ON HOMOLOGY, DOLD-THOM'S THEOREM, AND OPTIMAL TRANSPORTATION

J. H. MARTEL

This article aims to describe some connections between Optimal Transportation of Monge-Kantorovich and classical results of Algebraic Topology and especially homology and Dold-Thom theorem. In the author's experience, these connections are valuable computational tools which lead to a more constructive positive view of algebraic topology.

How to explore/experiment on spaces which exist only in your mind? How to avoid trivial (tautological) statements in geometry, and discover statements that are true and contingent, i.e. they *could* have been false.

Computers/physics is fundamentally discrete [Riemann?], e.g. WOLFRAM language is symbolic and its core objects are polynomials. In the course of my PhD I believe I experienced the same crisis that L.E.J. Brouwer lived against Hilbert's idealist logical positivism.

The physical circle is a collection of points, i.e. a discrete Borel-Radon measure. My goal was to find bridge between Topology (the realm of continuity) with Probability (the realm of discreteness). The basic example of passing from the discrete to the continuous is Krein-Milman's theorem: if F is convex closed subset, then F is the closed convex hull of its extreme points, that is $F = \overline{conv}(\mathscr{E}[F])$.

1.

Let X be a Riemannian manifold with boundary ∂X. In 1910 A.D. the fundamentals of algebraic topology are established by L.E.J. Brouwer using the singular homology functors in the following theorems:

(A) Invariant Domain Theorem:

(B) Fixed Point Theorem:

(C) No Retract Theorem: There do not exist continuous deformation retracts $X \rightsquigarrow \partial X$ when X is orientable.

In [year] Brouwer came to 'reject' his earlier work, and there developed a controversy between Brouwer's so-called Intuitionism and Hilbert's Formalism (a form of logical positivism centred at Gottingen). Which stands today?

Date: July 18, 2020.

The purpose of this article is to describe the greatest theorem you've never heard of, namely Dold-Thom's Theorem.

In principle, there are several variations on DT's original theorem [ref]. The variations arise in the different types of configuration spaces which can be defined on a given space. We remark that Almgren's thesis [ref] contains an important generalization of DT's original theorem, and the basis for the formal definition of "sweepouts" in geometric topology, c.f. [Guth-Width Inequality].

Further variants of DT's theorem can be found in [Segal, McDuff, ...]. However, from the author's perspective, all these variations suffer a severe defect, namely they are all basepoint dependant. This article was motivated by the author's desiring a basepoint-free version of DT's theorem. Indeed, reduced singular homology $\tilde{H}_*$ is basepoint independant. Fundamental groups, however, are intrinsically basepoint dependant. Our goal was to find version of DT for which the long exact sequence of relative homology [ref] is naturally isomorphic to the long exact sequence in homotopy [ref].

We remark the choice of basepoint on a space X is as complicated as Poincaré's fundamental group $\pi_1(X, pt)$. The fundamental fact of analysis situs is that canonical basepoints do not exist when π_1 is nontrivial.

A brief survey of configuration spaces follows. The basic models are:

(1) Hard disks (where collisions of point particles is prohibited);

(2) Hard disks with basepoint (where collisions except with an annihilating basepoint are prohibited);

(3) Water droplets (c.f. Gromov's category of finite probability spaces [ref], where collisions are encouraged, like water droplets occuring);

(4) Electrostatic droplets (e.g. signed Borel-Radon measures on X, except with existence of indivisible unit Millikan–).

In example (1) the configuration space forms a monoid without identity, and the disks are not allowed to collide or intersect or join, just as hard disks cannot be joined together.

In example (2), a basepoint pt can be chosen on X and this basepoint serves as zero element, thus rendering the monoid into an additive topological group. However, as we commented above, a choice of basepoint is noncanonical when $\pi_1 \neq 0$. The formal construction of earlier configuration spaces in the literature involves symmetric products with the diagonal removed, i.e. a limit of $X \times X \times \cdots - \Delta$. The conventional presentations of Dold-Thom's theorem, including the original [DT58], view configuration spaces in the hard-disk model, and arbitrary basepoint on X must be selected. This basepoint excludes the possibility of relating the conventional Dold-Thom theorem to relative homology.

Firstly, we remark that braid groups [ref] are indeed the fundamental groups of configuration spaces of hard disks on a background disk. [ref: Birman?]

Gromov's category of finite probability spaces [ref], where the objects are so-called reductions $f : \mu \to \nu$ between finite probability spaces μ, ν.

The category of (4) consists of finite electroneutral configurations, where the objects are again reductions $f : \mu \to \nu$.

The proper formalization of (4) into a topological abelian group is as follows.

Definition 1. Let X be a topological space.

Let $\mathbb{Z}(X)$ be the group of finitely-supported $\mathbb{Z}$-valued distributions on X, i.e.

$$\mathbb{Z}(X) := \{\sum n_x.x \mid \text{only finitely many nonzero } n_x \in \mathbb{Z}\}.$$

Let $\epsilon_X : \mathbb{Z}(X) \to \mathbb{Z}$, $\epsilon_X(\sum n_x x) = \sum n_x$ be the canonical augmentation map.

Let $AG_0(X) := \ker(\epsilon_X)$ be the kernel of the augmentation map.

Remark: if G is an abelian group, we define $G(X)$ and $AG_0(X; G)$ in the obvious way. However the cases of $G = \mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z}$ appear to be cases of most interest, and henceforth we suppress the 'G" from notation. Morever because $\mathbb{Z}$ is discrete, as opposed to $G = \mathbb{R}$, there exists unit indivisible charges: point charges (+), (-) are not infinitely divisible. This fails if we replaced $\mathbb{Z}$-valued distributions with $\mathbb{R}$-valued distributions, i.e. there exists unit charge quanta if and only G is discrete.

We observe that $\mathbb{Z}(X)$ is a topological abelian group, with zero element $\emptyset := \sum 0x = 0$ corresponding to the zero distribution, and whose zero element is essentially the zero element 0 of $\mathbb{Z}$. One might consider $\emptyset$ a type of "vacuum state on X". It follows that $AG_0(X)$ is a topological subgroup, consisting of all distributions with zero net charge. One naturally sees AG_0 as a discrete version of all signed Borel-Radon measures μ on X which integrate to zero, $\int_X 1.d\mu(x) = 0$, i.e. which are orthogonal to all constant functions on X.

The essence of DT theorems is to represent homology groups in terms of homotopy groups on configuration spaces. In otherwords to identify the homology functors as homotopy functors. The basic idea of this article is that the vacuum state $\emptyset$ serves as a type of "canonical basepoint on X", and this enables a natural equivalence between the long exact sequences of relative homology and homotopy.

To define the relative version of DT, let Y be a closed subset of X. Our goal is to define a relative configuration space $AG_0(X/Y)$, and the key identification will be the canonical isomorphism

$$AG_0(X/Y) = AG_0(X)/AG_0(Y).$$

In terms of net zero charged particle configurations, the idea is to view Y as a "reservoir" where any excess charge can "ground out". That is, spheres in the relative DT group $AG_(X/Y)$ basepointed at $\emptyset$ are recombinations which either neutralizes away from Y, or neutralizes by grounding out at Y.

Here is formal definition. Let Y be a closed subset of X. Then $\mathbb{Z}(Y)$ and $AG_0(Y)$ is a closed subgroup of $\mathbb{Z}(X)$ and there is a canonical quotient

$$\mathbb{Z}(X) \to \mathbb{Z}(X)/AG_0(Y).$$

Morever the augmentation map ϵ_X canonically descends to the quotient as a type of augmentation map

$$\epsilon_{X/Y} : \mathbb{Z}(X)/AG_0(Y) \to \mathbb{Z},$$

and we identify

$$AG_0(X/Y) := \ker(\epsilon_{X/Y}).$$

In the original terminology of [DT58], the DT theorems stated in terms of the existence of a natural "quasifibration" between the configuration spaces.

Dold-Thom Theorem - Relative Version 2. Let Y be closed subspace of X. Then the short exact sequence of topological abelian groups

$$0 \to AG_0(Y) \to AG_0(X) \to AG_0(X/Y) \to 0,$$

is a quasifibration inducing a long exact sequence of $\emptyset$-pointed homotopy groups

$$\cdots \to \pi_{*+1}(AG_0(Y), \emptyset) \to \pi_{*+1}(AG_0(X), \emptyset) \to \pi_{*+1}(AG_0(X/Y), \emptyset) \to \pi_*(AG_0(Y), \emptyset) \to \cdots$$

which is naturally equivalent to the long-exact sequence of relative homology groups

$$\cdot \to \tilde{H}_{*+1}(Y) \to \tilde{H}_{*+1}(X) \to H_{*+1}(X, Y) \to \tilde{H}_*(Y) \to \cdot.$$

Proof of (2). The standard proof is via Eilenberg–Steenrod axioms, and the canonical uniqueness of any functors satisfying such axioms. □

We make some remarks:

- In terms of category theory, DT says that if G is an abelian group, then the functor $X \mapsto \pi_*(AG_0(X;G), \emptyset)$ is naturally equivalent to the reduced singular homology functor $X \mapsto \tilde{H}_*(X;G)$ in the category TOP of basepoint-free topological spaces. Again, we emphasize that (2) is basepoint independant, with the vacuum state $\emptyset$ serving as "canonical basepoint" on X.
- Formally the homotopy groups $\pi_q(AG_0(X), \emptyset)$ consist of homotopy classes of pointed continuous maps $f : (S^q, pt) \to (AG_0(X), \emptyset)$. Identifying pt with the point at-infinity, we can thus model homotopy classes as compactly supported q-parameter family of distributions on $\mathbb{R}^q$, where "compactly supported" means the distribution is equal to vacuum state outside a compact subset.
- The topology on $AG_0(X)$ implies recombinations of $\emptyset$ satisfy a form of local conservation of charge [ref: Feynman]. Thus point charges (+) and (-) must be born from the same spatial position on X. [image] Therefore the recombinations require the point charges to continuously move on X, and without "teleportation". This is similar to the distinction between the standard L^1 and L^2 optimal transport.

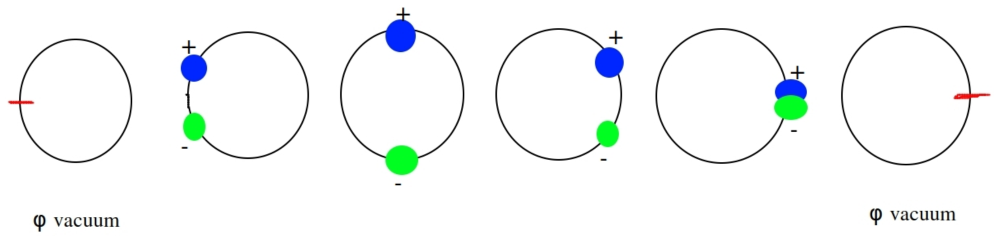

Figure 1. Basic 1-cycle on the one-dimensional circle S^1

Finally we motivate the connection between DT and optimal transportation. As consequence of (2) we have following

Lemma 3. *If Y is closed subset of X, then we have canonical isomorphism*

$$H_*(X,Y)/image(H_*(X)) = \ker(H_{*-1}(Y) \to H_{*-1}(X)),$$

where $image(H_*(X))$ is the image of $H_*(X)$ in the relative homology group $H_*(X,Y)$, and the morphism $H_{*-1}(Y) \to H_{*-1}(X)$ is induced by the inclusion $Y \hookrightarrow X$.

Proof. Long exactness in (2) implies

$$image(H_*(X)) = \ker(\delta)$$

and

$$image(\delta) = \ker(H_{*-1}(Y) \to H_{*-1}(X)).$$

But we have canonical isomorphism

$$H_*(X,Y)/\ker(\delta) = image(\delta),$$

and the result follows. □

While the isomorphism $H_*(X,Y) = H_{*-1}(Y)$ is canonical between homology groups, the morphisms are noncanonical on the level of singular chains.

Our interpretation of (3) is that relative cycles of X (mod Y) are represented either by chains w which are already cycles on X, or chains w whose chain boundaries ∂w are nontrivial cycles on Y but null homologous in X. In terms of DT, this says that if Y is a ground reservoir, then recombinations of $\emptyset$ will have parts which are recombinations in X separated from Y and other parts which are charges grounding out at Y. [image]

So with DT we represent homology cycles on X with spheres in $AG_0(X)$ and nontrivial recombinations of $\emptyset$. Thus k-spheres in $AG_0(X)$ basepointed at $\emptyset$ are

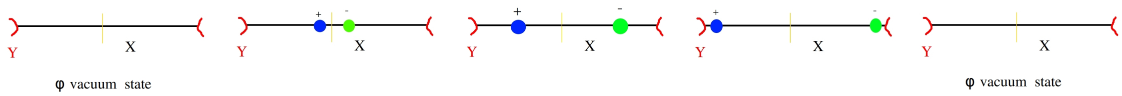

FIGURE 2. Nontrivial relative 1-cycle on $X = [0,1]$ with boundary $Y = \partial X = \{0,1\}$

represented by k-parameter continuous family of net zero charge measures, see (1) for basic example on one-dimensional circle.

For the relative homology groups, we can identify a component of $AG_0(X/Y)$ which is represented by "spontaneous" transports from $\emptyset$ to the reservoir Y, see (1) for basic example on one-dimensional disk with boundary reservoir.

Regularity: one of the key results in regularity of OT is McCann's [ref]: that MTW and nonnegative cross-sectional curvature. An observation of McCann: if the conditions of regularity are to be independant of all source/target measures, then the invariant is a tensor. McCann-Kim symplectic method interprets cross-sectional curvature as sectional curvature of ${}^tD\bar{D}c$, viewed as defining a signature $(\dim\sigma, \dim\tau)$ Riemannian pseudo-metric.

[Relate above regularity to: Steenrod Problem: embedded submanifolds...]

2.

[INCOMPLETE] Naively one tries to construct a single homology cycle, pursuing a form of Steenrod conjecture. There is another standard idea from differential topology to "constructing" prescribed homology cycles. [ref: Kirby].

For example, how does one readily construct nontrivial 2-cycles in a four dimensional torus T^4?

3. SWEEPOUTS

[Def: sweepout] Rather than constructing a single cycle, sweepouts of X are obtained by fibreing X into a family of subvarieties which assemble into the fundamental class of X. [Guth, Almgren]. Optimal transport provides a method for systematically producing parameter spaces and sweepouts of a given source space. The author was much influenced by article [McCann-Pass-Chiappori] and [martel].

Suppose (X,σ) is source space, (Y,τ) is target, and that the source is abundant with respect to the target $\sigma[X] \geq \tau[Y]$. If $c : X \times Y \to \mathbb{R}$ is a cost satisfying [hypotheses], then c-optimal semicouplings π define a contravariant functor $Z = Z_\pi : 2^Y \to 2^X$. [Z defined by maximizers of dual problem]. In case X, Y are compact, we can renormalize σ, τ to be probability measures on X, Y, respectively.

Lemma 4. *Let X be a Riemannian manifold without boundary. Let c satisfy (A0)–(A4), and let $Z = Z_\pi : 2^Y \to 2^X$ be Kantorovich's contravariant functor defined by $Z(Y_I) := \cap_{y \in Y_I} \partial^c \psi(y)$, where ψ is c-concave potential dual to π, and for every closed subset $Y_I \subset Y$. Then the cell $Z(Y_I)$ has topological boundary*

$$\partial_{top} Z(Y_I) = \cap_{Y_J \supset Y_I,\ Y_J \neq Y_I} Z(Y_J).$$

Corollary 5 (Hyp: let σ be volume. In the notation of Lemma [ref]**.** *, if the functor $Z = Z_\pi$ is supported on singletons in Y, then $Z : Y \to 2^X$ defines a Y-parameter sweepout of X.*

Proof. By Lemma [ref] the cells $Z(y)$, $y \in Y$, are submanifolds without boundary in X. The cells $Z(y)$, $y \in Y$, partition X. Finally, if y_0, y_1 are sufficiently close in Y, then we need show $Z(y_0)$, $Z(y_1)$ are homologous, and of small area. [INCOMPLETE] □

The contravariant functor $Z = Z_\pi$ is supported on singletons if and only if the transport map T is defined pointwise everywhere.

LEMMA: if Z supported on singletons and c satisfies [hyp], then transport T is Lipschitz continuous. [??] That is, uniqueness implies continuity.

The research of McCann, etc., shows the regularity of c-opt SC is primarily determined by the MTW tensor, which can be interpreted as the "sectional curvature" of a pseudo-Riemannian metric, c.f. cross-sectional curvature of McCann-Kim [ref]. The existence of discontinuities in the transport mapping leads to cells $Z(y)$ with nontrivial boundaries. In these cases we have a "sweepout" of X into subvarieties which sometimes have boundary.

Naive question: consider the Hopf mapping $S^3 \to S^2$, [ref]. Is there a natural cost $c : S^3 \times S^2 \to \mathbb{R}$ for which the Hopf map arises as c-optimal transport between the canonical measures on S^3, S^2, respectively? Remark: if the transport is regular, then we can guarantee that the fibres are closed one-dimensional manifolds (and therefore disjoint sums of circles). If we can ensure the fibres are connected, then can we recover the homotopy class of the Hopf map?

4.

[Two sweepouts on S^2] To illustrate the basic idea, we consider two different sweepouts of S^2 into 1-cycles. In both cases, we see an explosion of (+), (-) charges emerging from the equator. In the first case, the (+), (-) charges accumulate at the north, south poles, respectively, and then travelling in a group annihilate each other along some longitudinal line. In the second case, the (+), (-) charges emerge from the equator and pass through the north, south poles, respectively, and without stopping continue through the poles eventually annihilating again along the equator lines. Which of these two sweepouts is canonical? In our view, the second sweepout, wherein momentum is preserved, is the more canonical. This leads us to study some energy functionals on AG_0 and the idea of "momentum".

There is problem in extending the Newtonian particle idea of momentum (say, linear and angular) to the curved Riemannian manifolds. In General Relativity textbooks, the term "momentum" is replaced with [ref], e.g., [Hawking-Ellis]. According to [ref], the basic conservation laws are: linear momentum, angular momentum, and energy. Within GR there is great controversy in that the Einstein equations $G_{uv} := Ric_{uv} - g_{uv}R = \kappa T_{uv}$ are not divergence free on their domain of definition. More specifically, the Einstein *tensor equation* has nontrivial tensor divergence. However, in suitable coordinates, the tensor has vanishing coordinate divergence. [Ref: Crothers, ref].

Idea: replace momentum with a least action principle, the minimizers of which satisfy a conservation...

In the first approximation, one views probability measures as representing a cloud of particles. In the second approximation, one views the particles with their velocities, and thus a cloud possesses a total energy (something like kinetic energy). We further presume the particles have a potential energy relative to the background. Indeed, there will be an internal, self-interaction potential (which we might call the assembly energy of the system), and there will also be a potential relative to the background. [Question: is this distinction trivial or nontrivial?]

Now the vacuum state $\emptyset$ has zero energy, and an explosion of charges is a discontinuity of energy, wherein suddenly a configuration of (+), (-) particles are ejected.

Example: when a comet approaches the sun, the sun ejects mass/energy to reneutralize.

Reduction to ground state.

[Relaxation to ground state] [Juergen's Electric Sub Model] [Cobine: Gaseous Conductors]

[CME: Sun ejecting mass-energy to neutralize new potential?]

References

[DT58] A. Dold and R. Thom. "Quasifaserungen und Unendliche Symmetrische Produkte". In: *Annals of Mathematics* 67.2 (1958), pp. 239–281. URL: http://www.jstor.org/stable/1970005.