Disentangling Risk and Intertemporal Preferences with Costly Information Acquisition*

Jamie Hentall MacCuish†

Abstract

Risk aversion to many is synonymous with the curvature of utility over consumption. However, when information is costly, agents have a direct reason to dislike risk: the cost of reducing uncertainty. I investigate how costly attention impacts risk and intertemporal preferences by comparing two agents: one rationally inattentive and one standard and uninformed. Introducing rational inattention separates the elasticity of intertemporal substitution from relative risk aversion because intertemporal preferences are essentially unchanged, but risk preferences are not. I prove this for two-state two-period models. As an application, I show that this result may explain two finance puzzles.

KEYWORDS: Rational inattention, Stochastic choice, Risk aversion, Intertemporal preferences

JEL CLASSIFICATION: D81, D83, D90

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†University College London and Institute for Fiscal Studies, correspondence to jamie.maccuish.15@ucl.ac.uk

1 Introduction

Economists consider risk aversion synonymous with curvature, with respect to consumption, of utility or a risk-aggregator. Costly information acquisition creates a reason to be averse to risk even in the absence of curvature: the utility cost of reducing uncertainty. Aversion to risk unrelated to curvature raises the prospect of breaking the inverse reciprocals relation between relative risk aversion and the elasticity of intertemporal substitution imposed by standard time-additive expected utility. Other approaches to breaking this relation[1] do not consider this direct utility cost of risk but introduce parameters separately controlling within and across period curvature. For two-period two-state-of-the-world models, this paper shows that costly learning about reducible risk separates risk and intertemporal preferences. It separates them by introducing this additional motive to dislike risk while leaving intertemporal preference largely unchanged. This counterfactual inverse reciprocals relation is at the root of equity premium and the risk-free rate puzzles. In an application, I show that costly learning can solve these puzzles.

To investigate costly attention's impact on risk and intertemporal preference, I compare a rationally inattentive agent who can learn about reducible uncertainty with a standard uninformed agent who cannot. Like-for-like comparisons of their preferences are made before learning can occur. For the standard uninformed agent, the model framework is textbook expected utility. For the rationally inattentive, it is an application of Matějka and McKay (2015) for discrete choice and Jung et al. (2019) for continuous.

Rational inattention implies stochastic choice, and standard definitions of relative risk aversion and the elasticity of intertemporal substitution are ill-defined for stochastic choice as they are predicated on an agent faced with the same decision problem making the same choice. This is an interesting issue, although, for this paper, it more resembles a frustrating tangent. Section 3 considers an analyst who misattributes deviations from deterministic choice to measurement error. This generates a stochastic-choice adapted definition of risk aversion: a lottery is prefered to a sure-thing if the agent selects the lottery with a probability greater than one-half. This reflects in a stylised fashion approaches to dealing with the inference of parameters from noisy data.

Sections 4 and 5 apply the stochastic-choice adjusted definitions to the choices of the rationally inattentive and standard uninformed agents. Sections 4 compares their risk aversion by analysing their choices between binary lotteries and a sure-things when the rationally inattentive agent can learn about the outcome of the lottery by paying a utility cost. Propositions, 4 proves that the skewness of the lottery alone determines whether introducing costly attention increases or decreases risk aversion. Neither the cost of attention nor the curvature with respect to consumption affects the direction of change, but they do affect the magnitude. Section 5 compares their intertemporal substitution choices when able to self-insure against future learnable income risk using a risk-free in a two-period model. Proposition 5 proves that costly attention does not generally affect intertemporal preference.

Section 6 combines these results to show how costly information acquisition separates risk and intertemporal preferences. Previous work (Luo, 2010; Luo and Young, 2016) investigating the ability of rational inattention to explain the equity premium puzzle only considered zero skew uncertainty, so it missed rational inattention's ability to disentangle risk and time preferences. Section 6 concludes by calibrating a simple example to the US economy to demonstrate the potential of this mechanism to explain the equity premium and the risk-free rate puzzles.

[1] e.g. Epstein and Zin (1989) and Weil (1990) or Dillenberger et al. (2020)

The model explaining these puzzles assumes risk is learnable, whereas clearly, part of future stock return risk is not learnable. However, the financial literacy literature (e.g. Lusardi et al., 2015) has documented many facts about baseline probabilities of stock returns that consumers have not internalised. Hence, a component of individual uncertainty is reducible through learning. This paper investigates the implication of costly learning about reducible risk, so it abstracts away from the irreducible component.

Two related papers critique risk-aversion in stochastic choice models because the curvature of the utility function alone no longer predicts preferences over lotteries (Wilcox, 2011; Apesteguia and Ballester, 2018). Unlike them, I argue this is a desirable feature. Firstly, because it is an intuitive feature when costly information provides a separate reason to dislike risk. Secondly, because it means these models can offer independent predictions for empirically distinct objects.

Deriving results for tractable binary lotteries allows for illustrations that costly attention can separate risk and time preferences. This simplicity could raise doubts about broader validity. An online appendix, using results from Steiner et al. (2017), provides suggestive evidence for an equivalence between dynamic discrete choice rational inattention models and Hansen and Sargent (1995) risk-sensitive preferences, which Bommier et al. (2017) show are the only Krepes-Porteus recursive preference to separate risk and time preferences whilst preserving monotonicity with respect to first-order stochastic dominance. Hence, these desirable properties should extend to models of costly information acquisition, and rational inattention may provide an appealing alternative to some popular recursive preferences.

2 Decision Maker's Perspective: Model Framework

This section presents the model framework of a rationally inattentive agent who can engage in costly learning and a standard uninformed agent who cannot learn. The framework presented here nests specific models used later to analyse the two agents' risk and intertemporal preferences.

2.1 The Two Agents: Rationally Inattentive and Standard Uninformed

The standard uninformed agent controls a choice variable d subject to constraint $d \in \mathcal{C} \subset \mathbb{R}$, closed and bounded. She faces uncertainty about some outcome $z \sim Z$, with a known distribution; has utility $U(d,z)$ that is continuous with respect to the standard topology induced on $\mathcal{C}$; and chooses d to maximise expected utility. Her choices solve:

$$\max_{d \in \mathcal{C}} E[U(d,z)].$$

$U(d,z)$ may, or may not, be constructed from a discounted sum of within-period flow-utility functions, $U(d,z) = \sum_{t=1}^{T} \beta_t u(,d,z)$, but if $T > 1$ all uncertainty is resolved after the first decision is taken. This essentially static framework is complex enough to furnish examples that disentangle the risk and intertemporal preferences.

The rationally inattentive agent is identical, except she can receive a signal $x \sim X$ about the uncertain outcome Z by paying a utility cost. She chooses the distribution of the signal $f_{X|Z}(x|z)$, but her utility function is extended with an additive utility cost, higher for more informative signals. She makes her decision conditional on the draw from the noisy signal $d(x) \in \mathcal{C}$. Both agents have a correct prior.

The cost of information is assumed to be directly proportional to the mutual information between the signal X and the uncertain outcome Z: the expected reduction in uncertainty, measured by entropy, about Z from learning X. Hence, her decision problem becomes:

$$\max_{f_{X|Z}(.|.),d(x)\in\mathcal{C}} E[U(d,z)+\lambda I(X,Z)] \tag{1}$$

where,

$$I(X,Z)=H(Z)-E[H(Z|X)]$$

and H is the entropy function $H(X)=E[-\log(f_X)]$.

The choice of the signal process makes the rationally inattentive agent's actions stochastic. This paper focuses on the unconditional distribution of actions q ($d\sim q\in\Delta(\mathcal{C})$). This is because, as the standard uninformed receives no signal, only the unconditional distribution of actions q allows for like-for-like comparisons between the two agents. Additionally, after seeing the signal, the rationally inattentive agent is a standard agent with different beliefs, but unchanged preferences. So, nothing new is learned from studying her ex-post preferences.

Finding this unconditional distribution q is made possible by Matějka and McKay (2015) and Jung et al. (2019). I summarise the results from these papers used here:

Result 1. *The actions of the rationally inattentive agents that solve* (1) *have an unconditional distribution $q(d)$ that solves* (2)*.*

$$\max_{q\in\Delta(\mathcal{C})} E_Z\left[\log\left(E_q[\exp(u(d,z)/\lambda)]\right)\right] \tag{2}$$

3 Analyst's Perspective: Inferring Stochastic Choice Agents' Preferences

The textbook definition of preference, including risk and intertemporal preference, assumes that each time an agent faces a choice between A and B, she chooses identically. This does not describe a stochastic choice agent. Hence, I adapt the preference definition to make the like-for-like unconditional comparison between the two agents argued for in Section 2. This section proposes the following definition: A is prefered to B when the probability of choosing A exceeds $\frac{1}{2}$. *The reader who is happy with this definition without further justification loses little by skipping the rest of this section.*

The model justifying this definition is of an analyst who attributes all deviations from deterministic behaviour to measurement error. This captures, in a stylised way, approaches in the literature to inference with noisy data.

3.1 Model of the Analyst

An analyst observes stochastic choice data. That is repeated decisions $d\in\mathcal{C}$ from a single decision-maker for different decision problems $D=(\mathcal{C},X(d))$, where $\mathcal{C}$ is the observed choice set and X is the observed outcome resulting from a decision:

Definition 2 (Stochastic choice data)**.** This is a collection of decision problems $\{D_i=(\mathcal{C}_i,X_i(d))|i\in I\}$, each observed with a frequency given by measure $\mu_i\in\Delta(I)$, and a related set of stochastic choice functions $Q=\{q_{D_i}\in\Delta(\mathcal{C})\}$.

I first apply this definition to a decision maker who chooses between a lottery and a sure-thing amount. I then apply it to a decision maker who chooses savings in a risk-free asset given different interest rates. The first application captures preferences over risk and the second over intertemporal substitution. To illustrate the use of the definition, consider a saver facing different interest rates. Each different interest rate r defines a different decision problem $D_r = (\mathcal{C}, X_r(d))$. A given interest rate constitutes a fraction μ_r of all observations, and for each r, the full distribution of choices is observed q_{D_r}.

Any non-degenerate q_D is inconsistent with deterministic choice. When faced with stochastic choice data, most research does not abandon deterministic choice concepts, nor is it clear that doing so is desirable given the prevalence of measurement error and unobserved heterogeneity. I consider the measures of risk aversion and preference for intertemporal substitution an analyst would arrive at if she treated all deviation from deterministic behaviour as measurement error.

3.2 Risk Aversion

The Arrow-Pratt measure can be calculated for the rationally inattentive agents, but it does not predict her preference over lotteries. The certainty equivalent more directly relates to why we care about definitions of risk aversion: they encapsulate preferences between lotteries. Additionally, certainty equivalent is observable, whereas the Arrow-Partt measure is in terms of unobservables. Hence, I define risk aversion in terms of the certainty equivalent.

Setup The analyst has access to stochastic choice data. She observes the probability of accepting a lottery $q_D(x)$ when the agent is offered multiple different sure-thing alternatives x and the relative frequencies with which each x is observed as captured by measure μ_x. There is a lower amount x_l below which $q_D(x) = 0$ and a higher amount x_h above which $q_D(x) = 0$. The analyst believes there is a certainty equivalent amount $\pi \in (x_l, x_h)$ below which the sure-thing is rejected and above which it is accepted. She attributes the stochasticity of the agent's choices to measurement error that switches the binary outcome (accept, reject) with probability ϕ. If $d = 1$ represents accepting the lottery, she has a misspecified model of the DGP of $q(x)$ such that the observation j from $q(x)$ is given by $d_{j,x} = \mathbb{1}[x > \pi] + \varepsilon_j(\mathbb{1}[x < \pi] - \mathbb{1}[x > \pi])$ where $\varepsilon \sim Bernoulli(\phi)$. She wants to infer π.

Proposition 3. *$\hat{\pi}$ such that $q_D(\hat{\pi}) = \frac{1}{2}$ is the maximum likelihood estimator of π*

Proof. This analyst has a misspecified model where the j^{th} observed choice at sure-thing offer x is distributed:

$$d_{j,x} = \mathbb{1}[x > \pi] + \varepsilon_j(\mathbb{1}[x < \pi] - \mathbb{1}[x > \pi]), \tag{3}$$

where $\varepsilon \sim Bernouli(\phi)$ and $d = 1$ represent accpeting the lottery and $d = 0$ rejecting.

The likelihood contribution in her misspecified model from observing d when the sure-thing x is offered is:

$$f(d, x; \phi, \pi) = \phi^{y(d,x,\pi)}(1 - \phi)^{(1-y(d,x,\pi))},$$

where $y(d, x, \pi) = \mathbb{1}[x > \pi] + d - 2d\mathbb{1}[x > \pi]$. Then given the distribution of the sample μ_x, the log-likelihood function is:

$$l(\phi, \pi; q_D(x), \mu_x) = \int_{x_l}^{x_h} \sum_{d=0}^{1} q_{D=d}(x) \log\big(f(d, x; \phi, \pi)\big) d\mu_x$$

which gives,

$$l(\phi,\pi;q_D(x),\mu_x)=\log(\phi)\Big(\int_{x_l}^{x_h}\sum_{d=0}^{1}q_{D=d}(x)y(d,x,\pi)d\mu_x\Big)+\log(1-\phi)\Big(1-\int_{x_l}^{x_h}\sum_{d=0}^{1}q_{D=d}(x)y(d,x,\pi)d\mu_x\Big).$$

Treating π as known and maximising w.r.t ϕ gives MLE $\hat{\phi}(\pi)=\int_{x_l}^{x_h}\sum_{d=0}^{1}q_{D=d}(x)y(d,x,\pi)d\mu_x$ the FOC shows:

$$0=\frac{\partial l}{\partial\phi}=\frac{\int_{x_l}^{x_h}\sum_{d=0}^{1}q_{D=d}(x)y(d,x,\pi)d\mu_x}{\phi}-\frac{1-\int_{x_l}^{x_h}\sum_{d=0}^{1}q_{D=d}(x)y(d,x,\pi)d\mu_x}{1-\phi}.$$

Define $Y(x;\pi)=\sum_{d=0}^{1}q_{D=d}(x)y(d,x,\pi)$ and substitute $\hat{\phi}(\pi)$ into the log-likelihood function to find the total MLE of π, $\hat{\pi}$, gives:

$$l(\phi,\pi;q_D(x),\mu_x)=\log\Big(\int_{x_l}^{x_h}Y(x;\pi)d\mu_x\Big)\Big(\int_{x_l}^{x_h}Y(x;\pi)d\mu_x\Big)+\log\Big(1-\int_{x_l}^{x_h}Y(x;\pi)d\mu_x\Big)\Big(1-\int_{x_l}^{x_h}Y(x;\pi)d\mu_x\Big).$$

This is maximised at $\int_{x_l}^{x_h}Y(x;\pi)d\mu_x=0$ and $\int_{x_l}^{x_h}Y(x;\pi)d\mu_x=1$. Since each contribution, $y(d,x,\pi)$, is 0 or 1, these values represent the upper and lower attainable bounds of $\int_{x_l}^{x_h}Y(x;\pi)d\mu_x$, and the log-likelihood is maximised by getting closer to these bounds. The FOC to find the extremum is:

$$0=\frac{\partial}{\partial\pi}\int_{x_l}^{x_h}Y(x;\pi)d\mu_x=\frac{\partial}{\partial\pi}\Big(\int_{x_l}^{\pi}q_{D=1}(x)d\mu_x+\int_{\pi}^{x_h}1-q_{D=1}(x)d\mu_x\Big)$$

giving,

$$0=q_{D=1}(\pi)\mu(\pi)-(1-q_{D=1}(\pi))\mu(\pi)$$

implying $q_{D=1}(\pi)=\frac{1}{2}$, giving the result.

□

3.3 Elasticity of Intertemporal Substitution

Like risk aversion, the elasticity of intertemporal substitution has multiple definitions presupposing deterministic actions. Its definition as the semi-elasticity of consumption growth with respect to the interest rate is in terms of observables, so it is the one the analyst uses to estimate the elasticity (ρ):

$$\rho=\frac{d(\log(\frac{c_{t+1}}{c_t}))}{dr}$$

where r_t is the real interest rate and c_t consumption at time t. To estimate a single elasticity, the analyst assumes isoelastic preferences. With isoelastic preferences, a common method of estimating the elasticity of intertemporal substitution is the linear regression:

$$\log(\frac{c_{t+1}}{c_t})=\rho r_t+k+\varepsilon_t, \tag{4}$$

where k is a constant incorporating the individual's discount factor and any fixed effects, and ε_t is an error term. She assumes all deviations from deterministic actions result from measurement error. However, as regression is robust to measurement error in the dependent variable, where actions enter, and the independent variable r_t is observed without error, no adjustment for measurement error is needed. Hence, the analyst estimates ρ by running OLS on the observations of an individual's consumption and saving decision at the different interest rates in the stochastic choice dataset.

4 Risk Aversion

This section compares a standard uninformed agent's risk preferences with a rationally inattentive agent's by combining the model framework (Section 2) and the definition of risk aversion arrived at by the analyst (Section 3).

4.1 Model

An agent with wealth, w, is faced with the choice between a binary lottery $z \sim Z$ and some sure-thing amount.[2] The goal is to find the agent's certainty equivalent π. For a standard uninformed agent with strictly increasing consumption utility is $u(.)$, her certainty equivalent π solves:

$$u(w-\pi) = E_Z[u(w+z)].$$

The rationally inattentive agent can acquire information about the state of the world by paying a utility directly proportional to the mutual information between Z and the signal with constant of proportionality λ. Choosing between a sure-thing χ and the lottery Z, her decision problem is:

$$\max_{d(x)\in\{0,1\}, f_{X|Z}(x|z)\in\Delta} E[u(w+d.z-(1-d)\chi) - \lambda I(X,Z)].$$

The analyst, attributing deviation from deterministic action to measurement error, infers an agent is indifferent between a sure-thing and a lottery when she chooses both with equal probability. Therefore the rationally inattentive agent's certainty equivalent is the sure-thing π that makes her take each option equally.

4.2 Solution

As the choice between sure-thing and lottery is a discrete choice, we use the result from Matějka and McKay (2015) to deduce that the probability q of accepting the sure-thing solves:

$$\max_{q\in[0,1]} E_Z[\log(q\exp(u(w-\pi)/\lambda) + (1-q)\exp(u(w+z)/\lambda))].$$

Ignoring boundary conditions, the FOC with respect to q is:

$$E_Z[\frac{\exp(u(w-\pi)/\lambda) - \exp(u(w+z)/\lambda)}{q\exp(u(w-\pi)/\lambda) + (1-q)\exp(u(w+z)/\lambda)}] = 0 \tag{5}$$

As we are solving for the certainty equivalent amount, we only need to consider $q = 1/2$ (see Section 3), and so:

$$E_Z[\frac{\exp(u(w-\pi)/\lambda) - \exp(u(w+z)\lambda)}{\exp(u(w-\pi)/\lambda) + \exp(u(w+z)/\lambda)}] = 0. \tag{6}$$

[2]An online appendix derives the condition for a generic lottery.

To solve for the utility level that makes the agent choose the lottery and the sure-thing with equal probability, we define the certainty equivalent utility m as the utility resulting from consuming the certainty equivalent amount $m = \exp(u(w-\pi)/\lambda)$. Label the probability of the good state of the world π_G and that of the bad π_B, and label the exponentiated utility in the good state of the world $V_G = \exp(u(w+z_g)/\lambda)$ and in the bad $V_B = \exp(u(w+z_b)/\lambda)$. Then the equation becomes:

$$\pi_G \frac{V_G - m}{V_G + m} + \pi_B \frac{V_B - m}{V_B + m} = 0.$$

This is a solvable algebraic expression. Selecting the positive root, which the discriminant shows exists, leads to the following expression for m:

$$m = \frac{2E[V] - (V_G + V_B) + \sqrt{(V_G + V_B - 2E[V])^2 + 4V_G V_B}}{2}. \tag{7}$$

This characterises the certainty equivalent of the rationally inattentive agents. If the agent were unable to learn, she would be a standard uninformed agent, so $m = \exp(E[U]) = \exp(\pi_G u_G + \pi_B u_B) = V_G^{\pi_G} V_B^{\pi_B}$. Hence, the term:

$$\frac{2E[V] - (V_G + V_B) + \sqrt{(V_G + V_B - 2E[V])^2 + 4V_G V_B}}{2} - V_G^{\pi_G} V_B^{\pi_B} \tag{8}$$

tells us how much compensating utility the rationally inattentive agent requires compared with the standard uninformed agent to play the lottery[3].

4.3 Analysis

From (8), we want to infer qualitative information about certainty equivalent amounts, not to make interpersonal utility comparisons. Considering the two agents as alternative versions of a single person with the same utility from consumption $u(.)$, we can infer the direction of change in risk preferences from introducing costly attention without positing a form for $u(.)$. Since $u(.)$ is increasing, larger utility differences mean a larger certainty equivalent amount. Analysing (8) leads to the following proposition.

Proposition 4. *The rationally inattentive agent demands a smaller certainty equivalent than the standard uninformed agent when $\pi_g \in (0, \frac{1}{2})$, a larger certainty equivalent when $\pi_g \in (\frac{1}{2}, 1)$, and the same certainty equivalent when $\pi_g \in \{0, \frac{1}{2}, 1\}$.*

Proof. When $\pi_g = \frac{1}{2}$, $2E[V] = (V_G + V_B)$ and so:

$$m = \frac{\sqrt{4V_G V_B}}{2} = \sqrt{V_G V_B} = V_G^{\pi_G} V_B^{\pi_B},$$

hence (8) equals zero. When $\pi_g = 1$:

$$m = \frac{2V_G - (V_G + V_B) + \sqrt{(V_G + V_B - 2V_G)^2 + 4V_G V_B}}{2} = \frac{V_G - V_B + \sqrt{V_B^2 + V_G^2 + 2V_G V_B}}{2} = \frac{2V_G}{2},$$

[3]This normalises the standard uninformed agent's utility function by $1/\lambda$, but this is innocuous.

hence (8) becomes $V_G - V_G = 0$. By symmetrry when $\pi_g = 0$ (8) becomes $V_B - V_B = 0$.

Label (8) as a function of π_G, $f(\pi_G)$:

$$f(\pi_G) = \frac{2E[V] - (V_G + V_B) + \sqrt{(V_G + V_B - 2E[V])^2 + 4V_G V_B}}{2} - V_G^{\pi_G} V_B^{1-\pi_G}.$$

When $V_G > V_B$, f is an elementary function without singularity in $[0,1]$ and so is analytic on the domain. It is zero at $0, \frac{1}{2}, 1$, so we need to check it is negative on $(0, \frac{1}{2})$ and positive on $(\frac{1}{2}, 1)$.

$f(\pi_G)$ can be decomposed into the sum of a positive and negative function:

$$f(\pi_G) = \underbrace{E[V] - V_G^{\pi_G} V_B^{1-\pi_G}}_{g(\pi_G)} + \underbrace{\frac{\sqrt{((1-2\pi_G)V_G + (2\pi_G - 1)V_B)^2 + 4V_G V_B} - (V_G + V_B)}{2}}_{h(\pi_G)}.$$

As the difference between the arithmetic and geometric means, $g(\pi_G)$ is positive over [0,1]. To see $h(\pi_G)$ is negative rearrange it:

$$h(\pi_G) = \frac{\sqrt{(1 - 4\pi_G + 4\pi_G^2)V_G^2 + (1 - 4\pi_G + 4\pi_G^2)V_B^2 - 2(1 - 4\pi_G + 4\pi_G^2)V_G V_B + 4V_G V_B} - (V_G + V_B)}{2}$$

$$\Rightarrow h(\pi_G) = \frac{\sqrt{(V_G + V_B)^2 - 4(\pi_G - \pi_G^2)(V_G - V_B)^2} - (V_G + V_B)}{2} \leq \frac{\sqrt{(V_G + V_B)^2} - (V_G + V_B)}{2} = 0.$$

It is easy to check: $g(0) = h(0) = 0$, $g(1) = h(1) = 0$, both are unimodal ($g(\pi_G)$ hump shamped and $h(\pi_G)$ inverse hump shaped), and $h(\pi_G)$ is minimmised at $\pi_G = \frac{1}{2}$ (because $\frac{1}{2}$ maximising $\pi_G - \pi_G^2$). The FOC maximises $g(.)$ is:

$$0 = \frac{dg}{d\pi_G} = V_G - V_B - V_G^{\pi_G^\star} V_B^{1-\pi_G^\star}(\log(V_G) - \log(V_B))$$

$$\Rightarrow \pi_G^\star = \frac{\log(V_G - V_B) - \log(V_B \log(\frac{V_G}{V_G}))}{\log(\frac{V_G}{V_G})}.$$

For fixed V_B $\pi_G^\star$ is monotonic in V_G, converges to $\frac{1}{2}$ as $V_G \downarrow V_B$ and to 1 as $V_G \to \infty$. Since $V_G > V_B$ it follows $\pi_G^\star > \frac{1}{2}$.

We know that $f(\frac{1}{2}) = 0$, at which point h attains its minimum and starts increasing, and g is still increasing. Therefore f is postive over some $(\frac{1}{2}, \frac{1}{2} + \delta_1)$. Moreover, as h smoothly attains its minimum, there exists some region $(\frac{1}{2} - \delta_2, \frac{1}{2})$ such that $-\frac{dh}{d\pi_G} < \frac{dh}{d\pi_G}$ and so moving away from one-half into this region removes more from positive g than it does from negative h, and since $f(\frac{1}{2}) = 0$ it follows that f is negative over this region.

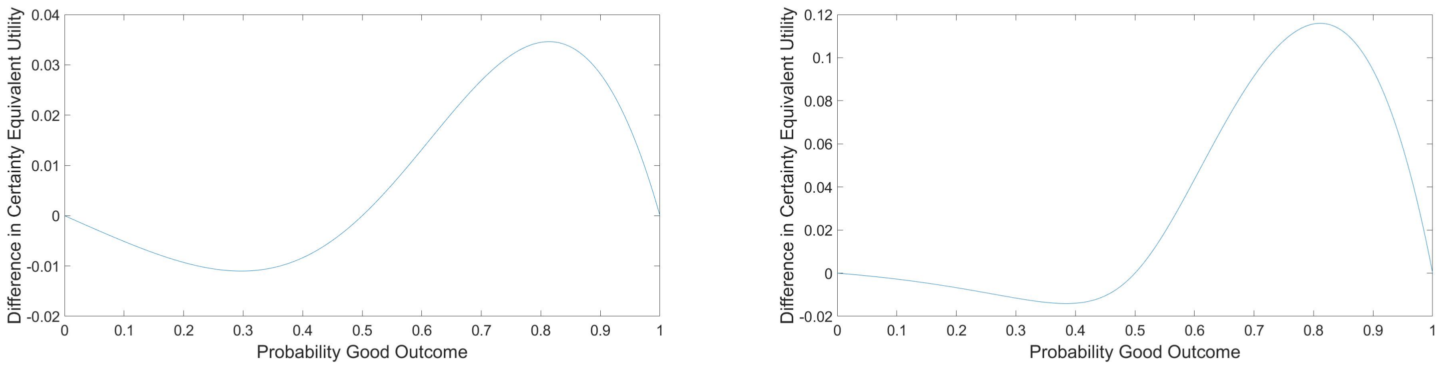
Next:

$$\frac{df}{d\pi_G} = \frac{(V_G - V_B)^2(2\pi_G - 1)}{\sqrt{(V_G + V_B)^2 - 4(\pi_G - \pi_G^2)(V_G - V_B)^2}} + (V_G - V_B) - (\frac{V_G}{V_B})^{\pi_G} V_B \log(\frac{V_G}{V_B}),$$

$$\Rightarrow \frac{df}{d\pi_G}(1) = \frac{(V_G - V_B)^2}{(V_G + V_B)} + (V_G - V_B) - V_G \log(\frac{V_G}{V_B}).$$

For given V_G, this quantity is strictly increasing in V_B, and so it is maximised when V_B is maximised. $V_B < V_G$ and if we set it to its upper bound $V_B = V_G$ we get $\frac{df}{d\pi_G}(1) = 0$. This is an untenable upper bound, and it is strictly increasing in V_B for any V_B, therefore $\frac{df}{d\pi_G}(1) < 0$.

Figure 1: Comparison of Certainty Equivalent Utility



Similarly,

$$\frac{df}{d\pi_G}(0) = -\frac{(V_G - V_B)^2}{(V_G + V_B)} + (V_G - V_B) - V_B \log(\frac{V_G}{V_B})$$

Given V_B, this quantity is strictly decreasing in V_G, and so it is maximised with V_G minimised. $V_G > V_B$ and if we set it to it's lowever bound $V_G = V_B$ we get $\frac{df}{d\pi_G}(0) = 0$. This is an unattainable upper bound and it is strictly decreasing in V_G for any V_B, therefore $\frac{df}{d\pi_G}(0) < 0$

As $f(0) = f(\frac{1}{2}) = f(1) = 0$; f starts and ends negative over $(0, \frac{1}{2})$; and starts and ends positive $(\frac{1}{2}, 1)$, f has at least two turning points implying at least one point where $\frac{d^2 f}{d\pi_G^2} = 0$. If f is not negative (positive) over $(0, \frac{1}{2})$ $((\frac{1}{2}, 1))$, this would require at least two other turning point in the derivative of f , hence two more points where $\frac{d^2 f}{d\pi_G^2} = 0$. However,

$$0 = \frac{d^2 f}{d\pi_G^2} = \frac{8(V_G - V_B)^2 V_G V_B}{((V_G + V_B)^2 - 4(\pi_G - \pi_G^2)(V_G - V_B)^2)^{\frac{3}{2}}} - (\frac{V_G}{V_B})^{\pi}_G V_B \log^2(\frac{V_G}{V_B})$$

$$\Rightarrow \pi_G \log(\frac{V_G}{V_B}) + \frac{3}{2}\log((V_G + V_B)^2 - 4(\pi_G - \pi_G^2)(V_G - V_B)^2) = \log(8(V_G - V_B)^2 V_G \log^{-2}(\frac{V_G}{V_B}))$$

The RHS of this is constant; the LHS is the sum of a positive linear increasing term and a negative inverse hump-shaped term. Therefore the LHS is either monotonic or has one turning point; in either case, it has at most two solutions.

□

Proposition 4 shows that whether the rationally inattentive agents display more or less risk aversion than the standard uninformed agent depends solely on the nature of uncertainty faced (its skewness) - not on the cost of attention or any other preference parameters.

The left panel of Figure 1 illustrates Proposition 4. As proved, the rationally inattentive agent's certainty equivalent (utility) is lower when the probability of the good outcome is less than one-half and higher when it is greater. The right panel of Figure 1 shows that when the cost of attention λ is decreased, the shape of the curve is changed, becoming more skewed around $\frac{1}{2}$ and the size of the maximal difference in certainty equivalent increases. If the consumption utility function is unchanged, this represents an increase in the difference of certainty equivalent amounts. Therefore the size of the rationally inattentive agent's certainty equivalent is determined by the curvature of her utility $u(.)$ over consumption, her cost of attention λ, and the nature of uncertainty.

5 Elasticity of Intertemporal Substitution

This section investigates the implication of costly attention for the Elasticity of Intertemporal Substitution using self-insurance decisions in a dynamic model.

5.1 Model

There are two periods in the model, after which the agent dies with certainty, receiving a terminal value of 0. In period 1, the agent has known and certain income y_1. In period 2, the agent's income is either y_{2b} or y_{2g} occurring with probabilities π_b and π_g where $y_{2b} < y_{2g}$ and $y_1 > y_{2b}$. In period 1, the agent can save in a risk-free asset a with a gross rate of return R. The agent gets utility from consumption and discounts the future at rate β. The agent's consumption flow utility function $u(.) \in \mathcal{C}^2(\mathbb{R}_{>0})$, is strictly increasing, convex, and satisfies an Inada conditions $\lim_{c\to 0} u'(c) = 0$.

The rationally inattentive agent can learn about y_2 by paying an additively separable utility cost, proportional to the mutual information between the signal and y_2 with constant of proportionality λ. Her maximisation problem is:

$$\max_{a(x)\in[0,1], f_{X|Z}(x|z)\in\Delta} E[u(y_1 - a) + \beta u(Ra + y_2)) - \lambda I(X, Z)].$$

I impose a no-borrowing condition ($a > 0$), but the results hold for any borrowing constraint that prevents Ponzi schemes.

5.2 Solution

Self-insurance in a risk-free asset is a continuous choice. Jung et al. (2019) prove that, faced with a continuous choice, the rationally inattentive agent simplifies by only considering a finite subset, and we can learn about how she simplifies from the exponentiated-utility space curve:

$$\mathcal{K} = (\exp(u(y_1 - a) + \beta u(Ra + y_{2b}))^{1/\lambda}, \exp(u(y_1 - a) + \beta u(Ra + y_{2g}))^{1/\lambda})\ \forall a \in [0, y_1],$$

because the marginal probability distribution q of a is found by solving:

$$\max_{q\in\Delta([0,y_1])} \{\pi_b \log(E_q[q(a)\exp(u(y_1 - a) + \beta u(Ra + y_{2b}))^{1/\lambda}]) + \pi_g \log(E_q[q(a)\exp(u(y_1 - a) + \beta u(Ra + y_{2g}))^{1/\lambda}])\}. \quad (9)$$

5.3 Analyis

By choosing $q \in \Delta([0, y_1])$, the rationally inattentive agent can attain in expectation any point in the convex hull of $\mathcal{K}$. Since there are two states of the world, at most, two actions are needed to achieve this. Therefore she either gathers no information and behaves like the standard uninformed agent or randomises[4] over two actions. Since she aims to maximise the log-sum-exp objective in (9) which is a strictly increasing function in both its arguments, she only randomises if the upper convex hull of $\mathcal{K}$ lies strictly above $\mathcal{K}$. Otherwise, she chooses the same utility-maximising point as the standard

[4] I use randomise to describe non-degenerate action distributions. This differs from usage in the game-theory literature because the randomisation device is partially informative.

Figure 2: Feasible Set Exponentiated Utility Space

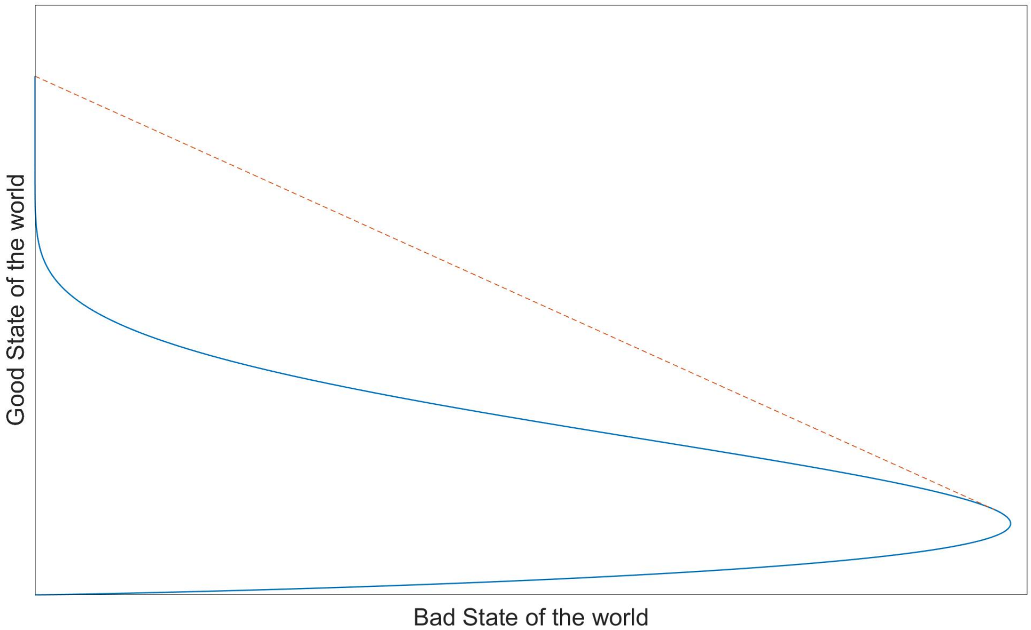
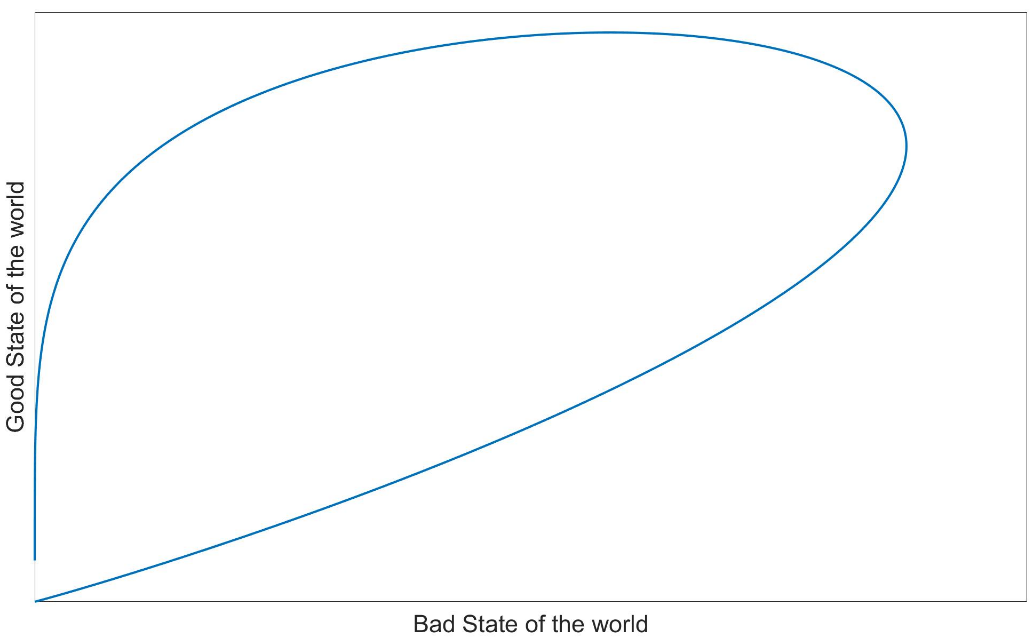


Figure 3: Feasible Set Exponentiated Utility Space



uninformed agent. Figure 2 shows an example where the upper convex hull lies strictly above $\mathcal{K}$, so it offers possibilities to increase net utility. Figure 3 shows an example where it does not.

Part 1 of Proposition 5 confirms that if the cost of information is sufficiently high, the agent gathers no information, and if sufficiently low enough, she gathers some. Part 2 shows that the rationally inattentive agent has the same preference for intertemporal substitution as the standard uniformed agent unless selecting the borrowing constraint is optimal in the good state of the world. It holds because, conditional on the signal, the rationally inattentive agent is a standard agent with different beliefs. So if her utility is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the analyst observations could be generated by multiple standard agents with the same γ. If saving nothing in the good state of the world is not optimal, the agent is on her Euler Equation, and the elasticity of intertemporal substitution is unchanged by this change in beliefs. If saving nothing in the good state of the world is optimal, rational inattention can increase the frequency with which the agent selects the borrowing constraint, pushing up the elasticity of intertemporal substitution because she can reduce the precautionary saving motive through learning.

Proposition 5. *1. If the cost of attention is high enough, the rationally inattentive agent has degenerate unconditional choice distribution q; if it is low enough, q assigns positive probability to two choices.*

2. If $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the elasticity of intertemporal substitution inferred by the analyst $\hat{\rho}$ is $\hat{\rho} = \gamma^{-1}$, unless when $y_2 = y_{2g}$ with probability 1, it is optimal to save nothing in which case $\hat{\rho} \leq \gamma^{-1}$

Proof. When the upper envelope of $\mathcal{K}$ is concave, nothing is gained by randomising over two savings levels, and the rationally inattentive agent makes the same choice as the standard uninformed agent; thus, her elasticity of intertemporal substitution is the same $\hat{\rho} = \gamma^{-1}$, proving part 2 when $\mathcal{K}$ is concave.

So, we need to identify when the upper envelope is convex and a non-degenerate solution q is possible. Doing this leads to a proof of part 1.

Let $V_i(a) = \exp(U_i(a)/\lambda)$ where $U_i(a) = u(y_1 - a) + \beta u(Ra + y_{2i})$ for $i \in B, G$.

To analyse the shape of $\mathcal{K}$ let's follow the curve traced out in exponentiated utility space by:

$$f(a) = (V_B(a), V_G(a)),$$

starting at the end $a = y_1$, and decreasing a. To facilitate discussion, associate the exponentiated utility in the bad state of the world with the x-axis and in the good with the y–axis.

The Inada conditions mean $f(a)$ is strictly increasing away from $f(y_1)$ in both dimensions because the marginal gains from increasing consumption today at the expense of tomorrow are infinite in both states of the world. Eventually, we reach the optimal saving level in the bad state of the world $a^\star_B$. If $a^\star_B = 0$, there are no gains from randomising and $q(a^\star_B) = 1$, so the propsition is true. So assume $a^\star_B = 0$ is interior. From $a^\star_B = 0$, $V_B(a)$ starts decreasing as we continue to decrease a (its derivtive turns postive). Hence from this point, $f(a)$ doubles back on itself as its payoff in the good state of the world continues to increase but decreases in the bad. No point between $(a^\star_B, y_1]$ can be on the upper convex hull because their supporting lines (tangent) lie below another point on $\mathcal{K}$, $a^\star_B + \epsilon$.

Continuing to decrease a, $f(a)$ remains aligned along the $x = -y$ plane until it reaches the optimal saving in the good state of the world $a^\star_G$, which may be at the borrowing constraint $a = 0$ or an interior point. If $a^\star_G$ is interior, then as we continue to decrease a past $a_G^\star$, the payoff in the good state of the world starts to decrease, and so $f(a)$ begins to move downward aligned with the $x = y$ plane. No point between $[0, a_G^\star)$ can be on the upper convex hull because their supporting lines (tangent) lie below another point on $\mathcal{K}$, $a_G^\star - \epsilon$.

Therefore, only points in $[a_G^\star, a_B^\star]$ can be in the upper convex hull, but they will only be so if they lie on a convex portion of the curve (i.e. all points $[a_G^\star, a_B^\star]$ are on the upper envelope, so we need to check convexity).

We implicictly defferntiate f to find $\frac{dV_G}{dV_B}$ and $\frac{d^2V_G}{dV_B^2}$:

$$\frac{dV_G}{dV_B} = \frac{U_G' \exp(U_G/\lambda)}{U_B' \exp(U_B/\lambda)},$$

$$\frac{d^2V_G}{dV_B^2} = \frac{\lambda^{-1} \exp((U_G + U_B)/\lambda)\Big(U_B'(U_G'' + \lambda^{-1}(U_G')^2) - U_G'(U_B'' + \lambda^{-1}(U_B')^2)\Big)}{(U_B' \exp(U_B/\lambda))^3}.$$

U_B is increasing over $[a_G^\star, a_B^\star]$ and so the sign of$\frac{d^2V_G}{dV_B^2}$ is compltely determined by:

$$\Delta(a) := \Big(U_B'(U_G'' + \lambda^{-1}(U_G')^2) - U_G'(U_B'' + \lambda^{-1}(U_B')^2)\Big).$$

U_B and U_G are both concave so U_B'' and U_G'' are both negative. As U_B' is postive and U_G' negative over $[a_G^\star, a_B^\star]$ is follows that $\forall a \in (a_G^\star, a_B^\star)$:

$$\lim_{\lambda\to\infty} \Delta(a) = (U_B' U_G'' - U_G' U_B'')\Big) < 0,$$

$$\lim_{\lambda\to 0} \Delta(a) = \lim_{\lambda\to 0} \Big(U_B'\lambda^{-1}(U_G')^2) - U_G'\lambda^{-1}(U_B')^2\Big) > 0.$$

So when the cost of attention is sufficiently large, the upper envelope of $\mathcal{K}$ is concave, so the rationally inattentive agent behaves like the standard uninformed agent. When the cost of attention is sufficiently small, she gathers information and takes two actions with non-zero probability. This completes the proof of part 1.

For part 2, conditional on a given signal, the rationally inattentive agent is a standard-utility maximiser with different beliefs. Since the rationally attentive agent's curvature of utility over consumption is unchanged, what the analyst observes is equivalent to multiple standard agents with different beliefs but the same utility curvature. When R is varied, this shifts both beliefs and savings of the rationally inattentive agent, but if $a_G^\star$ is interior, whatever combination of beliefs and savings she ends up at these will be interior and so satisfy the Euler equation. Therefore all observations of the rationally inattentive (a, R) agent lie on a curve that implies the same elasticity of intertemporal substitution as the standard uninformed agent.

If $a_G^\star = 0$, then it may form one of the two points of support, and, as with standard agents when they are at the borrowing constraint, this point will not respond to decreases in the interest rate implying a lower elasticity of intertemporal substitution. Although the standard uniformed agent can also be at the borrowing constraint, she only chooses this if it is the unique expected-utility maximising, whereas the rationally inattentive agent may randomise over this point of support as long as $a_G^\star = 0$. So, if $a_G^\star = 0$, rational inattention may increase observations at the borrowing constraint and hence the elasticity of intertemporal substitution.

□

6 Combining Risk and Intertemporal Preferences

This section combines the results on relative risk aversion and the Elasticity of intertemporal substitution to show that rational inattention can break the usual inverse reciprocity between them.

If the agent in Section 5 had the option to buy an insurance contract against the income risk instead of the option to self-insure with a risk-free asset, the model would be an application of the risk model from Section 4. Thus, changing the agent's choice allows for a comparison of the impact of rational inattention on risk and intertemporal preferences.

Proposition 5 shows that, unless borrowing constraints bind in the good state of the world, the elasticity of intertemporal substitution is unaffected by rational inattention. Risk preference, however, is explained by Proposition 4 and is different from textbook risk preferences except for a handful of edge cases that depend solely on the nature of uncertainty. Hence the reciprocal coupling.

6.1 Implication for Finance Puzzles

The difficulty in separating intertemporal and risk preference is central to the equity premium puzzle and the risk-free rate puzzle.

6.1.1 Equity Premium Puzzle

The equity premium puzzle (Mehra and Prescott, 1985) is that generating the large observed excess returns of stock over bonds requires unrealistic levels of risk aversion given the low risk of stocks and their poor insurance values against income risk. A large literature attempting to explain it exists (see, Kocherlakota, 1996); however, the aim here is not to evaluate the merits of other explanations but to document how rational inattention can explain this puzzle by separating intertemporal and risk preference. As a full portfolio selection model with consumption growth uncertainty is beyond the analytically solvable models analysed in this paper, I provide suggestive evidence using a simpler model.

Model The model of Section 4 was presented as a model of insurance purchase but can be conceived of as a choice between a risky and risk-free asset. It provides a stylised illustration of the equity premium puzzle considering only the extensive margin choice between stocks and bonds.

Consider a version of the model of Section 4.3 in which the agent has a choice between investing all wealth w in a risk-free asset with returns r^f or investing a fixed fraction α in a risky asset with uncertain return r having a binary distribution with support $\{r_b, r_g\}$, $r_b < r_g$. We can find the risk-free rate r^f that would make the standard uninformed agent indifferent to the lottery by solving:

$$\frac{((1-\alpha)r^f w + \alpha r^f w)^{1-\gamma}}{1-\gamma} = E[\frac{((1-\alpha)r^f w + \alpha r w)^{1-\gamma}}{1-\gamma}].$$

Since, in a representative agent economy, the agent must be indifferent between assets for both to exist in equilibrium, checking this indifference condition gives a simplified way of investigating the equity premium puzzle using this binary model. Section 4 shows that to be indifferent a rationally inattentive agent requires a risk-free rate that solves:

$$\exp \frac{((1-\alpha)r^f w + \alpha r^f w)^{1-\gamma}}{\lambda(1-\gamma)} = \frac{2E[V] - (V_G + V_B) \pm \sqrt{(V_G + V_B - 2E[V])^2 + 4V_G V_B}}{2},$$

where $V_i = \frac{((1-\alpha)r^f w + \alpha r_i w)^{1-\gamma}}{\lambda(1-\gamma)}$ for $i \in \{G, B\}$.

Calibration As a binary distribution has three free parameters, it can be calibrated to match the first three moments of US stock return data. I calibrate skewness to the mean firm-level returns documented in Albuquerque (2012)[5], giving a standardised third central moment of 0.531. I take a mean stock return of 0.081, and its standard deviation of 0.156 from Campbel (2003). Fraction of wealth in the stock market I take from Luo (2010) as $\alpha = 0.22$ and, as all wealth gets consumed in this static model, I set w to the conservative value of $20,000.

Results With this calibration, the standard uninformed agent needs a utility curvature of $\gamma = 34.3$ for indifference between the risk-free and risky asset. With the curvature of utility at the relatively low value of $\gamma = 2$, the rationally inattentive can be made indifferent between the risk-free and the risky return by lowering her cost of information acquisition.

6.1.2 Risk-free Rate Puzzle

The risk-free rate puzzle is that the relatively rapid consumption growth over the life cycle, given the low returns on safe assets, implies an implausibly high elasticity of intertemporal substitution. In standard models, where inverse reciprocity holds, this puts a limit on explaining the equity premium puzzle by simply increasing risk-aversion.

Taking the calibrated model of Section 6.1.1, we can generate indifference between holding the risky and risk-free assets with a much lower curvature of the utility $\gamma = 0.5$. This is the type of value typically associated with solving the risk-free rate puzzle. Hence, the ability of rational inattention to separate these parameters offers a potentially simple solution to these two puzzles jointly: set the curvature of utility to solve the risk-free rate puzzle and the cost of attention to match the equity premium.

7 Conclusion

This paper has shown how rational inattention can separate the elasticity of intertemporal substitution from relative risk aversion within a time-additive expected-utility framework. This is because costly attention creates an additional reason to dislike learnable risk, namely the cost of reducing uncertainty. Calibrating simple models to stylised facts of the US economy suggests that this ability to disentangle risk and intertemporal preferences may help explain the equity premium and risk-free rate puzzles. This paper use entropy-based cost of attention, but these results may extend to other methods of modelling costly attention (e.g. Gabaix, 2014; Caplin et al., 2022). The parallels between the rationally inattentive agents' choice between information gathering and self-insurance and the choice between self-insurance and self-protection (e.g. Ehrlich and Becker, 1972) suggest a promising avenue of future research.

[5]The online appendix discusses using aggregate skewness.

References

Albuquerque, R. (2012, 5). Skewness in stock returns: Reconciling the evidence on firm versus aggregate returns. *Review of Financial Studies 25*, 1630–1673.

Apesteguia, J. and M. A. Ballester (2018). Monotone stochastic choice models: The case of risk and time preferences. *Journal of Political Economy 126*, 74–106.

Bommier, A., A. Kochov, and F. L. Grand (2017). On monotone recursive preferences. *Econometrica 85*, 1433–1466.

Campbel, J. Y. (2003). Consumption-based asset pricing. *Handbook of the Economics of Finance*, 803–887.

Caplin, A., M. Dean, and J. Leahy (2022). Rationally inattentive behavior: Characterizing and generalizing shannon entropy. *Journal of Political Economy 130*(6), 1676–1715.

Dillenberger, D., D. Gottlieb, and P. Ortoleva (2020). Stochastic impatience and the separation of time and risk preferences.

Ehrlich, I. and G. S. Becker (1972). Market insurance, self-insurance, and self-protection. *Journal of Political Economy 80*(4), 623–648.

Epstein, L. G. and S. E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*, 937–969.

Gabaix, X. (2014). A sparsity-based model of bounded rationality. *The Quarterly Journal of Economics 129*(4), 1661–1710.

Hansen, L. P. and T. J. Sargent (1995, 5). Discounted linear exponential quadratic gaussian control. *IEEE Transactions on Automatic Control 40*, 968–971.

Jung, J., J. H. J. Kim, F. Matějka, and C. Sims (2019, 11). Discrete actions in information-constrained decision problems. *Review of Economic Studies 86*, 2643–2667.

Kocherlakota, N. R. (1996). The equity premium: It's still a puzzle. *Source: Journal of Economic Literature 34*, 42–71.

Luo, Y. (2010, 10). Rational inattention, long-run consumption risk, and portfolio choice. *Review of Economic Dynamics 13*, 843–860.

Luo, Y. and E. R. Young (2016). Long-run consumption risk and asset allocation under recursive utility and rational inattention *. *Journal of Money, Credit and Banking 48*, 325–362.

Lusardi, A., P.-C. Michaud, and O. S. Mitchell (2015). Optimal financial knowledge and wealth inequiality. *Journal of Political Economy 125*, 1–48.

Matějka, F. and A. McKay (2015). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review 105*, 272–298.

Mehra, R. and E. C. Prescott (1985). The equity premium a puzzle. *Journal of Monetary Economics 15*, 145–161.

Steiner, J., C. Stewart, and F. Matějka (2017). Rational inattention dynamics: Inertia and delay in decision-making. *Econometrica 85*, 521–553.

Weil, P. (1990). Nonexpected utility in macroeconomics. *Source: The Quarterly Journal of Economics 105*, 29–42.

Wilcox, N. T. (2011, 5). 'stochastically more risk averse:' a contextual theory of stochastic discrete choice under risk. *Journal of Econometrics 162*, 89–104.