, 1), (3, 2), (3, 3)\}$

(i) Let $\boldsymbol{A = \{1, 2, 3\}}$. $\boldsymbol{f : A \times A \to \mathbb{Z} \times \mathbb{Z}}$, where $\boldsymbol{f(x, y) = (x, y + 1)}$.

$A \times A = (x, y) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
Range $= (x, y + 1) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

(l) Let $\boldsymbol{A = \{1, 2, 3\}}$. $\boldsymbol{f : P(A) \to P(A)}$. For $\boldsymbol{X \subseteq A, f(X) = X - \{1\}}$.

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$
Range $= \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$

Question 4

Exercise 4.2.2

(c) $\boldsymbol{h : \mathbb{Z} \to \mathbb{Z}. h(x) = x^3}$

The function is one-to-one but not onto. There is no integer x such that $h(x) = 2$.

(g) $\boldsymbol{f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 1, 2y)}$

The function is one-to-one but not onto. There is no integer y such that $f(x, y) = (x, 1)$ where x is an integer.

(k) $\boldsymbol{f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+, f(x, y) = 2^x + y}$**.**

The function is not one-to-one as $f(1, 7) = f(2, 5) = 9$. The function is not onto either as there is no integer pair of (x, y) such that $f(x, y) = 1$

Exercise 4.2.4

(b) $\boldsymbol{f : \{0, 1\}^3 \to \{0, 1\}^3}$**. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1.**

The function is not one-to-one as $f(001) = f(101) = 101$.The function is not onto either as the range would not have any 3-digit numbers starting with 0.

(c) $\boldsymbol{f : \{0, 1\}^3 \to \{0, 1\}^3}$**. The output of f is obtained by taking the input string and reversing the bits.**

The function is both one-to-one and onto.

(d) $\boldsymbol{f : \{0, 1\}^3 \to \{0, 1\}^4}$**. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string.**

The function is one-to-one but not onto. There is no $s \in \{0.1\}^3$ such that $f(s) = 1000$.

(g) Let A $\boldsymbol{= \{1, 2, 3, 4, 5, 6, 7, 8\}}$ **and** $\boldsymbol{B = \{1\}. f : P(A) \to P(A)}$**. For** $\boldsymbol{X \subseteq A}$**,** $\boldsymbol{f(X) = X - B}$**.**

The function is not one-to-one as $f(\{\emptyset\}) = f(\{1\}) = \{\emptyset\}$. The function is not onto as there is no $x \in P(A)$ such that $f(x) = \{1\}$.

Give an example of a function from the set of integers to the set of positive integers that is:

(a) one-to-one, but not onto.

$$f : \mathbb{Z} \to \mathbb{Z}^+ \ f(x) = 2^x$$

(b) onto, but not one-to-one.

$$f : \mathbb{Z} \to \mathbb{Z}^+ \ f(x) = |x| + 1$$

(c) one-to-one and onto.

$$f(x) = \begin{cases} 2|x| \text{ if } x < 0 \\ x + 1 \text{ if } x \geq 0 \end{cases}$$

(d) neither one-to-one nor onto.

$$f : \mathbb{Z} \to \mathbb{Z}^+ \; f(x) = \lceil x^2 \rceil$$

Question 5

Exercise 4.3.2

(c) $\boldsymbol{f : \mathbb{R} \to \mathbb{R}\ f(x) = 2x + 3}$

The function is bijective. The inverse of the function is well-defined.

$$f^{-1}(x) = \frac{x-3}{2}$$

(d) Let be defined to be the set $\boldsymbol{\{1, 2, 3, 4, 5, 6, 7, 8\}}$
$\boldsymbol{f : P(A) \to \{1, 2, 3, 4, 5, 6, 7, 8\}}$
For $\boldsymbol{x \subseteq A,\ f(x) = |x|}$.

The function is not one-to-one. The inverse of the function is not well-defined.

(g) $f : \{0,1\}^3 \to \{0,1\}^3$**. The output of f is obtained by taking the input string and reversing the bits.**

The function is bijective. The inverse of the function is well-defined.

$$f^{-1} = \{0,1\}^3$$

(i) $\boldsymbol{f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}\ f(x, y) = (x + 5, y - 2)}$

The function is bijective. The inverse of the funcion is well-defined.

$$f^{-1}(x, y) = (x - 5, y + 2)$$

Exercise 4.4.8

$$f(x) = 2x + 3$$
$$g(x) = 5x + 7$$
$$h(x) = x^2 + 1$$

(c) $\boldsymbol{f \circ h}$

$$f(h(x)) = f(x^2 + 1) = 2(x^2 + 1) + 3 = 2x^2 + 5$$

(d) $\boldsymbol{h \circ f}$

$$h(f(x)) = h(2x + 3) = (2x + 3)^2 + 1 = 4x^2 + 12x + 10$$

Exercise 4.4.2

$$f(x) = x^2$$
$$g(x) = 2^x$$
$$h(x) = \left\lceil \frac{x}{5} \right\rceil$$

(b) Evaluate $\boldsymbol{(f \circ h)(52)}$

$$f(h(52)) = f(11) = 11^2 = 144$$

(c) Evaluate $(\boldsymbol{g}\circ\boldsymbol{h}\circ\boldsymbol{f})(\mathbf{4})$

$$g(h(f(4))) = g(h(16)) = g(4) = 2^4 = 16$$

(d) Give a mathematical expression for $\boldsymbol{h}\circ\boldsymbol{f}$.

$$h(f(x)) = \left\lceil \frac{x^2}{5} \right\rceil$$

Exercise 4.4.6

$f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
$g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
$h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, h(011) = 010.

(c) What is $(\boldsymbol{h}\circ\boldsymbol{f})(\mathbf{010})$?

$$h(f(010)) = h(110) = 110$$

(d) What is the range of $\boldsymbol{h}\circ\boldsymbol{f}$?

$$\text{Range of } f = \{100,\ 101,\ 110,\ 111\}$$
$$\text{Range of } h\circ f = \{100,\ 110\}$$

(e)What is the range of $\boldsymbol{g}\circ\boldsymbol{f}$?

$$\text{Range of } f = \{100,\ 101,\ 110,\ 111\}$$
$$\text{Range of } g\circ f = \{001,\ 101,\ 011,\ 111\}$$

(Extra Credit) Exercise 4.4.4

(c) Is it possible that $\boldsymbol{f}$ is not one-to-one and $\boldsymbol{g}\circ\boldsymbol{f}$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

No. If f is not one-to-one, then $g\circ f$ is not one-to-one. Let $x_1 \in X$ and $x_2 \in X$ such that $x_1 \neq x_2$. if $g\circ f$ is one-to-one, $g(f(x_1)) \neq g(f(x_2))$, hence, $f(x_1) \neq f(x_2)$. Therefore, if $g\circ f$ is one-to-one, f must be one-to-one.

(d) Is it possible that g is not one-to-one and $g\circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Yes. $g\circ f$ can be one-to-one when g is not one-to-one. For example, Assume $f = \{A \to B \mid a, b, c\}$ and $g = \{B \to C \mid 1, 2, 3, 4\}$. The function f is defined as $f(a) = 1, f(b) = 2, f(c) = 4$ and the function g is defined as

$g(1) = x, g(2) = y, g(3) = w, g(4) = x$. So, the function g is not one-to-one whereas the compound function $g \circ f(x)$ is one-to-one.