Periodic Modeling

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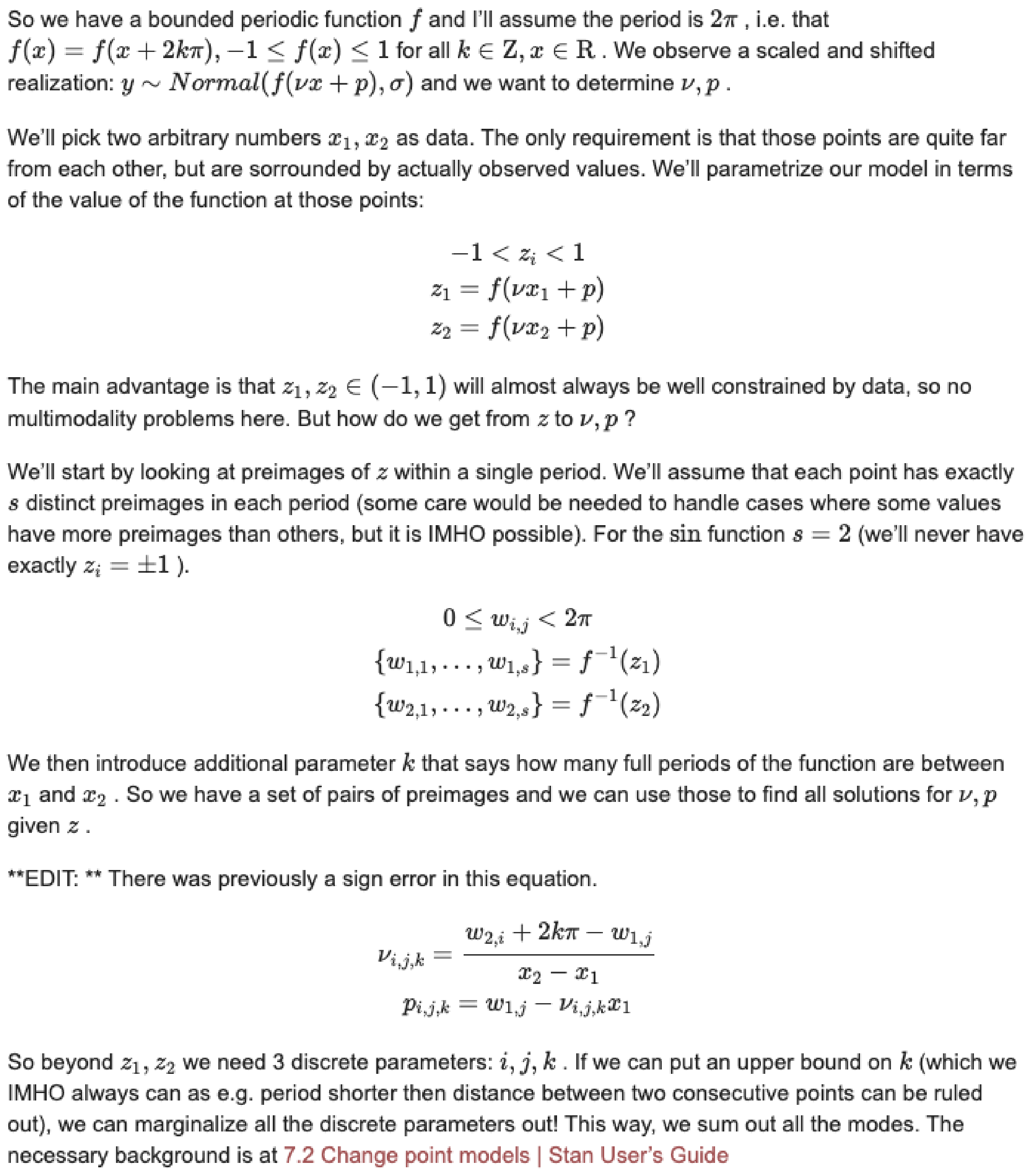
https://discourse.mc-stan.org/t/mathy-folks-thoughts-on-the-asymptotic-representation-of-this-k-hmm-model-for-periodic-signals/22038/24

25 April 2021

Clear Out Console Script

```
cat("\014")
```



So we have a bounded periodic function f and I'll assume the period is 2π , i.e. that $f(x) = f(x + 2k\pi), -1 \leq f(x) \leq 1$ for all $k \in \mathrm{Z}, x \in \mathrm{R}$. We observe a scaled and shifted realization: $y \sim Normal(f(\nu x + p), \sigma)$ and we want to determine ν, p .

We'll pick two arbitrary numbers x_1, x_2 as data. The only requirement is that those points are quite far from each other, but are sorrounded by actually observed values. We'll parametrize our model in terms of the value of the function at those points:

$$\begin{aligned} -1 < z_i < 1 \\ z_1 = f(\nu x_1 + p) \\ z_2 = f(\nu x_2 + p) \end{aligned}$$

The main advantage is that $z_1, z_2 \in (-1, 1)$ will almost always be well constrained by data, so no multimodality problems here. But how do we get from z to ν, p ?

We'll start by looking at preimages of z within a single period. We'll assume that each point has exactly s distinct preimages in each period (some care would be needed to handle cases where some values have more preimages than others, but it is IMHO possible). For the $\sin$ function $s = 2$ (we'll never have exactly $z_i = \pm 1$).

$$\begin{aligned} 0 \leq w_{i,j} < 2\pi \\ \{w_{1,1}, \dots, w_{1,s}\} = f^{-1}(z_1) \\ \{w_{2,1}, \dots, w_{2,s}\} = f^{-1}(z_2) \end{aligned}$$

We then introduce additional parameter k that says how many full periods of the function are between x_1 and x_2 . So we have a set of pairs of preimages and we can use those to find all solutions for ν, p given z .

**EDIT: ** There was previously a sign error in this equation.

$$\begin{aligned} \nu_{i,j,k} = \frac{w_{2,i} + 2k\pi - w_{1,j}}{x_2 - x_1} \\ p_{i,j,k} = w_{1,j} - \nu_{i,j,k} x_1 \end{aligned}$$

So beyond z_1, z_2 we need 3 discrete parameters: i, j, k . If we can put an upper bound on k (which we IMHO always can as e.g. period shorter then distance between two consecutive points can be ruled out), we can marginalize all the discrete parameters out! This way, we sum out all the modes. The necessary background is at 7.2 Change point models | Stan User's Guide

Figure 1: Description.

Setting Work Directory

```
setwd("~/Dropbox/My Mac (Jonathan's MacBook Pro)/Desktop")
getwd()
```

```
## [1] "/Users/jonathan.h.morgan/Dropbox/My Mac (Jonathan's MacBook Pro)/Desktop"
```

Options

```
options(stringsAsFactors = FALSE)
options(mc.cores = parallel::detectCores())
options(mc.cores = 4)
```

PACKAGES

```
library(cmdstanr)     #Used to run cmdstan from R
```

```
## This is cmdstanr version 0.3.0

## - Online documentation and vignettes at mc-stan.org/cmdstanr

## - CmdStan path set to: /Users/jonathan.h.morgan/.cmdstanr/cmdstan-2.26.1

## - Use set_cmdstan_path() to change the path
```

```
library(tidyverse)
```

```
## -- Attaching packages --------------------------------------- tidyverse 1.3.1 --

## v ggplot2 3.3.3     v purrr   0.3.4
## v tibble  3.1.1     v dplyr   1.0.5
## v tidyr   1.1.3     v stringr 1.4.0
## v readr   1.4.0     v forcats 0.5.1

## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

FUNCTIONS

Plot Utilities

```
source("~/R Resources/R Plotting Utilities & Resources/R Plot Utilities_20ct2020.R")
```

```
## ------------------------------------------------------------------------------

## You have loaded plyr after dplyr - this is likely to cause problems.
## If you need functions from both plyr and dplyr, please load plyr first, then dplyr:
## library(plyr); library(dplyr)

## ------------------------------------------------------------------------------

##
## Attaching package: 'plyr'
## The following objects are masked from 'package:dplyr':
##
##     arrange, count, desc, failwith, id, mutate, rename, summarise,
##     summarize
```

```
## The following object is masked from 'package:purrr':
##
##     compact
```

```
## Package 'sm', version 2.2-5.6: type help(sm) for summary information
```

```
## Loading required package: foreign
```

```
## Included Plot Elements and Templates
##
## Base R Continuous Color Legend
## Base R Discrete Legend
## Base R Pie Chart
## Correlation Plots: Corrgram and Base R Correlation Plot Using the Psych Package
## Base R Scatter Plot with Thrills
## Base R Grouped Bar Chart
## Base R Stacked Bar Chart
## Base R Shaded Density Plot with Mathematical Annotations
## Base R Convex Hull Function Plot
## Base R Histogram with CDF
## Base R Kolmogorov-Smirnov Plot
## Base R CCDF Plot
## ggplot2 Violin Plot with Boxplot
## Base R Split Violin Plot
## Pirate Plot
## Base R Beeswarm Plot Overlay Plot with Dot Plot Subplot
## Base R Bubble Plot
## Stylized ggplot Panel Plot
## Base R Panel Plot
## Base R Jittered Group Scatter Plot with Extras
## Ternary Plot with Base R Overlay and Subplots
## Tab Plot for Exploring Missing Data
## Classic Base R Heat Maps
## Binned Continuous Variable Heat Map
## Base R Continuous Heat Map with Overlays
## Base R Contour Scatter Plot
## Contour Network Plot
## Base R 3D Polar Plot
## Chord Plot
## Arc Diagrams implemented using the arcdiagram and ggplot2
## Alluvial Plot
## Hägerstrand Time-Space Cube
## Regression Diagnostics: Base R and ggfortify
## PCA Diagrams and Steps
## Panel Plot Comparing Multiple Model Fits
## Stylized Fitted Line and Confidence Interval: Base R
## Stylized ggplot2 Smoothed Regression Line with Leverage Lines
## Dataplot Style 4Plot
## Dataplot Style 6Plot
## Johnson-Neyman Interval for 2-Way Interactions
## Johnson-Neyman Intervals: Testing Slope Homogeneity for Two Groups
## Model Assessment: Repeated Density Ratios
## AUROC Analysis
## Complex Model Assessment: Taylor Diagram
```

```
source("~/R Resources/R Plotting Utilities & Resources/util.R")
```

```
## The following object is masked from package:graphics:
##
##     legend
```

Checking that I am pointing to cmdstan

```
set_cmdstan_path("/Users/jonathan.h.morgan/cmdstan")
```

```
## CmdStan path set to: /Users/jonathan.h.morgan/cmdstan
```

```
cmdstan_path()
```

```
## [1] "/Users/jonathan.h.morgan/cmdstan"
```

```
cmdstan_version()
```

```
## [1] "2.26.1"
```

CREATING SYNTHETIC DATA

Parameters

```
n <- 30
frequency <- 13
phase <- 0.3
noise <- 0.5
amplitude <- 1.4
```

Constructing Data

```
x <- runif(n, max = 2 * pi)
f <- amplitude * sin(frequency * x + phase)
y <- rnorm(n, mean = f, sd = noise)
x_fixed = c(pi / 4, 7/4 * pi)
```

Creating Input List

```
data <- list(
  n = n,
  y = y,
  x = x,
  x_fixed = x_fixed,
  max_k = 10
)
```

ESTIMATING STAN MODEL & SAVING DATA OBJECTS

Setting Seed

```
set.seed(10271998)
```

Setting Temporary Directory

```
temp <- tempdir()
```

Compiling Model

```
mod <- cmdstan_model("~/Dropbox/My Mac (Jonathan's MacBook Pro)/Desktop/periodic.stan")
```

```
## Model executable is up to date!
```

Checking Model

```
mod$print()
```

```
## //  Example Periodic Function
## //  Martin Modrak: https://discourse.mc-stan.org/t/mathy-folks-thoughts-on-the-asymptotic-representa
## //  25 April 2021
##
## data{
##    int n ;
##    vector[n] y ;
##    vector[n] x ;
##    int<lower=0> max_k;
##    real x_fixed[2];
## }
##
## transformed data {
##   real log_prior[max_k + 1] = rep_array(-log(max_k + 1.0), max_k + 1); //prior prob 1/(max_k + 1)
## }
##
## parameters{
##    real<lower=0> noise ;
##    real<lower=0> amplitude;
##    real<lower=-1, upper=1> z[2];
## }
##
## transformed parameters{
##   real frequency[2, 2, max_k + 1];
##   real phase[2, 2, max_k + 1];
##   real log_lik[2, 2, max_k + 1];
##   {
##     real asinz[2] = asin(z);
##     real w[2,2];
##     w[1,]= { asinz[1], pi() - asinz[1]};
##     w[2,]= { asinz[2], pi() - asinz[2]};
##     for(i in 1:2) {
##       for(j in 1:2) {
##         for(k in 1:(max_k + 1)) {
##           frequency[i, j, k] = (w[2, i] + 2 * (k-1) * pi() - w[1, j]) / (x_fixed[2] - x_fixed[1]);
##           phase[i, j , k] = w[1,i] - frequency[i, j, k] * x_fixed[1];
##           {
##             vector[n] f = amplitude * sin(frequency[i, j, k] * x + phase[i, j, k]);
##             log_lik[i, j, k] = normal_lpdf(y | f, noise)
##               + log_prior[k]
##               -log(2.0) //Uniform prior over the 2 options
##               ;
##           }
##         }
##       }
##     }
##   }
## }
##
## model{
##    noise ~ weibull(2,1) ; //peaked at ~.8, zero-at-zero, ~2% mass >2
##    amplitude ~ weibull(2,1) ; //ditto
##    target += log_sum_exp(to_array_1d(log_lik));
## }
##
## generated quantities {
##   // Discrete sampling is inefficient, directly computing weights
##   // And taking weighted expectation values would work better
##   int index = categorical_logit_rng(to_vector(to_array_1d(log_lik)));
##   real freq_chosen = to_array_1d(frequency)[index];
##   real phase_chosen = to_array_1d(phase)[index];
##   //Force phase to lie between -pi + pi the stupid way
##   while(phase_chosen < -pi()) {
##     phase_chosen += 2 * pi();
##   }
##   while(phase_chosen > pi()) {
##     phase_chosen -= 2 * pi();
##   }
## }
```

Sampling: Non-Centered Random Effects

```r
fit <- mod$sample(
  data = data,
  output_dir = temp,
  seed = 123,
  chains = 4,
  thin = 1,
  parallel_chains = 4,
  iter_warmup = 2000,
  iter_sampling = 2000,
  adapt_delta = 0.99,
  max_treedepth = 15,
  refresh = 500
)
```

```
## Running MCMC with 4 parallel chains...
##
## Chain 1 Iteration:    1 / 4000 [  0%]  (Warmup)
## Chain 2 Iteration:    1 / 4000 [  0%]  (Warmup)

## Chain 2 Informational Message: The current Metropolis proposal is about to be rejected because of th

## Chain 2 Exception: weibull_lpdf: Random variable is inf, but must be finite! (in '/var/folders/dl/_9

## Chain 2 If this warning occurs sporadically, such as for highly constrained variable types like cova

## Chain 2 but if this warning occurs often then your model may be either severely ill-conditioned or m

## Chain 2

## Chain 3 Iteration:    1 / 4000 [  0%]  (Warmup)

## Chain 3 Informational Message: The current Metropolis proposal is about to be rejected because of th

## Chain 3 Exception: normal_lpdf: Location parameter[1] is -inf, but must be finite! (in '/var/folders

## Chain 3 If this warning occurs sporadically, such as for highly constrained variable types like cova

## Chain 3 but if this warning occurs often then your model may be either severely ill-conditioned or m

## Chain 3

## Chain 4 Iteration:    1 / 4000 [  0%]  (Warmup)

## Chain 4 Informational Message: The current Metropolis proposal is about to be rejected because of th

## Chain 4 Exception: weibull_lpdf: Random variable is inf, but must be finite! (in '/var/folders/dl/_9

## Chain 4 If this warning occurs sporadically, such as for highly constrained variable types like cova

## Chain 4 but if this warning occurs often then your model may be either severely ill-conditioned or m

## Chain 4

## Chain 1 Iteration:  500 / 4000 [ 12%]  (Warmup)
## Chain 2 Iteration:  500 / 4000 [ 12%]  (Warmup)
## Chain 4 Iteration:  500 / 4000 [ 12%]  (Warmup)
## Chain 3 Iteration:  500 / 4000 [ 12%]  (Warmup)
## Chain 1 Iteration: 1000 / 4000 [ 25%]  (Warmup)
## Chain 2 Iteration: 1000 / 4000 [ 25%]  (Warmup)
## Chain 3 Iteration: 1000 / 4000 [ 25%]  (Warmup)
## Chain 4 Iteration: 1000 / 4000 [ 25%]  (Warmup)
## Chain 1 Iteration: 1500 / 4000 [ 37%]  (Warmup)
## Chain 2 Iteration: 1500 / 4000 [ 37%]  (Warmup)
## Chain 3 Iteration: 1500 / 4000 [ 37%]  (Warmup)
## Chain 4 Iteration: 1500 / 4000 [ 37%]  (Warmup)
## Chain 1 Iteration: 2000 / 4000 [ 50%]  (Warmup)
## Chain 1 Iteration: 2001 / 4000 [ 50%]  (Sampling)
## Chain 2 Iteration: 2000 / 4000 [ 50%]  (Warmup)
## Chain 2 Iteration: 2001 / 4000 [ 50%]  (Sampling)
## Chain 3 Iteration: 2000 / 4000 [ 50%]  (Warmup)
## Chain 3 Iteration: 2001 / 4000 [ 50%]  (Sampling)
## Chain 4 Iteration: 2000 / 4000 [ 50%]  (Warmup)
## Chain 4 Iteration: 2001 / 4000 [ 50%]  (Sampling)
## Chain 1 Iteration: 2500 / 4000 [ 62%]  (Sampling)
## Chain 2 Iteration: 2500 / 4000 [ 62%]  (Sampling)
## Chain 3 Iteration: 2500 / 4000 [ 62%]  (Sampling)
## Chain 4 Iteration: 2500 / 4000 [ 62%]  (Sampling)
## Chain 2 Iteration: 3000 / 4000 [ 75%]  (Sampling)
## Chain 3 Iteration: 3000 / 4000 [ 75%]  (Sampling)
## Chain 1 Iteration: 3000 / 4000 [ 75%]  (Sampling)
## Chain 4 Iteration: 3000 / 4000 [ 75%]  (Sampling)
## Chain 2 Iteration: 3500 / 4000 [ 87%]  (Sampling)
## Chain 3 Iteration: 3500 / 4000 [ 87%]  (Sampling)
## Chain 1 Iteration: 3500 / 4000 [ 87%]  (Sampling)
## Chain 4 Iteration: 3500 / 4000 [ 87%]  (Sampling)
## Chain 3 Iteration: 4000 / 4000 [100%]  (Sampling)
## Chain 3 finished in 12.2 seconds.
## Chain 2 Iteration: 4000 / 4000 [100%]  (Sampling)
## Chain 2 finished in 12.4 seconds.
## Chain 1 Iteration: 4000 / 4000 [100%]  (Sampling)
## Chain 1 finished in 12.6 seconds.
## Chain 4 Iteration: 4000 / 4000 [100%]  (Sampling)
## Chain 4 finished in 13.2 seconds.
##
## All 4 chains finished successfully.
## Mean chain execution time: 12.6 seconds.
## Total execution time: 13.3 seconds.
```

Checking that the Model Passes: E-BFMI, Treedepth, Sample Size, & R-Hat Cursory Checks

```
fit$cmdstan_diagnose()
```

```
## Processing csv files: /var/folders/dl/_9shwqkn19s9d3lvpyfcmkf00000gn/T/Rtmp6xUnEo/periodic-20210426
##
## Checking sampler transitions treedepth.
## Treedepth satisfactory for all transitions.
##
## Checking sampler transitions for divergences.
## No divergent transitions found.
##
## Checking E-BFMI - sampler transitions HMC potential energy.
## E-BFMI satisfactory for all transitions.
##
## Effective sample size satisfactory.
##
## Split R-hat values satisfactory all parameters.
##
## Processing complete, no problems detected.
```

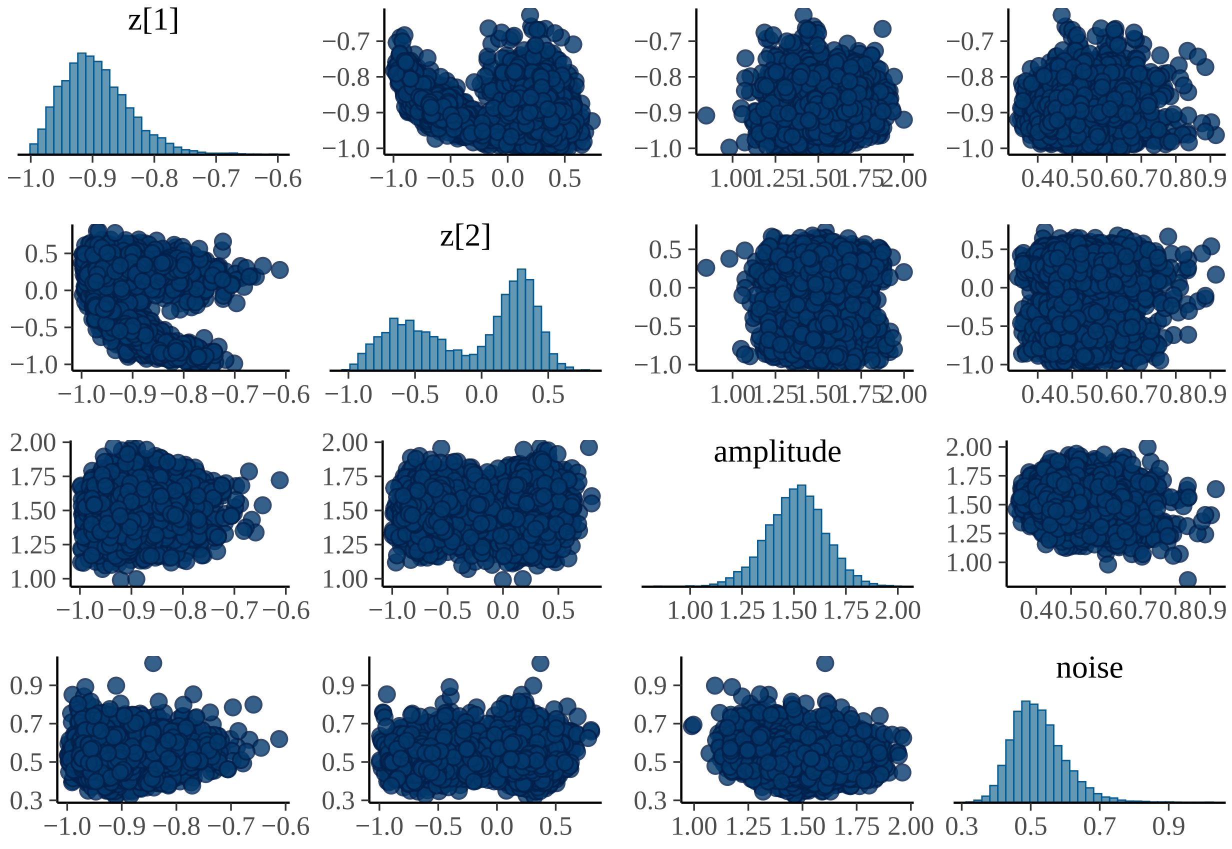
PLOTTING

Creating Data Objects

```
fit$summary(c("noise", "z", "freq_chosen", "phase_chosen", "amplitude", "index"))
```

```
## # A tibble: 7 x 10
##   variable     mean  median     sd    mad      q5    q95  rhat ess_bulk ess_tail
##   <chr>       <dbl>   <dbl>  <dbl>  <dbl>   <dbl>  <dbl> <dbl>    <dbl>    <dbl>
## 1 noise       0.522   0.514 0.0758 0.0705  0.414   0.659  1.00    4043.    4397.
## 2 z[1]       -0.894  -0.900 0.0539 0.0519 -0.971  -0.795  1.00    3311.    2290.
## 3 z[2]       -0.0789  0.0899 0.438  0.464  -0.799   0.462  1.00     614.    2719.
## 4 freq_cho~ 13.0     13.0    0.0461 0.0454 13.0     13.1    1.00    4415.    4331.
## 5 phase_ch~  0.311   0.310  0.152  0.150   0.0621  0.564  1.00    3304.    2459.
## 6 amplitude  1.51    1.51   0.138  0.135   1.28    1.74   1.00    4226.    3606.
## 7 index     38.6    44      5.97   0      32      44      1.01     398.      NA
```

```
bayesplot::mcmc_pairs(fit$draws(), pars = c("z[1]","z[2]", "amplitude", "noise"))
```

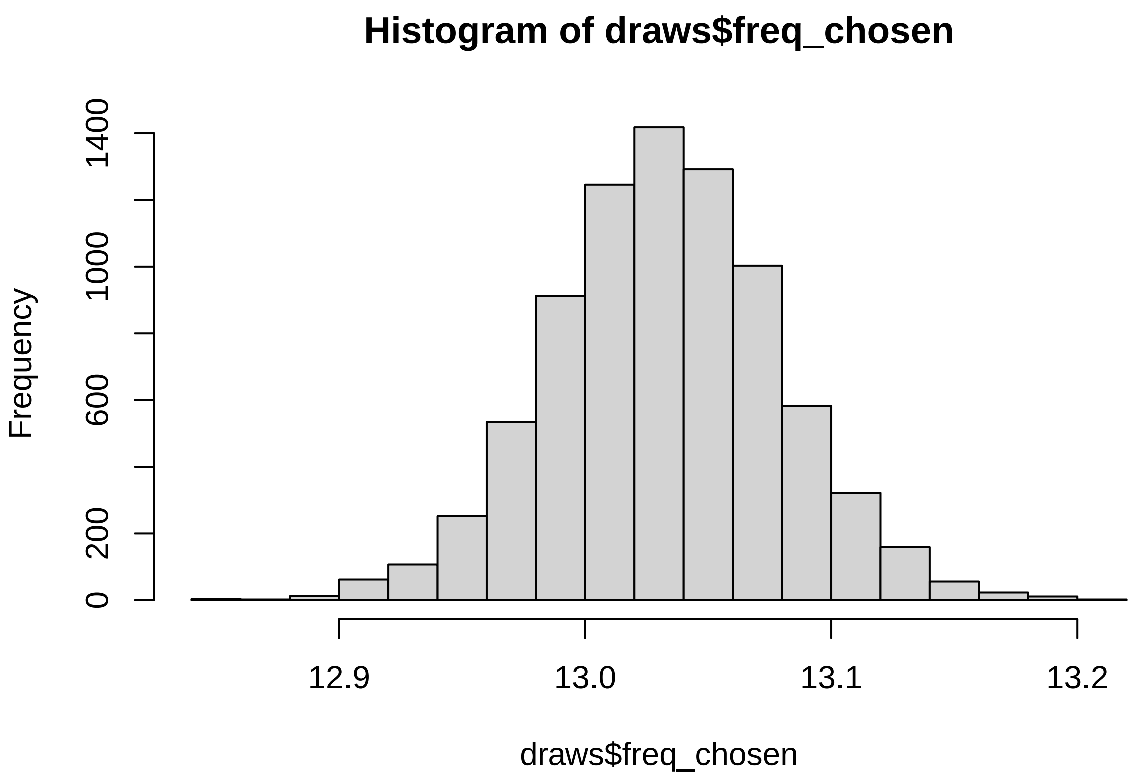


Pulling Out Draws for Frequency, Phase, and Amplitude

```
draws <- as.data.frame(posterior::as_draws_matrix(fit$draws(c("freq_chosen", "phase_chosen", "amplitude
```

Checking the frequency

```
hist(draws$freq_chosen)
```



# Constructing True Model

```
x_adj <- seq(0, 2 * pi, length.out = 101)
y_adj <- amplitude * sin(frequency * x_adj + phase)
true <- as.data.frame(cbind(x_adj,y_adj))
colnames(true) <- c('x', 'y')
rm(x_adj,y_adj)
```

Observed Data

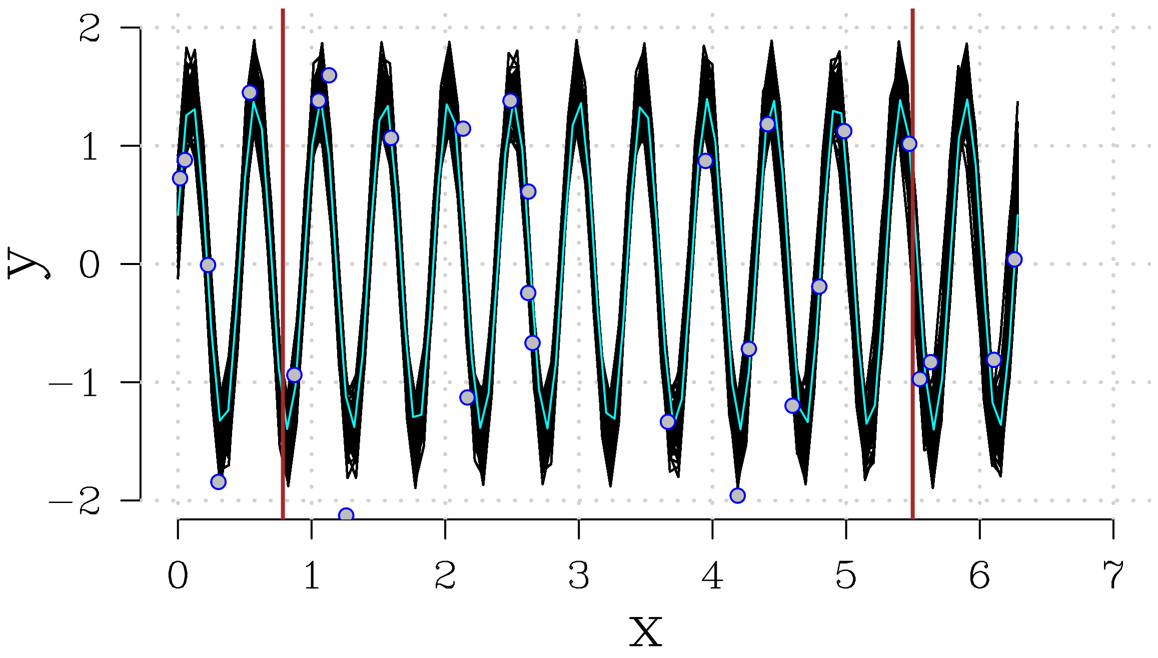
```
observed <- tibble(x = x, y = y)
```

Constructing Draws

```
z <-  tibble(x = seq(0, 2 * pi, length.out = 100)) %>% crossing(as_tibble(draws) %>% mutate(id = 1:dim(
  mutate(y = amplitude * sin(x * freq_chosen + phase_chosen)) %>%
  filter(id %in% sample(unique(id), 100))

rm(amplitude,f,frequency,n,noise, phase, temp,x,x_fixed,y)
```

Plotting

```
x11(width=10.6806, height=7.30556)
periodic_visual <- function() {
  x_values <- pretty(z$x)
  y_values <- pretty(z$y)
  x_fixed = c(pi / 4, 7/4 * pi)

  plot(0, type='n', xlab=' ', ylab=' ', xlim=c(min(x_values), max(x_values)), ylim=c(min(y_values), max
  grid(lwd = 2)

  ids <- sort(z$id)
  for(i in seq_along(z$id)){
    iteration <- z[(z$id == ids[[i]]), ]
    lines(iteration$x, iteration$y, col="black")
  }

  lines(true, col="cyan")

  abline(v=x_fixed[[1]],col='brown',lwd=2, lty=1)
  abline(v=x_fixed[[2]],col='brown',lwd=2, lty=1)

  points(observed, pch=21, bg="grey", col="blue")

  mtext(side = 1, text = 'x', col = "black", line = 2.5, cex = 2, family='HersheySerif')
  mtext(side = 2, text = 'y', col = "black", line = 2.5, cex = 2, family='HersheySerif')
}

g <- cowplot::as_grob(periodic_visual)
p_1 <- cowplot::ggdraw(g)

p_1
```



NOTES

Parameters where is not fully contrained

```r
set.seed(24855)
n <- 8
frequency <- 4
phase <- -0.5
noise <- 0.5
amplitude <- 1.4
```

Constructing Data

```r
x <- runif(n, max = 2 * pi)
f <- amplitude * sin(frequency * x + phase)
y <- rnorm(n, mean = f, sd = noise)
x_fixed = c(pi / 4, 7/4 * pi)
```

Creating Input List

```r
data <- list(
  n = n,
  y = y,
  x = x,
  x_fixed = x_fixed,
  max_k = 10
)
```