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1. Background

Throughout this article (X, d) designates a noncompact, complete, finite-dimensional Alexandrov space. So X is a length space satisfying $\kappa \geq 0$ sectional curvature conditions. There are many equivalent formulations, e.g. every quadruple (a, b, c, d) in X satisfies Toponogov's comparison condition [Vil09, Ch.26, pp.738], [MT07, pp.53–55]. See also [Petrunin-Alexander-Kapovitch].

The subject of this article is singular Alexandrov spaces in finite-dimensions. The metric definition of singular point depends on the isometry-type of the space of directions of X at x. A singular Alexandrov space has a locus of nonregular points, e.g. the edges and vertices on a cube are nonregular points. Singular spaces however have an everywhere well-defined space of directions, i.e. well-defined tangent cones at every point. The space of directions is isomorphic to $\mathbf{S}^{d-1}$ at the regular points, and otherwise an Alexandrov space of dimension $d-1$ with sectional curvature $\kappa \geq 1$. [Need insert specific definition and construction of metric space of directions].

The difficulty in extending arguments from [Mar] to singular Alexandrov spaces is that gradients and gradient projections are not so easy to define on the singular spaces. Thus we need to know enough about metric space of directions to sufficiently define gradient flows along vector fields. Our specific constructions depend on gradient flow towards the poles of a vector field denoted $\eta(x, avg)$. We find $\eta(x, avg)$ is the gradient of an averaged potential f_{avg}. Retractions deeper into the singularity structure requires ∇^Z gradients over closed singular subvarieties Z of X. Thus if f is function on X, then we will require the gradient of the restriction $f|Z$ of f to Z, having $\nabla^Z f = proj_Z \nabla f$. This gradient projection needs be carefully defined on the singular points.

2. Ends of Open Alexandrov Spaces

This section is more background, and allows us to define horofunctions and the sup function b, which is useful for our applications.

Date: April 22, 2022.

Definition 1. For arbitrary basepoint $x_0 \in X$, let $M = M(x_0)$ be the set of all geodesics λ in X satisfying:
(i) the geodesic λ passes through x_0;
(ii) the geodesic λ is distance minimizing over every compact subinterval;
(iii) the geodesic is maximally nonextendible.

Lemma 2. *If X is noncompact, then $M = M(x)$ is nonempty for every $x \in X$.*

Proof. [MT07, Lemma 2.1] □

In otherwords there exists distance minimizing asymptotic geodesic rays. The set M of geodesics contains evidently three types: the geodesics $\lambda \in M$ are either
(a) compact; or
(b) noncompact and doubly-ended; or
(c) noncompact and singly-ended.
We abbreviate $M^* \subset M$ as the subset of noncompact geodesics.
For every $\lambda \in M^*$, let $h_\lambda : X \to \mathbb{R}$ be the unique horofunction satisfying $h_\lambda(x_0) = 0$ and defined by the formula

$$h_{\lambda,x_0}(x) := \lim_{t\to+\infty} d(\lambda(t), x) - t.$$

We observe $h_{\lambda,x_0}(x) \geq -d(x, x_0)$ and that h_{λ,x_0} diverges to $+\infty$ along the geodesic λ. Our curvature hypothesis $\kappa \geq 0$ implies h_λ is geodesically-convex function and the sublevel sets $\{h_{\lambda,x_0} \leq T\}$ are totally convex subsets of X for all $T \in \mathbb{R}$. (The same definitions imply h_λ is totally-concave in nonpositive curvature $\kappa \leq 0$).

Lemma 3. *If the parameter $t : M^* \to \mathbb{R}$ is sufficiently small ($t \approx 0$) and finite for every $\lambda \in M^*$, then the excision*

$$X_0 := X[t] := X - \cup_\lambda H_{\lambda,t}$$

is a nontrivial compact totally convex subset of X.

Proof. The hypothesis of $\kappa \geq 0$ implies the excision $X - H_{\lambda,t}$ is a totally convex subset of X. If the parameter is sufficiently small pointwise, then the intersection $\cap_{\lambda\in M^*} X - H_{\lambda,t}$ is nonempty and totally convex. Moreover the completeness of X implies all the geodesics in X_0 are compact, and Lemma (2) implies X_0 is a compact subset. □

The excision $X_0 := X[t]$ constructed above is a totally-convex subset with boundary ∂X_0 cellulated by the boundaries $\partial H_{\lambda,t}$ of the excised horoballs $H_{\lambda,t}$.

Lemma 4 (Error?)**.** *The excision X_0 defined above has boundary ∂X_0 with Hausdorff dimension*

$$\dim_{\mathscr{H}}(\partial X_0) = \dim_{\mathscr{H}}(X_0) - 1.$$

Proof. □

The above proof is related to the fact that the boundary ∂C of bounded convex subsets C has $\dim(\partial C) = \dim(C) - 1$.

Lemma 5. *If the parameter $t : M^* \to \mathbb{R}$ is sufficiently small, then the inclusion $X_0 \hookrightarrow X$ is a homotopy-isomorphism, and there exists a continuous strong deformation retract $X \rightsquigarrow X_0$.*

Proof. □

The above result can be established abstractly, for example using Borel-Serre's approach of absolute neighborhood retracts (ANR). However these are non constructive, and our goal has always been to positively explicitly construct retracts and homotopy-isomorphisms. Explicit retractions were constructed by Sharafutdinov [ref, two papers], and used by Cheeger-Gromoll and Perelman [ref]. We briefly recall the construction. One begins with $C = C_0$ a convex set. If ∂C is empty, then we stop. Otherwise ∂C is nonempty. Let C_1 be defined $C_1 := argmax_{x \in C} d(x, \partial C)$. There is continuous deformation retract $C \rightsquigarrow C_1$. Now C_1 is a convex subset. We repeat the above process. If ∂C_1 is nonempty, then we construct the retraction of C_1 onto $C_2 := argmax_{x \in C_1} d(x, \partial C_1)$. The process eventually terminates because the argmax domains $C_1, C_2, \ldots$ are nonempty, convex, and have strictly decreasing dimension $\dim C_{i+1} \leq \dim C_i - 1$. Roughly the process seems to terminate in a point if we begin with a compact convex set, for the only compact convex set with empty topological boundary is the point singleton $\{pt\}$. Otherwise open convex spaces need not have any boundaries. So the final image of the Sharafutdinov-type retractions is a convex subset with empty topological boundary, i.e. a convex subset $C' \subset C$ such that $\partial C' = \emptyset$.

The above lemma [ref] describes the topological dimension of the boundary ∂X_0, but this dimension need not coincide with it's homotopic dimension! Thus we approach the question of constructing homotopy retracts and homotopy reductions into higher codimensions. A general method for learning and representing homotopy-reductions of large codimension, and thereby obtaining small-dimensional models of topological spaces, is achieved in Kantorovich's singularity functors $Z : 2^{\partial X_0} \to 2^{X_0}$ as constructed in [Mar]. We elaborate below.

3. Horosphere Distance

We begin with a variational relation between horospherical distances and the metric distance on X. For pairs $x, y \in X$, compare $d(x, y)$ to the distances between horospheres $h_\lambda(x) - h_\lambda(y)$. The horosphere distance is evidently independant of the basepoint x_0 defining $h_\lambda = h_{\lambda,x_0}$. The triangle inequality implies

$$\sup_{\lambda\in M^*} h_\lambda(x) - h_\lambda(y) \leq d(x,y). \tag{1}$$

It is important to emphasize that the comparison (1) is not necessarily an equality, depending on the geometry of X. This is related to the notorious fact that metric balls are frequently *nonconvex* subsets in Alexandrov geometry, and especially at large distances. But the inequality (1) suggests replacing the Alexandrov distance $d(x,y)$ with a function of the type

$$b(x,y) := \sup_{\lambda\in M^*} |h_\lambda(x) - h_\lambda(y)|. \tag{2}$$

Now the function b is not jointly convex in the variables x, y. However we first observe that $x \mapsto h_\lambda(x)$ is geodesically convex; fixing y the function $\sup_\lambda (h_\lambda(x) - h_\lambda(y))^2/2$ is again convex in the x variable. [Error? Need be very careful here.]

The goal is to obtain a sentence like "the b-metric balls are d-convex". In this sense, the degenerate Alexandrov distance b at least has the property of all its metric balls being d-convex. Thus there is a tradeoff between the symmetry and nondegeneracy, and the convexity of metric balls. The subject of the present article is to treat $b(x,y)$ as a type of *degenerate distance function* on $b : X \times X \to \mathbb{R}$. However $b : X \times X \to \mathbb{R}$ does not necessarily satisfy the metric geometry definition of "distance". For example:

- b is not symmetric unless the geodesics λ are all double-ended;
- b is nondegenerate unless M^* contains sufficiently many nearly orthogonal directions;

E.g., if M^* is a singleton, then $b(x,y) = |h_\lambda(x) - h_\lambda(y)| = 0$ if x, y both occupy the same horosphere centred at λ.

However b does satisfy an important distance property, namely the triangle inequality:

- b is 1-Lipschitz and satisfies triangle inequality. [Insert proof].

4. Splitting Along a Double-Sided Line

Let γ be an isometrically embedded infinite line $\gamma : (-\infty, +\infty) \to X$. For large parameter t, let $X[t]$ be the excision of X obtained by scooping out the pair of halfspaces $\{h_\lambda \geq t\}$ and $\{h_{\lambda,x} \leq -t\}$. So $X[t]$ is convex subset of X, with boundary $\partial X[t]$ consisting of two horospheric boundaries $Y_+ := \{h_{\lambda,x} = t\}$ and $Y_- := \{h_{\lambda,x} = -t\}$. These boundary components are disjoint when $t > 0$.

[Following cost is special case of above cost, where λ_x is supported on two-points y_+, y_- for $x \in X$.]

The geodesic convexity of $X[t]$ implies the existence of the following well-defined correspondance $\Gamma : Y_+ \times Y_- \times [0,1] \to X[t]$, where $\Gamma(y, y', s) := y_s := [y, y']_s$ is (any) point in $X[t]$ satisfying $d(y, y_s) = sd(y, y')$ and $d(y', y_s) = (1-s)d(y, y')$.

Lemma 6. *Let $t > 0$ be real parameter, and define Y_+, Y_- as above. The correspondance $\Gamma : Y_+ \times Y_- \times [0,1] \to X[t]$ is surjective.*

So every point $x \in X[t]$ lies on a geodesic segment with endpoints on Y_+ and Y_-. For general Alexandrov spaces this representation is nonunique, and a given point x is incident to numerous geodesic segments. In a complete finite-dimensional Alexandrov space the geodesics have a nonsplitting property which implies, for every pair $(y, x) \in Y_+ \times X[t]$, there exists unique geodesic γ joining y, x in $X[t]$.

Now we define a general cost function $v : X[t] \times \partial X[t] \to \mathbb{R}$ by the formula

$$c(x, y_+) = (d(x, y_+)^{-2} . \frac{1}{2} + d(x, y_-)^{-2}) . d(y_+, y_-),$$

where $y_+ \in Y_+$ is joined along a geodesic γ passing through x, and which intersects Y_- at y_-, and where $|y_+ y_-|$ represents the distance between y_+, y_-. Likewise we define

$$c(x, y_-) = (d(x, y_+)^{-2} + d(x, y_-)^{-2} . \frac{1}{2}) . d(y_+, y_-)^2.$$

So we obtain a cost function $c : X[t] \times \partial X[t] \to \mathbb{R}$.

Proposition 7. *The cost c is continuous in (x, y). For every $x \in X[t]$ the gradient $\nabla_x c(x, y)$ is continuous and the rule $Y \to T_x X[t]$ defines injective mapping $y \mapsto \nabla_x c(x, y)$.*

Proof. Fix $x \in X[t]$. Then

$$\nabla_x c(x, y_+) = d(y_+, y_-)^2 . (\frac{1}{2} \nabla_x q(x, y_+)^{-2} + \nabla_x q(x, y_-)^{-2}).$$

Our hypotheses on y_-, y_+ imply the gradients $\nabla_x q(x, y_+)^{-2}$ and $\nabla_x q(x, y_-)^{-2}$ are parallel and pointing in opposite directions. Thus the gradient vectors make angle $\angle = \pi$ at x. This implies the gradients $\nabla_x c(x, y_+)$ and $\nabla_x c(x, y'_+)$ are distinct when y_+ and y'_+ are distinct and the angle $0 < \angle(y_+ x y'_+) < \pi$. In case $y_+ x y'_+$ lie on a geodesic segment and $y'_+ = y_-$, then definition [ss] implies $\nabla_x c(x, y_+)$ and $\nabla_x c(x, y'_+)$ are distinct. ☐

The proposition [ss] says the cost c satisfies (Twist) throughout the source $X[t]$. Therefore every choice of source measure $\sigma = dx$ on $X[t]$, and target measure $\tau = dy$ on $\partial X[t]$ admits a unique c-optimal semicoupling π on $X[t] \times \partial X[t]$.

Theorem 8. *Let σ be source measure on $X[t]$, and τ a target measure on $\partial X[t]$, and with cost c defined in [ss]. If $mass(\sigma)/mass(\tau) \approx 1^+$, then the active domain A of* *the unique c-optimal semicoupling is a strong deformation retract of* $X[t]$*. Moreover the active domain* A *splits isometrically* $A = (-t, +t) \times Z$*, where* $Z := Z_2$ *consists of all source points* $x \in X[t]$ *such that* $\partial^c \psi^c(x) \geq 2$*.*

Incomplete. We fix the parameters t, a. Let π be c-optimal semicoupling, uniquely defined by [ss]. When the parameter a is sufficiently close to 1^+, need show Theorem A from [ss: thesis] can be applied. Next need prove subdifferentials are geodesically-convex. Then Z_2 is totally-geodesic subvariety of X. Next need prove isometric-splitting $A = (-t, t) \times Z_2$. □

Thus we find two parameters t and $a_t := \int_{X[t]} \sigma / \int_Y \tau$. As a decreases 1^+, then the active domain $A = A_t$ of the c-optimal semicoupling fills the domain $X[t]$. As $t \to +\infty$ the isometric splitting

$$A_t = (-t, +t) \times Z$$

converges to an isometric splitting

$$(-\infty, +\infty) \times Z.$$

Thus we recover Toponogov's splitting principle for complete finite-dimensional Alexandrov mm-spaces.

5. Nonnegative Ricci Curvature

When (X, d) has nonnegative Ricci-curvature $Ric(X, d) \geq 0$, then Cheeger-Gromoll proved the following generalization.

Theorem 9 (Cheeger-Gromoll)**.** *If* (X, d) *has nonnegative Ricci curvature, then for every ray* γ *in* X *the horofunction* $h_\gamma : X \to \mathbb{R}$ *is superharmonic. I.e. the divergence of the gradient flow* $div(\nabla_x h_\gamma)$ *is nonnegative* (≥ 0) *throughout* X*.*

Question: does the superharmonic property imply (i) well defined outward normals? (ii) well defined visibility and visibility kernel? (iii) visibility cost satisfies (Twist)?

Can the argument extend to nonnegative Ricci curvature? Need the subdifferentials be geodesically-convex in $Ric \geq 0$. We can further imagine (X, d) satisfies mm-Ricci curvature bound, controlling the divergence (!) of the exponential maps. Prove cheeger-gromoll splitting for $Ric \geq 0$? The sublevels are not convex: compare basic distinction: convex versus mean-convex sublevels.

6. Loeper Mountain Pass

Suppose x_0, x_1 are distinct points in $\partial^c\psi(y_0)$. Let us examine the conditions necessary to guarantee that geodesic midpoint $x_{1/2} := [x_0, x_1]_{1/2}$ also occupies the subdifferential $\partial^c\psi(y_0)$. Because $\psi^{cc} = \psi$,, we require the inequality

$$\psi(y_*) - \psi(y_0) \leq c(x_{1/2}, y_*) - c(x, y_0)$$

for all $y_* \in Y$. But x_0, x_1 simultaneously in $\partial^c\psi(y_0)$ implies the above inequality holds if and only if

$$c(x_{1/2}, y_*) - c(x_{1/2}, y_0) \geq \min[c(x_0, y_*) - c(x_0, y_0), c(x_1, y_*) - c(x_1, y_0)],$$

for all $y_* \in Y$. This is Loeper's condition [ref: Loeper, Villani], and generalized in the Ma-Trudinger-Wang condition [ref].

Proposition 10. *Describe active domain, with respect to expanding almost-orthonormal frame (burst expanding to infinity). horospheric excisions. then kantorovich-retraction defines distance nonincreasing (?) deformations $A = Z_1 \to Z_2 \to$ to an index Z_J, where J is maximal Halfspace conditions (and also existence of topological boundary).*

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