# MT105A Study Guide

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## What will be on the exam?

The exam will be split into two parts, Section A and Section B, for 60 and 20 marks, respectively. The exam is always eight questions long - six questions in the first part and two in the second. ~~I would suggest doing Section B first.~~ In hindsight, do Section A before Section B, because the second section will definitely have a question designed to stump most students, and time is so limited that there isn't a moment to ruminate on it. Leave a question like that until the end.

Section B will present two 'wordy', multi-part questions, that will definitely cover the topics of optimization and constrained optimization, and may possibly cover compound interest or something else.

The optimization problems in Section B will almost definitely be economics problems related to cost, utility and/or profit. The questions may be related to an individual or a firm and the firm may be producing goods as a monopoly or not. Every exam I have looked at has a constrained optimization problem in which the student has to use the method of Lagrange multipliers. Many of the problems will ask the student to solve a general equation of optimization, with variables in place of real numbers, and often the final solution is provided and the instructions are to demonstrate the steps to arrive at that solution.

The key to the exam is to know all of the variations of the optimization problems and to ace this part of the test.

Section A will have six questions that will be much more brief, and cover topics such as:

- integration
- solving variables using a matrix method
- compound interest
- the Hessian test for critical points
- curve sketching
- anything else covered in the course material

If you can successfuly answer questions relating to the first five topics then you will have no problem with Section A.

## Standard derivatives to memorize:

| $f(x)$ | $f'(x)$ |
|---|---|
| $x^k$ | $kx^{k-1}$ |
| $e^x$ | $e^x$ |
| $\ln(x)$ | $\frac{1}{x}$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |

### 1. The sum rule:

If $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$

### 2. The product rule:

If $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$

### 3. The quotient rule:

If $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

### 4. The chain rule:

If $h(x) = s(r(x))$ then $h'(x) = s'(r(x))r'(x)$

## Standard integrals to memorize:

| $f(x)$ | $\int f(x)dx$ |
|---|---|
| $x^n$, $n \neq -1$ | $\frac{x^{n+1}}{n+1} + c$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + c$ |
| $\ln\lvert x\rvert$ | $x\ln(x) - x + c$ |
| $e^x$ | $e^x + c$ |
| $e^{\alpha x}$ | $\frac{1}{\alpha}e^{\alpha x} + c$ |
| $\sin(x)$ | $-\cos(x) + c$ |
| $\cos(x)$ | $\sin(x) + c$ |

**1. The addition rule:**

$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$

**2. The constant rule:**

$\int k \times f(x)dx = k \times \int f(x)dx$

**3. Integration by parts:**

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

The formula replaces one integral (that on the left) with another (that on the right); the intention is that the one on the right is a simpler integral to evaluate.

**4. Integration by substituion:**

When we change the variable by putting $x = x(u)$ in the integral of $f(x)$ then we must replace $dx$ by $(\frac{dx}{du})du$

To determine $\int f(x)dx$ we can solve $\int f(x(u))x'(u)du$ and then substitute $x$ back for $x(u)$

For definite integrals, $\int_{x=a}^{x=b} f(x)dx = \int_{u=\alpha}^{u=\beta} f(x(u))x'(u)du$ where $\alpha$ and $\beta$ are the values of $u$ that correspond to $x = a$ and $x = b$

## Partial fractions:

An example:

$\frac{x}{x^2+5x+4} = \frac{x}{(x+1)(x+4)} = \frac{A_1}{(x+1)} + \frac{A_2}{(x+4)}$

Cross multiply to obtain:

$A_1(x+4) + A_2(x+1) = x$

let $x = -4 \implies A_2(-4+1) = -4 \implies A_2 = \frac{4}{3}$

let $x = -1 \implies A_1(-1+4) = -1 \implies A_1 = -\frac{1}{3}$

Therefore, the partial fracion is:

$\frac{4}{3(x+4)} - \frac{1}{3(x+1)}$

## The second partial derivative test:

The second partial derivative test is a method used in multivariable calculus to determine if a critical point of a function is a local minimum, maxium or saddle point.

First find the critical points, where the partial derivatives for each variable is zero.

The Hessian function is: $H(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

If, at a critical point, we have:

$H < 0$, then that point is a *saddle point.*

$H > 0$ and $f_{xx} > 0$ then that point is a *local minimum.*

$H > 0$ and $f_{xx} < 0$ then that point is a *local maximum.*

If $H = 0$ then the test fails!

Note that for single variable functions, the second part of the Hessian test defines the nature of the critical points, (ie. $f_{xx} > 0$ is the minimum, $f_{xx} < 0$ is the maximum).

## Economic questions:

Total profit:

$\pi(q) = TR(q) - TC(q)$

Revenue is always quantity $\times$ price. This may be in the form of a revenue equation, expressed in quantity or price (usually price), which must be multiplied by the second term.

For example, for two products, $X$ and $Y$ in quantities $x$ and $y$, the total revenue will be:

$TR(x, y) = xp_x + yp_y$

Total cost:

$TC(q) = VC(q) + FC(q)$

Total cost equals variable cost plus fixed cost. If you solve total cost through integration, then the constant will be the fixed cost.

Average cost:

$AC(q) = \frac{TC(q)}{q}$

Average variable cost:

$AVC(q) = \frac{VC(q)}{q}$

Marginal revenue:

The revenue derived from the production of one more unit:

$MR(q) = TR'(q)$

Marginal cost:

The cost incurred from the production of one more unit:

$MC(q) = TC'(q)$

## Optimization of profit or utility:

Optimizing a firm's profit or an individual's utility is almost guaranteed to be a question on the exam.

A common question is to be given a demand function in terms of $q$ (quantity) and $p$ (price) and asked to maximize profit. Many questions involve a firm with a monopoly on one or two goods, and in this case the firm does not price discriminate (ie. they are the price maker).

For example, if a firm has a marginal cost function of $MC(q) = 5q^4 - 2q - 15q^2$, a fixed cost of $FC = 10$, and a demand function of $p + q = 20$, then to find the production level, q, that maximizes profit:

Integrate the marginal cost function to determine the total cost function:

$TC(q) = q^5 - q^2 - 5q^3 + c$

And recall that $c$ is the fixed cost, 10.

Rewrite the demand functon in terms of $q$, so $p = 20 - q$, and remember that revenue is quantity $\times$ price.

Therefore:

$\pi(q) = TR(q) - TC(q)$

$\pi(q) = 20q - q^2 - q^5 + q^2 + 5q^3 - 10$

$\pi(q) = 20q - q^5 + 5q^3 - 10$

Solve the first derivative of this equation to find the critical point:

$\pi'(q) = q^2(5q^2 - 15) - 20$

And we determine that $q = 2$, because we cannot produce a negative quantity of goods ($q = -2$).

For the exam you can assume that this is the local maximum, but to check, solve for the second derivative and perform the derivative test as described above.

**Constrained optimization using the Lagrange multiplier method:**

**Compound interest:**

**The Gaussian Elimination method for solving linear systems:**