1.

Throughout this article we let (X, d) designate a noncompact, complete, finite-dimensional Alexandrov space. So X is a length space satisfying $\kappa \geq 0$ sectional curvature conditions. In otherwords every quadruple (a, b, c, d) in X satisfies Toponogov's comparison condition, c.f. [**Vil1**], [**morgan2007**]. For a basepoint $x_0 \in X$, let $M = M(x_0)$ be the set of all geodesics λ in X satisfying:

(i) the geodesic λ passes through x_0;
(ii) the geodesic λ is distance minimizing over every compact subinterval;
(iii) the geodesic is maximally nonextendible.

Lemma 1 (**.** *morgan2007] If X is noncompact, then M^* is nonempty.*

I.e. there exists such distance minimizing asymptotic geodesic rays. The set M of geodesics contains evidently three types: the geodesics λ are either

(a) compact; or
(b) noncompact and doubly-ended; or
(c) noncompact and singly-ended.

We abbreviate $M^* \subset M$ as the subset of noncompact geodesics. In case M^* contains a doubly-ended geodesic, then our arguments below will establish Gromoll-Cheeger's Splitting theorem [**morgan2007**]; otherwise we use the geodesics of M^* to establish the Cheeger-Gromoll-Perelman's Soul theorem [**morgan2007**] for singular Alexandrov spaces. Our purpose is to demonstrate how our methods of Kantorovich singularity and optimal semicouplings from [**martel**] establishes both theorems nearly simultaneously.

For every $\lambda \in M^*$, let $h_\lambda : X \to \mathbb{R}$ be the unique horofunction satisfying $h_\lambda(x_0) = 0$, and defined by the familiar formula

$$h_{\lambda,x_0}(x) := \lim_{t \to +\infty} d(\lambda(t), x) - t.$$

We observe $h_{\lambda,x_0}(x) \geq -d(x, x_0)$ and that h_{λ,x_0} diverges to $+\infty$ along the geodesic λ. Our curvature hypothesis $\kappa \geq 0$ implies h_λ is geodesically-convex function and the sublevel sets $\{h_{\lambda,x_0} \leq T\}$ are totally convex subsets of X for all $T \in \mathbb{R}$. (The

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same definitions imply h_λ is totally-concave in nonpositive curvature $\kappa \leq 0$). If the geodesic λ is doubly-ended, then h_λ will be symmetric with respect to x_0 and approaches values $\pm\infty$ as arc-parameter $\lambda(s)$ diverges to $s \to \pm\infty$. For $\lambda \in M^*$, let $t = t(\lambda) \in \mathbb{R}$ be a real number for which $1/|t|$ is sufficiently large.

Lemma 2. *If the parameter $t : M^* \to \mathbb{R}$ is sufficiently small ($t \approx 0$), then the excision $X_0 := X[t]$ is a nontrivial compact totally-convex subset of X.*

Proof. The hypothesis of $\kappa \geq 0$ implies the excision $X - H_{\lambda,t}$ is a totally convex subset of X. Therefore the intersections $\cap_{\lambda \in M^*} X - H_{\lambda,t}$ is a totally convex subset. Moreover the completeness of X implies all the geodesics in X_0 are compact, and Lemma [ref] implies X_0 is a compact subset. □

The excision X_0 is a compact totally-convex subset with boundary ∂X_0. The boundary ∂X_0 is "cellulated" by the boundaries $\partial H_{\lambda,t}$ of the excised horoballs. Moreover the following claim may be established:

Lemma 3. *The inclusion $X_0 \hookrightarrow X$ is a homotopy-isomorphism. That is, there exists a continuous strong deformation retract $X \rightsquigarrow X_0$.*

Proof. □

The above constructions lead us to our semicoupling program. The excision X_0 has a canonical Hausdorff measure $\sigma := \mathscr{H}_{X_0} = \mathscr{H}_X 1_{X_0}$, and the excision boundary ∂X_0 has canonical Hausdorff measure $\tau := \mathscr{H}_{\partial X_0}$. The measures σ, τ are designated our source and target measures, respectively.

2.

For pairs $x_1, x_2 \in X$, we may compare $d(x_1, x_2)$ to the signed distances between horospheres $h_\lambda(x_1) - h_\lambda(x_2)$, which we observe is independant of the basepoint x_0 defining $h_\lambda = h_{\lambda,x_0}$. Indeed the triangle inequality implies

$$\sup_\lambda \{|h_\lambda(x_1) - h_\lambda(x_2)|\} \leq d(x_1, x_2). \tag{1}$$

The inequality (1) is rather interesting for our purposes. Indeed the convexity of h_λ has the following elementary consequence: for every $x_2 \in X$, the function

$$b_{x_2}(x) = b(x) := \sup_\lambda \{|h_\lambda(x) - h_\lambda(x_2)|\} \text{ is geodesically convex,}$$

defined $b : X \to \mathbb{R}_{\geq 0}$. This is familiar consequence of fact that the supremum of a family of convex functions is again convex.

It is important to emphasize that the comparison (1) is not necessarily an equality. Indeed this related to familiar fact that metric balls (i.e. sublevels of distance functions) are possibly nonconvex subsets in Alexandrov geometries. However (1) suggests the possibility of *replacing* the distance function $d(x,y)$ with a function of the type

$$(2) \qquad b(x,y) := \sup_{\lambda \in M^*} \{|h_\lambda(x) - h_\lambda(y)|\}.$$

The key idea of this present article is to treat $b(x,y)$ as a type of "distance" on X, although we will eventually restrict b to the subset $X_0 \times \partial X_0$ defined earlier in **??**. As defined above, b represents a possibly degenerate distance function on X. For instance, if M^* is a singleton, then $b(x,y) = |h_\lambda(x) - h_\lambda(y)| = 0$ if and only if x, y both occupy the same horosphere centred at λ.

3.

The following technical lemma is important to our results:

Lemma 4. *The* c^**-subdifferentials* $\partial^{c^*}\psi(y) \hookrightarrow X_0$ *of* c^**-concave potentials* $\psi : \partial X_0 \to \mathbb{R} \cup \{-\infty\}$ *are geodesically convex subsets for every* $y \in \partial X_0$.

Proof. □

Examples: pointed cone. cone over a compact polyhedron. cones with warped product metric.

Remark. A difficulty in extending our arguments from [ref][thesis] to singular Alexandrov spaces is that gradients and gradient projections, are not easy to define on the singular spaces. I.e., our deformation retracts are via gradient flow towards poles of a vector field $\eta(x, avg)$. Moreover retractions deeper into the singularity structure requires ∇^Z gradients over closed singular subvarieties Z of X. I.e., if f is function, then we require the gradient of the restriction $f|Z$ of f to Z, having $\nabla^Z f = proj_Z \nabla f$.

The cone C is important test case. There is a circle's worth of asymptotic rays (diverging to infinity), and the horospheric excisions create a boundary ∂C_0 homeomorphic to S^1. But the semicoupling singularity will homotopy-reduce C_0 to a circle(?) or a point(?). The cone vertex $\{pt\}$ on C is however the desired "soul".

Heuristic: almost-all geodesic segments on C are disjoint from vertex. Therefore optimal semicouplings almost-surely are supported on geodesics disjoint from vertices. Recall definition of "extreme set" of Alexandrov space. I.e., maximal length geodesics (which are everywhere distance minimizing) intersect the extreme set?

Question: suppose nontrivial "asymptotic burst points" always exist on a complete and noncompact finite-dimensional Alexandrov space (X,d). Then the excision $X[t]$ centred on the asymptotic burst points is a geodesically convex compact subset of X.

4. Splitting

Let γ be an isometrically embedded infinite line $\gamma : (-\infty, +\infty) \to X$. For large parameter t, let $X[t]$ be the excision of X obtained by scooping out the pair of halfspaces $\{h_\lambda \geq t\}$ and $\{h_{\lambda,x} \leq -t\}$. So $X[t]$ is convex subset of X, with boundary $\partial X[t]$ consisting of two horospheric boundaries $Y_+ := \{h_{\lambda,x} = t\}$ and $Y_- := \{h_{\lambda,x} = -t\}$. These boundary components are disjoint when $t > 0$.

[Following cost is special case of above cost, where λ_x is supported on two-points y_+, y_- for $x \in X$.]

The geodesic convexity of $X[t]$ implies the existence of the following well-defined correspondance $\Gamma : Y_+ \times Y_- \times [0,1] \to X[t]$, where $\Gamma(y, y', s) := y_s := [y, y']_s$ is (any) point in $X[t]$ satisfying $d(y, y_s) = sd(y, y')$ and $d(y', y_s) = (1-s)d(y, y')$.

Lemma 5. *Let $t > 0$ be real parameter, and define Y_+, Y_- as above. The correspondance $\Gamma : Y_+ \times Y_- \times [0,1] \to X[t]$ is surjective.*

So every point $x \in X[t]$ lies on a geodesic segment with endpoints on Y_+ and Y_-. For general Alexandrov spaces this representation is nonunique, and a given point x is incident to numerous geodesic segments. In a complete finite-dimensional Alexandrov space the geodesics have a nonsplitting property which implies, for every pair $(y, x) \in Y_+ \times X[t]$, there exists unique geodesic γ joining y, x in $X[t]$.

Now we define a general cost function $v : X[t] \times \partial X[t] \to \mathbb{R}$ by the formula

$$c(x, y_+) = (d(x, y_+)^{-2}.\frac{1}{2} + d(x, y_-)^{-2}).d(y_+, y_-),$$

where $y_+ \in Y_+$ is joined along a geodesic γ passing through x, and which intersects Y_- at y_-, and where $|y_+ y_-|$ represents the distance between y_+, y_-. Likewise we define

$$c(x, y_-) = (d(x, y_+)^{-2} + d(x, y_-)^{-2}.\frac{1}{2}).d(y_+, y_-)^2.$$

So we obtain a cost function $c : X[t] \times \partial X[t] \to \mathbb{R}$.

Proposition 6. *The cost c is continuous in (x, y). For every $x \in X[t]$ the gradient $\nabla_x c(x, y)$ is continuous and the rule $Y \to T_x X[t]$ defines injective mapping $y \mapsto \nabla_x c(x, y)$.*

Proof. Fix $x \in X[t]$. Then

$$\nabla_x c(x, y_+) = d(y_+, y_-)^2.(\frac{1}{2}\nabla_x q(x, y_+)^{-2} + \nabla_x q(x, y_-)^{-2}).$$

Our hypotheses on y_-, y_+ imply the gradients $\nabla_x q(x, y_+)^{-2}$ and $\nabla_x q(x, y_-)^{-2}$ are parallel and pointing in opposite directions. Thus the gradient vectors make angle $\angle = \pi$ at x. This implies the gradients $\nabla_x c(x, y_+)$ and $\nabla_x c(x, y'_+)$ are distinct when y_+ and y'_+ are distinct and the angle $0 < \angle(y_+ x y'_+) < \pi$. In case $y_+ x y'_+$ lie on a geodesic segment and $y'_+ = y_-$, then definition [ss] implies $\nabla_x c(x, y_+)$ and $\nabla_x c(x, y'_+)$ are distinct. □

The proposition [ss] says the cost c satisfies (Twist) throughout the source $X[t]$. Therefore every choice of source measure $\sigma = dx$ on $X[t]$, and target measure $\tau = dy$ on $\partial X[t]$ admits a unique c-optimal semicoupling π on $X[t] \times \partial X[t]$.

Theorem 7. *Let σ be source measure on $X[t]$, and τ a target measure on $\partial X[t]$, and with cost c defined in [ss]. If $mass(\sigma)/mass(\tau) \approx 1^+$, then the active domain A of the unique c-optimal semicoupling is a strong deformation retract of $X[t]$. Moreover the active domain A splits isometrically $A = (-t, +t) \times Z$, where $Z := Z_2$ consists of all source points $x \in X[t]$ such that $\partial^c \psi^c(x) \geq 2$.*

Incomplete. We fix the parameters t, a. Let π be c-optimal semicoupling, uniquely defined by [ss]. When the parameter a is sufficiently close to 1^+, need show Theorem A from [ss: thesis] can be applied. Next need prove subdifferentials are geodesically-convex. Then Z_2 is totally-geodesic subvariety of X. Next need prove isometric-splitting $A = (-t, t) \times Z_2$. □

Thus we find two parameters t and $a_t := \int_{X[t]} \sigma / \int_Y \tau$. As a decreases 1^+, then the active domain $A = A_t$ of the c-optimal semicoupling fills the domain $X[t]$. As $t \to +\infty$ the isometric splitting

$$A_t = (-t, +t) \times Z$$

converges to an isometric splitting

$$(-\infty, +\infty) \times Z.$$

Thus we recover Toponogov's splitting principle for complete finite-dimensional Alexandrov mm-spaces.

5. Nonnegative Ricci Curvature

When (X, d) has nonnegative Ricci-curvature $Ric(X, d) \geq 0$, then Cheeger-Gromoll proved the following generalization.

Theorem 8 (Cheeger-Gromoll)**.** *If (X, d) has nonnegative Ricci curvature, then for every ray γ in X the horofunction $h_\gamma : X \to \mathbb{R}$ is superharmonic. I.e. the divergence of the gradient flow $div(\nabla_x h_\gamma)$ is nonnegative (≥ 0) throughout X.*

Question: does the superharmonic property imply (i) well defined outward normals? (ii) well defined visibility and visibility kernel? (iii) visibility cost satisfies (Twist)?

Can the argument extend to nonnegative Ricci curvature? Need the subdifferentials be geodesically-convex in $Ric \geq 0$. We can further imagine (X, d) satisfies mm-Ricci curvature bound, controlling the divergence (!) of the exponential maps. Prove cheeger-gromoll splitting for $Ric \geq 0$? The sublevels are not convex: compare basic distinction: convex versus mean-convex sublevels.

6. Stability of Splittings and Excisions

Goal: Show splittings/excision constructions converge/vary continuously in GH-topology.

Lemma 9. *If we have GH mm-convergent sequence (X_j, σ_j) to (X, σ), then the pointed asymptotic explosion maps can be canonically lifted to the approximants X_j. The target measures supported on the excision boundaries $\{\tau\}$ will mm-converge to τ on X.*

Along a convergent GH-sequence, we want to speak of some 'convergent' mm-cellular structure. Require a 'convergent' semicoupling problem, relative to the electro-costs $c|\tau$, between convergent excision models $X'[t], \partial X'[t]$. Need prove the excisions can be 'naturally' lifted and defined along the convergent sequence.

Prop: replace a convergent sequence X_j with measured excision models $X[t]_j \times \partial X[t]_j$.

7. Loeper Mountain Pass

Suppose x_0, x_1 are distinct points in $\partial^c\psi(y_0)$. Let us examine the conditions necessary to guarantee that geodesic midpoint $x_{1/2} := [x_0, x_1]_{1/2}$ also occupies the subdifferential $\partial^c\psi(y_0)$. Because $\psi^{cc} = \psi$,, we require the inequality

$$\psi(y_*) - \psi(y_0) \leq c(x_{1/2}, y_*) - c(x, y_0)$$

for all $y_* \in Y$. But x_0, x_1 simultaneously in $\partial^c\psi(y_0)$ implies the above inequality holds if and only if

$$c(x_{1/2}, y_*) - c(x_{1/2}, y_0) \geq \min[c(x_0, y_*) - c(x_0, y_0), c(x_1, y_*) - c(x_1, y_0)],$$

for all $y_* \in Y$. This is Loeper's condition [ref: Loeper, Villani], and generalized in the Ma-Trudinger-Wang condition [ref].

When we imagine the fermi-electron costs, then the fat/round triangle inequality shows us that the midpoint $x_{1/2}$ between x_0, x_1 will experience a weaker force with respect to the target points y_* and y_0. So if x_0, x_1 are simultaneously activated and transported to target y_*, then the mass at $x_{1/2}$ can be more efficiently routed to y_*, unless the mass at y_* has already been consumed and transported elsewhere. XX Incomplete XX.

So we could indeed imagine the Kantorovich Singularities as geodesically convex subsets, or at least locally with respect to the subdifferential cellular structure $\partial^c\psi^c(x')$.

Proposition 10. *Describe active domain, with respect to expanding almost-orthonormal frame (burst expanding to infinity). horospheric excisions. then kantorovich-retraction defines distance nonincreasing (?) deformations* $A = Z_1 \to Z_2 \to$ *to an index* Z_J*, where* J *is maximal Halfspace conditions (and also existence of topological boundary).*