OPTIMAL TRANSPORTATION AND SWEEPOUTS

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Abstract.

Contents

1.

The purpose of this article is to construct interesting sweepouts of a given source Riemannian manifold (X, σ) using optimal transportation methods and variational data. If (Y, τ) is a target space satisfying $\dim(X) \geq \dim(Y)$ and $\int_X \sigma \geq \int_Y \tau$, then costs $c : X \times Y \to \mathbb{R}$ which satisfy sufficient assumptions [ref] admit unique c-optimal semicouplings and potentials solving Kantorovich's dual max program, namely c-concave potentials ψ on the target Y. The c-concave potential, or specifically it's c-subdifferential, generates a Y-parameter family of subsets $Z(y) := \partial^c \psi(y)$. Our goal is to study $y \mapsto Z(y)$ as a *topological sweepout* of X in the sense of [Almgren, Guth]. This requires hypotheses on c, τ, such that:

- the cells $Z(y) \hookrightarrow X$ are closed cycles on X, i.e. $Z(y)$ has vanishing boundary; and
- the Y-parameter sweepout $y \mapsto Z(y)$ of X is continuous in the flat Almgren topology: namely if $y_0, y_1 \in Y$ are sufficiently close, then $Z(y_0)$, $Z(y_1)$ bound a Lipschitz chain C of arbitrarily small area with $\partial C = Z(y_1) - Z(y_0)$.

We begin with the following elementary lemma which is a combination of results from [MCP].

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Lemma 1. *Suppose $c, D_y c$ satisfy [assumptions]. Then $Z(y')$ is a smooth submanifold of X if $y' \in dom(D\psi)$, and if $\nabla_y \psi(y')$ is a regular value of $\nabla_y c(\cdot, y') : X \to T_{y'}Y$. That is, if $D^2_{xy}c(x', y')$ has full rank for every $x' \in X$ satisfying $\nabla_y c(x', y') = \nabla_y \psi(y')$.*

The c-concave potential ψ is locally Lipschitz when c satisfies (A0)–(A5). See [ref]. This implies $dom(D\psi)$ has full measure in $dom(\psi)$. We can weaken the hypothesis that $y' \in dom(D\psi)$ by extending the definition of *degeneracy* of $D^2_{xy}c$ to the nonsmooth case.

If $c \in C^2$, then for every $y' \in Y$, the set of regular values of $D_y c(\cdot, y') : X \to T_{y'}Y$ has full Lebesgue measure in $T_{\bar{y}}Y$. [Error? Need more regularity on $D_y c$?]

Lemma 2. *If $\bar{k} = D_y c(x, \bar{y}) \in T_{\bar{y}}Y$ is a regular value for some $x \in X$, and if c is nondegenerate throughout $X(\bar{y}, \bar{k})$, then $X(\bar{y}, \bar{k})$ is the intersection of X with an $(n-d)$-dimensional submanifold of X.*

Proof. The first sentence is application of Sard's theorem. The second follows from Lemma [ref]. □

Therefore the regular values of $D_y c(\cdot, y)$ are generic in X, and consequently the fibres $Z(y)$ are globally smooth submanifolds of X for almost every $y \in Y$, and representing cycles in X with $\partial_{top} Z(y) = \emptyset$.

Returning to sweepouts, we find that most sweepouts require *some* fibres to be singular, and it's very difficult – if not impossible – to generate sweepouts $Z(y)$ is a smooth submanifold for every $y \in Y$.

Lemma 3. *Suppose the cost c satisfies assumptions (A0)–(A5). Let ψ be a c-concave potential on Y. Then the Y-parameter family $y \mapsto Z(y)$ is continuous in the flat Almgren topology.*

Proof. Under the above assumptions we know c-concave potentials ψ are locally Lipschitz on $dom(\psi)$ [ref: Prop 1. in TopSing]. Let y_0, y_1 be points in $dom(\psi)$ and suppose $y_0 \to y_1$. Then $\psi(y_0) \to \psi(y_1)$. Therefore the minimum value of $x \mapsto c(x, y_0) + \phi(x)$ $(= \psi(y_0))$ converges to the minimum value of $x \mapsto c(x, y_1) + \phi(x)$ $(= \psi(y_1))$. Thus we need show that the minimizers of $c(x, y_0) + \phi(x)$ converge in flat Almgren topology to the minimizers of $c(x, y_1) + \phi(x)$. By [ref: Prop 2. in TopSing] we know the c-convex potential ϕ is locally Lipschitz on $dom(\phi)$. Hence the function $(x, y) \mapsto c(x, y) + \phi(x)$ is locally Lipschitz, and that is sufficient to prove $Z(y_0)$ converges to $Z(y_1)$ in flat Almgren topology. □

N.B. Without ϕ, ψ being locally Lipschitz, it would be difficult to control the cell in the limit $\lim_{y \to y_1} Z(y)$.

2. Regularity of Optimal Transports

In equal dimensions the regularity of optimal transportation is well-developed thanks to the work of Y-H. Kim, R.J. McCann, Ma-Trudinger-Wang, et. al.. Typically when X, Y are equidimensional, the cost assumptions are symmetric in the source and target variables. However in our applications we specifically focus on transport in nonequal dimensions, especially $dim(X) >> dim(Y)$. Thus we specifically distinguish between source points (X, σ) and target points (Y, τ). In applications the source space (X, σ) is given, and there is some freedom in choosing target spaces (Y, τ).

The key assumptions require the cost c to be C^2 on $X \times Y$, proper, and satisfying a (Twist) condition. The assumptions imply every c-concave function $\psi : Y \to \mathbb{R} \cup \{+\infty\}$ is Lipschitz and semiconcave, and differentiable τ-almost everywhere on Y. Therefore $dom(\nabla_x \phi)$ is a full measure subset of the activated domain $A \hookrightarrow$. The active domain A is naturally defined as $dom(\phi)$ and with a closure equal to $dom(D\phi)$. Furthermore, $D^2\psi$ is defined τ-almost everywhere on $dom(D\psi)$.

Following McCann-Pass [ref] we define the following two subsets:

$$X_1(y) := \{x \in X \mid \nabla_y c(x, y) = \nabla_y \psi(y)\},$$

and

$$X_2(y) := \{x \in X_1(y) \mid \nabla^2_{yy} c(x, y) - \nabla^2_{yy} \psi(y) \geq 0\}.$$

By definition we see

$$\partial^c \psi(y) \subset X_2(y) \subset X_1(y).$$

The definition of c-concavity implies $x \in \partial^c \psi(y)$ if and only if equality is attained in $-\phi(x) + \psi(y) \leq c(x, y)$, where $\phi = \psi^c$. But then $x \in X_1$ if $y \in dom D\psi$, and $x \in X_2(y)$ if $y \in dom D^2\psi$. Moreover X_2 is a closed subset of X_1. If $D^2_{xy} c(x, y)$ is nonsingular for every $x \in X_1(y)$, then the Inverse Function Theorem implies $X_1(y)$ is a smooth submanifold of X.

Example. We now illustrate following the approach of [McCann-Pass]. Transporting (X, σ) to a one-dimensional target measure $(\mathbb{R}, \tau)$. Suppose $c : X \times \mathbb{R}) \to \mathbb{R}$ is a cost. If $\nabla^2_{xy} c(\cdot, y)$ is nonvanishing, i.e. $\frac{\partial c}{\partial y}(\cdot, y)$ has only regular values, then all the fibres of $x \mapsto \frac{\partial c}{\partial y}(\cdot, y)$ define smooth hypersurfaces in X. The c-optimal transport map from σ to τ then has the following form. For given $y \in \mathbb{R}$, define $k = k(y) \in \mathbb{R}$ such that

$$\int_{\{x \in X \ \mid \ \frac{\partial c}{\partial y}(x,y) \leq k\}} d\sigma(x) = \int_{-\infty}^{k} d\tau(y).$$

The c-optimal transport map $F : (X, \sigma) \to (Y, \tau)$ is then defined for a given $\bar{x} \in X$ by $F(\bar{x}) = \bar{y}$, where $\bar{y}$ is the unique element satisfying

$$\sigma[\{x \in X \mid \frac{\partial c}{\partial y}(x, \bar{y}) \leq \frac{\partial c}{\partial y}(\bar{x}, \bar{y})\}] = \tau[(-\infty, \frac{\partial c}{\partial y}(\bar{x}, \bar{y}))].$$

[Insert result on regularity?]

3. Comparison with Width Inequalities

3.1. **Gromov-Guth Width Inequalities.** Given X, let $\{z_t\}$ be a sweepout of X into k-dimensional cycles. We require the sweepout to be continuous with respect to Almgren's flat chain topology. [Insert defn] Consider the maximum volume $\max_t vol_k(z_t)$ of the cycles in $\{z_t\}$, and let

$$width_k(X) := \min_{\{z_t\}} \max_t vol_k(z_t),$$

where the minimum ranges over all k-sweepouts of X. Estimates on $width_k$ imply every k-sweepout contains at least one cycle of "large" volume. The basic estimate of $width_k$ is the following comparison:

Gromov-Guth's Width Inequality 1. Let (X, g) be a compact oriented n-dimensional Riemannian manifold. Then there exists an absolute constant $C = C(n)$ depending only on the dimension such that

$$width_k(X, g)^{1/k} \leq C.vol_n(X)^{1/n}.$$

Now our task is to interpret $width_k$ in terms of optimal transportation.

On a k-dimensional target Y, consider the min-max quantity

$$\alpha_Y := \min_g \max_y \frac{g(y)}{\int_Y g},$$

where the minimum ranges over all positive continuous functions $g : Y \to \mathbb{R}_{\geq 0}$, and the maximum ranges over $y \in Y$. One readily finds the minimum is attained for the uniform distribution on Y, in which case $\alpha_Y = (vol_k(Y))^{-1}$.

Suppose c satisfies the assumptions [A]. Then π describes a measurable decomposition of X into $(n-k)$-subvarieties $\partial^c \psi(y)$. By construction, the coarea formula yields the formula

$$g(y) = \int_{F^{-1}(y)} \frac{1}{JF(x)} \cdot f(x) d\mathscr{H}^{n-k}(x). \tag{1}$$

We review the definition of the Jacobian term $JF(x)$. Consider the derivative DF of F. Then $JF(x)$ is the square root of the sum of squares of all $k \times k$-minors of DF where $k = dim(Y)$. Equivalently one has $JF(x) = \sqrt{det(DF \,.\, {}^t DF)}$.

Our goal is to evaluate the volumes of the cells $\partial^c\psi(y)$, specifically the integral

$$\int_{\partial^c\psi(y)} 1.f(x)d\mathscr{H}^{n-k}(x) = vol_{f\mathscr{H}^{n-k}}[F^{-1}(y)] = vol_{f\mathscr{H}^{n-k}}[\partial^c\psi(y)].$$

Note that we need integrate the $(n-k)$-dimensional Hausdorff measure multiplied by the density $f(x)$. Now the question is how to relate the integrals along the fibres $\partial^c\psi(y)$ when the integrands are 1 and JF^{-1}, respectively. In the simplest case, the Jacobian is constant along the fibres, although this is not typical.

For applications to the width inequalities, we study the maximum $\max_y vol_{f\mathscr{H}^{n-k}}[\partial^c\psi(y)]$, and the min-max

$$\beta_{\sigma,c} := \min_\tau \max_y vol_{f\mathscr{H}^{n-k}}[\partial^c\psi(y)] = \min_{\psi \text{ c-concave}} \max_y vol_{f\mathscr{H}^{n-k}}[\partial^c\psi(y)].$$

We include the subscripts "σ, c" in β because the subdifferential $\partial^c\psi(y)$ depends on the geometry of the cost c and source σ relative to τ. N.B. we can equivalently define β by minimizing over all c-concave potentials $\psi^{cc} = \psi$ on the target space Y, since there is essentially a unique correspondance between c-concave potentials on Y and c-optimal transports from source σ to target τ.

But can we compare $width_k(X, g)$ with $\beta_{\sigma,c}$? An elementary application of Cauchy-Schwartz inequality yields

$$\int_{\partial^c\psi(y)} 1 \le g(y)^{1/2} \cdot (\int_{\partial^c\psi(y)} JF)^{1/2},$$

where the integration is with respect to $f\mathscr{H}^{n-k}$ along the fibres. Evidently equality holds if and only if JF is constant along the fibres. N.B. the above inequality is only useful when $\int_{\partial^c\psi(y)} JF < +\infty$. When the dimension of Y is zero, then the density of the target $g(y)$ is directly related to the volume of the cells $\partial^c\psi(y)$. In general case when $dim(Y) > 0$ the Jacobian term is not so easily controlled.