HW 7

Q3 - Q7

Joo Hyun Lee
jhl504@nyu.edu
NYUCSBR Summer 24-week

Question 3

Exercise 8.2.2

(b) $f(n) = n^3 + 3n^2 + 4$**. Prove that** $f = \Theta(n^3)$**.**

$f(n) = \Theta(n^3)$ if and only if $f(n) = \mathrm{O}(n^3)$ and $f(n) = \Omega(n^3)$. Assume $g(n) = n^3$. We will prove that $f(n) = \Theta(g(n))$.

For $c = 1$ and $n_0 = 1$, $n \geq 1$ and $n^3 \leq n^3 + 3n^2 + 4$. So, $1 \cdot g(n) \leq f(n)$. Therefore, $f(n) = \Omega(g(n)) = \Omega(n^3)$ for all $n \geq n_0$.

For $c = 8$ and $n_0 = 1$, $n \geq 1$ and $n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3 = 8n^3$. So, $f(n) \leq 8 \cdot g(n)$. Therefore, $f(n) = \mathrm{O}(g(n)) = \mathrm{O}(n^3)$.

As $\Omega(n^3) \leq f(n) \leq \mathrm{O}(n^3)$, $f = \Theta(n^3)$.

Exercise 8.3.5

(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with p = 0)

The code has 2 inner while loops nested in a outer while loop. The inner loops are counted from both ends: the variable 'i' starts from the beginning of the sequence and gets incremented while the variable starts from end of the sequence and gets decremented. While the variables being incremented and/or decremeneted, if $a_i \geq p$ or $a_j \leq p$, the inner while loops end and the two numbers are swapped. The outerloop repeats until $i \geq j$.

(b) What is the total number of times that the lines "i := i + 1" or "j := j - 1" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

The number of times that the lines executed is $\frac{n}{2}$ for a sequence of length n. The number depends on the values of the numbers in the sequence. To maximize the "i := i + 1" line, all the numbers in the input must be negative, and for "j := j - 1", all the number in the input must be positive.

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

The total number of times of the swap operation being executed depends on the actual values of the numbers in the sequence. The maximum number of times of the swap operation is $\frac{n}{2}$ and the minimum number of times is 0.

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

The lower bound for the time complexity of the algorithm is $\Omega(n)$. As the maximum number of execution is $\frac{n}{2}$, it is not important to consider the worst-case input.

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

The matching upper bound would be $O(n)$.

Question 4

Exercise 5.1.2

$$\text{Digits} = \{\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\ \}$$
$$\text{Letters} = \{\ a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\ \}$$
$$\text{Special characters} = \{*, \&, \$, \#\}$$

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

Let A = {strings of length 7}, B = {strings of length 8}, and C = {strings of length 9}. The cardinality of the sets are:
$|A| = 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 = 40^7$,
$|B| = 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 = 40^8$, and
$|C| = 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 = 40^9$.
Because the sets are disjoint, $A \cup B \cup C = |A| + |B| + |C| = 40^7 + 40^8 + 40^9$.

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

Let A = {strings of length 7}, B = {strings of length 8}, and C = {strings of length 9}. The cardinality of the sests are:
$|A| = 14 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 = 14 \times 40^6$,
$|B| = 14 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 = 14 \times 40^7$, and
$|C| = 14 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 \times 40 = 14 \times 40^8$. Because the set are disjoint, $A \cup B \cup C = |A| + |B| + |C| = 14\big(40^6 + 40^7 + 40^8\big)$.

Exercise 5.3.2

(a) How many strings are there over the set {a, b, c} that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

$$3 \times 2^9$$

Exercise 5.3.3

Digit-Letter-Letter-Letter-Letter-Digit-Digit

(b) How many license plate numbers are possible if no digit appears more than once?

$$10 \times 26 \times 26 \times 26 \times 26 \times 9 \times 8 = 329022720$$

(c) How many license plate numbers are possible if no digit or letter appears more than once?

$$10 \times 26 \times 25 \times 24 \times 23 \times 9 \times 8 = 258336000$$

Exercise 5.2.3

Let B = $\{0, 1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of $1's$. Note that zero is an even number, so a string with zero $1's$ (i.e., a string that is all $0's$) has an even number of $1's$.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

Given that E_{10} is th set of 10-bit binary strings that have an even number of $1's$, half of the strings woud have odd number of $1's$ and the other half would have even number of $1's$. So the cardinality of E_{10} is:

$$|E_{10}| = \frac{2^{10}}{2} = 2^9.$$

The cardinality of B^9 is:

$$\left|B^9\right| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$$

Since, $|E_{10}| = \left|B^9\right|$, they are bijection to each other.

(b) What is $|E_{10}|$?

$|E_{10}| = \frac{2^{10}}{2} = 2^9$.

Question 5

Exercise 5.4.2

(a) How many different phone numbers are possible?

$$10^4 \times 2 = 20000$$

(b) How many different phone numbers are there in which the last four digits are all different?

$$10 \times 9 \times 8 \times 7 \times 2 = 10080$$

Exercise 5.5.3

(a) No restrictions.

Let A be the set of 10-bit strings. There are 2 choices (0 or 1) for every digit. So $|A| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10}$.

(b) The string starts with 001.

Let A be the set of 10-bit strings starting with 001. There are 2 choices (0 or 1) for the last 7 digits. So, $|A| = 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$.

(c) The string starts with 001 or 10.

Let A be the set of 10-bit strings starting with 001 and B be the set of 10-bit strings starting with 10. So, $|A| = 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$ and $|B| = 1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$. Since the sets are disjoint, $|A \cup B| = |A| + |B| = 2^7 + 2^8 = 2^7(2+1) = 3 \times 2^7$.

(d) The first two bits are the same as the last two bits.

Let A be the set of 10-bit strings starting and ending with 00, B be the set of 10-bit strings starting and ending with 01, C be the set of 10-bit strings starting with 10, and D be the 10-bit strings starting and ending with 11. So, $|A| = |B| = |C| = |D| = 1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 \times 1 = 2^6$. Since the sets are disjoin from each other, $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| = 4 \times 2^6 = 2^8$.

(e) The string has exactly six 0′s.

Let A be the set of 10-bit strings having exactly six 0′s. $|A| = C(10, 6) = \frac{10!}{6!4!} = 210$.

(f) The string has exactly six 0′s and the first bit is 1.

Let A be the set of 10-bit strings having exactly six 0′s and starting with 1. So, $|A| = 1 \times C(9, 6) = 84$.

(g) There is exactly one 1 in the first half and exactly three 1′s in the second half.

Let A be the 5-bit strings having exactly one 1 and B be the 5-bit strings having three 1′s. $|A| = C(5, 1)$ and $|B| = C(5, 3)$. Therefore, $|A| \times |B| = 5 \times 10 = 50$.

Exercise 5.5.5

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

$C(30, 10) \times C(35, 10) = \binom{30}{10}\binom{35}{10}$.

Exericse 5.5.8

(c) How many five-card hands are made entirely of hearts and diamonds?

There are 13 hearts and 13 diamonds in a deck of 52 cards. 5 cards must be chosen from these 26 cards. Therefore, it is $C(26, 5) = 65780$.

(d) How many five-card hands have four cards of the same rank?

There are 13 ranks in a deck of 52 cards, and 1 card is chosen out of the 13 ranks, which gives $C(13, 1)$. The fifth card is chosen from the rest 48 cards, it gives $C(48, 1)$. Therefore, it is $C(13, 1) \times C(48, 1) = 13 \times 48 = 624$.

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?

First, a rank is chosen, which is represented as $C(13, 1)$. Then, 3 cards are chosen in that rank, which is represented as $C(4, 3)$. Then another rank is chosen, which is represented as $C(12, 1)$, and within that rank, 2 cards are chosen, which is represented as $C(4, 2)$. Therefore, it gives $C(13, 1) \times C(4, 3) \times C(12, 1) \times C(4, 2) = 13 \times 4 \times 12 \times 6 = 3744$.

(f) How many five-card hands do not have any two cards of the same rank?

There are $C(13, 5)$ ways to choose 5 different ranks of cards. There are 4 cards in each rank. So, the total number of five cards that don't have any two cards of teh same rank is: $C(13, 5) \times 4^5 = 1287 \times 1024 = 1,317,888$.

Exercise 5.6.6

44 are Demonstrators and 56 are Repudiators

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

5 members are chosen among 44 Demonstrators and 5 members are chosen among 56 Repudiators. This gives $C(44, 5) \times C(56, 5) = 414,834,421,848$.

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

Choosing a speaker from Demonstrators is $C(44,1)$ and a vice speaker from the same party is $C(43,1)$. Choosing a speaker from Repudiators is $C(56,1)$ and a vice speaker from the same party is $C(55,1)$. Therefore, the number of ways to select two speakers and two vice speakers is:
$C(44,1) \times C(43,1) \times C(56,1) \times C(55,1) = 44 \times 43 \times 56 \times 55 = 5,827,360.$

Question 6

Exercise 5.7.2

A 5-card hand is drawn from a deck of standard playing cards.

(a) How many 5-card hands have at least one club?

Let A be the set of outcomes of choosing 5 cards that have at least one club.
$|U| = C(52, 5)$
$\left|\overline{A}\right| = C(39, 5)$
$|A| = |U| - \left|\overline{A}\right| = C(52, 5) - C(39, 5) = 2,598,960 - 575,757 = 2,023,203.$

(b) How many 5-card hands have at least two cards with the same rank?

Let A be the set of outcome of choosing 5 cards that have at least two cards with the same rank.
$|U| = C(52, 5)$
$\left|\overline{A}\right| = C(13, 5) \times 4^5$
$|A| = |U| - \left|\overline{A}\right| = C(52, 5) - C(13, 5) * 4^5 = 2,598,960 - 1,287 \times 1,024 = 1,281,072.$

Exercise 5.8.4

20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

since there are 20 books, each of which can take on values between 1 and 5 books, there must be 5^{20} possible ways. $5^{20} = 95,367,431,640,625$

(b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

$\frac{20!}{4!4!4!4!4!} = \frac{20!}{4!^5} = 305,540,235,000$

Question 7

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

(a) 4

Zero. There is no on-to-one function since the cardinality of the domain is greater than that of target.

(b) 5

$P(5, 5) = 5! = 120$

(c) 6

$P(6, 5) = \frac{6!}{1!} = 6! = 720$

(d) 7

$P(7, 5) = \frac{7!}{2!} = 2520$