# DOLD THOM, SWEEPOUTS, AND OPTIMAL TRANSPORT

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Abstract. This article introduces a generalization of Dold Thom's theorem (DT) which more completely demonstrates the natural equivalence between singular homology and spherical homotopy functors. DT represents singular reduced homology on $X$ with spherical maps on the configuration space $A(X)$ canonically basepointed at the zero state 0. The DT theorem, as developed from our perspective, provides an alternative pathway to F.J. Almgren's varifold approach to sweepouts by way of Monge-Kantorovich optimal transport.

## 1. Dold Thom

Let $X$ be a topological space and $G$ a discrete topological group, e.g. $G = \mathbf{Z}$ or $G = \mathbf{Z}/2$.

**Definition 1** (Dold-Thom Group)**.** *Let $G(X)$ be the group of finitely supported $G$-valued distributions on $X$, i.e.*

$$G(X) := \{\sum n_x x \mid \textit{only finitely many nonzero } n_x \in G\}.$$

*Let $\epsilon_X : G(X) \to G$ defined by $\epsilon_X(\sum n_x x) = \sum n_x$ be the canonical augmentation map.*

*Let $A(X) := \ker(\epsilon_X)$ be the kernel of the augmentation map $\epsilon_X$.*

We observe that $G(X)$ and $A(X)$ are topological abelian groups with canonical zero element $\emptyset = \sum 0x = 0$. This zero element $\emptyset$ corresponds to the zero distribution representing the constant $0_G$-valued distribution on $X$. One naturally sees $A(X)$ as a discretization of the space of signed Borel Radon measures $\mu$ on $X$ satisfying $\int_X d\mu(x) = 0$. The topology on $A(X)$ implies that recombinations of $\emptyset$, i.e. continuous paths $f : (S^1, pt) \to (A(X), \emptyset)$ satisfy local conservation of charge.

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That is, the supports of the distributions $f_t$ in $A(X)$ are discrete and continuous.

The purpose of Dold-Thom's theorem is to represent homology groups as homotopy groups on configuration spaces. The idea is that the vacuum state $\emptyset$ is a canonical basepoint on $A(X)$. This is key to the natural equivalence between the basepoint independant homology functors, and the basepointed homotopy groups. This remark perhaps appears trivial, but it's crucial. Our version of DT includes a relative version which demonstrates a natural equivalence between the *long exact sequence in relative homology*, and the *long exact sequence of homotopy groups*. This result does not appear in the literature.

Let $Y$ be a closed subset of $X$. Our goal is to define a relative configuration space $A(X/Y)$, and the key identification will be the canonical isomorphism

$$A(X/Y) = A(X)/A(Y).$$

In terms of net zero charged particle configurations, the idea is to view $Y$ as a *reservoir* where excess charges can "ground out". That is, spheres in the relative DT group $A(X/Y)$ basepointed at $\emptyset$ are recombinations which either neutralize away from $Y$, or neutralize at $Y$.

Here is the formal definition. If $Y$ is a closed subset of $X$, then $G(Y)$ and $A(Y)$ is a closed subgroup of $G(X)$ and there is a canonical quotient $G(X) \to G(X)/A(Y)$. Morever the augmentation map $\epsilon_X$ canonically descends to the quotient as a type of augmentation map

$$\epsilon_{X/Y} : G(X)/A(Y) \to G,$$

and we identify

$$A(X/Y) := \ker(\epsilon_{X/Y}).$$

We follow the original approach of [2], which theorems are stated in terms of natural quasifibrations between the configuration spaces. We begin with their original technical definition.

**Definition 2** (Quasifibrations)**.** *A continuous map $f : X \to Y$ between topological spaces $X, Y$ is a* quasifibration *if the canonical inclusion of fibres $f^{-1}(y)$ into the homotopy fibre of $f$ is a weak homotopy equivalence for every $y \in Y$.*

Here is the main statement of the Relative Dold-Thom Theorem. We suppress the basepoint $\emptyset$ from the notation below.

**Theorem 3** (Relative Dold-Thom Theorem)**.** *Let $Y$ be closed subspace of $X$. The short exact sequence of topological abelian groups*

$$0 \to A(Y) \to A(X) \to A(X/Y) \to 0,$$

*is a quasifibration inducing a long exact sequence of $\emptyset$-pointed homotopy groups*

$$\cdots \to \pi_{*+1}A(Y) \to \pi_{*+1}A(X) \to \pi_{*+1}A(X/Y) \to \pi_{*}A(Y) \to \cdots$$

*which is naturally isomorphic to the long exact sequence of relative homology groups*

$$\cdots \to \tilde{H}_{*+1}(Y) \to \tilde{H}_{*+1}(X) \to H_{*+1}(X, Y) \to \tilde{H}_{*}(Y) \to \cdots.$$

*Proof.* Following the original argument of Dold-Thom, the theorem reduces to verifying that the functors $X \mapsto \pi_* A(X)$ satisfy the Eilenberg-Steenrod axioms. By a standard argument it follows that the functors $X \mapsto \pi_*(A(X), \emptyset)$ and $X \mapsto \tilde{H}_*(X)$ are naturally equivalent. $\square$

Formally the homotopy groups $\pi_q(A(X), \emptyset)$ consist of homotopy classes of pointed continuous maps $f : (S^q, pt) \to (A(X), \emptyset)$. Identifying $pt$ with the point at-infinity, we can thus model homotopy classes as compactly supported $q$-parameter family of distributions on $\mathbf{R}^q$, where "compactly supported" means the distribution is equal to vacuum state outside a compact subset.

An immediate consequence of Dold Thom is that we find new topological models for the Eilenberg Maclane $K(G, n)$ classifying spaces.

**Corollary 4.** *If $Y$ is a Moore space, e.g. $Y = \mathbf{S}^q$ is a q-sphere, then $A(Y; G)$ is a model of an Eilenberg-Maclane classifying space $K(G, q)$.*

For example, we find $K(\mathbf{Z}, 2)$ is modelled by $A(S^2)$, as opposed to the usual $\mathbf{C}P^\infty$ model. Identifying $S^2$ with the Riemann sphere $\mathbf{C}P^1$, we can identify $A(S^2)$ with the set of rational functions on $\mathbf{C}P^1$. The correspondance from meromorphic functions on $S^2$ to $A(S^2)$ is given by assigning to every meromorphic function $f$ the signed group of it's divisors, i.e. poles and zeros of $f$. More generally we find $K(G, n)$ modelled by $A(S^n; G)$ for every integer $n \geq 1$ and abelian group $G$.

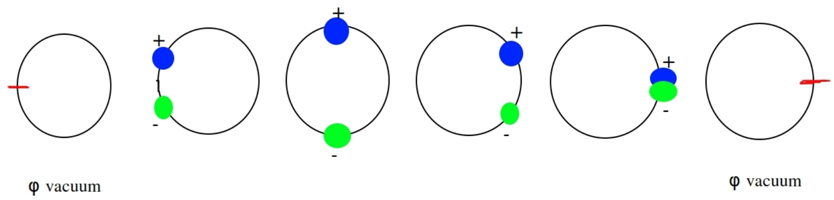


FIGURE 1. Nontrivial 1-cycle on $S^1$ represented as a loop in $A(X)$

We remark that Corollary 4 implies all the higher homotopy groups $\pi_k(A(S^n), \emptyset)$ are *trivial* for $k > n$. By contrast, it's well known that the higher homotopy groups of spheres $\pi_k(S^n, pt)$ are nontrivial for large $k$, c.f. [8].

The exactness of the long exact sequences in 3 implies

**Lemma 5.** *If $Y$ is closed subset of $X$, then we have canonical isomorphism*

$$H_*(X, Y)/image(H_*(X)) = \ker(H_{*-1}(Y) \to H_{*-1}(X)),$$

*where $image(H_*(X))$ is the image of $H_*(X)$ in the relative homology group $H_*(X, Y)$, and the morphism $H_{*-1}(Y) \to H_{*-1}(X)$ is induced by the inclusion $Y \hookrightarrow X$.*

*Proof.* Long exactness in Dold Thom's theorem 3 implies $image(H_*(X)) = \ker(\delta)$ and $image(\delta) = \ker(H_{*-1}(Y) \to H_{*-1}(X))$. But we have canonical isomorphism $H_*(X, Y)/\ker(\delta) = image(\delta)$, and the result follows. ∎

When the source space $X$ is contractible, then we find the isomorphism $H_*(X, Y) \approx H_{*-1}(Y)$ is canonical between homology groups. However the morphism is *noncanonical* on the singular chain groups. The construction of such a morphism requires a well-defined "filling" operation by which the cycles on $Y$ can be filled to relative cycles on $X$ mod $Y$. We see that relative cycles on $X$ (mod $Y$) are represented either by chains $w$ which are already cycles $\partial w = 0$ on $X$, or chains $w$ whose chain boundaries $\partial w$ are nontrivial on $Y$ but null homologous in $X$. In terms of DT, this says that if $Y$ is a ground reservoir for charges on $X$, then recombinations of $\emptyset$ will have parts which are recombinations in $X$ separated from $Y$ and other parts representing charges grounding out at $Y$. For example, any excess charge on a conductive plate will either ground out at the boundary, or recombine and neutralize in the interior. For the relative homology groups, we can identify the component of $A(X/Y)$ which is represented by "spontaneous" transports from $\emptyset$ to the reservoir $Y$. These relative cycles on the one-dimensional disk with boundary are represented in the figure below.

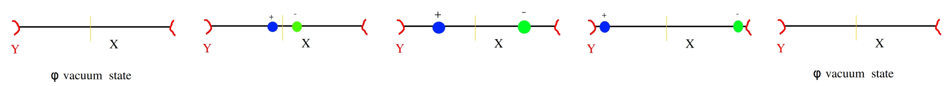


FIGURE 2. Nontrivial relative 1-cycle on the interval

## 2. Sweepouts: Implicit and Explicit Representations via Dold-Thom

In this section we introduce the idea of using DT (3) as a representation theorem to generate explicit sweepouts on a source space $X$. This problem originates with the classical Steenrod Realization problem in algebraic topology [9], namely how to constructexplicit singular chains $z \in C_q(X)$ which represent nonzero cycles $[z] \neq 0$ in $H_q(X)$. We interpret Dold-Thom (3) as providing two options: an *implicit* and an *explicit* representation of chains and cycles.

(i) Implicit representations are given by topological maps $f : X \to A(S^q)$, which define $(n-q)$-dimensional sweepouts $z = f^{-1}(y)$ for regular values $y \in A(S^q)$.

(ii) Explicit representations are given by topological maps $f : (S^q, pt) \to (A(X), \emptyset)$ of $q$-cycles as per the definition of DT (3).

The implicit representation (i) is well known construction in algebraic topology [6, 9]. Corollary 4 provides a new classifying space model for abelian coefficient groups $K(G, q) := A(S^q, G)$. The classical obstruction theory of Steenrod, Eilenberg, Maclane, etc., describes the natural equivalence between free homotopy classes of continuous maps $f : X \to A(S^q; G)$ and singular cohomology groups

$$[X, A(S^q; G)] \simeq H^q(X; G).$$

The main observation is that regular fibres $f^{-1}(y)$ are homologically nontrivial $(n-q)$-cycles in $X$. As we vary $y \in A(S^q)$, we obtain a possibly singular family of $(n-q)$-cycles $y \mapsto f^{-1}(y)$ on $X$. Thus $f$ provides an implicit representation of cycles on $X$. In practice the difficulty is constructing nontrivial classifying map $f : X \to A(S^q)$.

The explicit representation (ii) is similar to F. J. Almgren's original sweepout construction [1]. Almgren defines a sweepout as a $k$-parameter family of $q$-cycles $f_y$ which assembles "glues" to a $(k+q)$-cycle on $X$. Almgren begins by defining $Z(k,n)$ the group of relative $k$-cycles on the $n$-ball $(B^n, \partial B^n)$. This means $Z(k,n)$ consists of formal linear combinations of Lipschitz $k$-chains, which are maps from the $k$-simplex into $B^n$. Moreover Almgren topologizes $Z(k,n)$ using the flat chain topology.

A key result is Almgren's Glueing Isomorphism [1, 3]:

$$H_q(Z(k,n)) \approx H_{q+k}(B^n, \partial B^n).$$

Proving Almgren's isomorphism requires formally defining the glueing homomorphism, which takes the following form according to Dold-Thom.

**Lemma 6.** *The topological hom set $Hom := Hom((S^q, pt), (A(X), \emptyset))$ is canonically basepointed at the zero morphism $S^q \to \emptyset$. The Dold-Thom Theorem 3 implies natural isomorphisms*

$$\pi_k(Hom, \emptyset) \approx \pi_{q+k}(A(X), \emptyset) \approx H_{n+q}(X).$$

*Proof.* The natural isomorphisms are obtained by the topological smash product between pointed spheres

$$(S^q, pt) \wedge (S^p, pt') \approx (S^{p+q}, pt'').$$

Here $pt''$ is the image of $pt \times pt'$ in the smash product. The basepoint hypotheses implies that the Cartesian product factors through the smash product as required. □

In this fashion we recover Almgren's Glueing Isomorphism using DT and the smash product.

Another application of the smash product is the following natural description of cohomological cup product $\cup$ in terms of smash products and intersections of sweepouts.

**Lemma 7.** *The bilinear cup product*

$$\cup : H^p(X) \times H^q(X) \to H^{p+q}(X), \quad (f,g) \mapsto f \cup g$$

*is naturally equivalent to the pointed smash product of homotopy classes*

$$[X, A(S^p)] \times [X, A(S^q)] \to [X, A(S^{p+q})], \quad (f, g) \mapsto f \wedge g.$$

The lemma implies the squaring map $f \mapsto f \cup f$ for $f \in H^q(X)$ is naturally equivalent to the smash $f \wedge f$ for $f : X \to A(S^q)$. This suggests the following question: *What is the definition of Steenrod squares according to Dold-Thom?* Recall the Steenrod squares are cohomological operations

$$Sq^i : H^q(X, Y) \to H^{q+i}(X, Y)$$

defined for general topological spaces $X, Y$. The operations $Sq^i$ can be uniquely characterized axiomatically, but there is a topological construction of A. Hatcher [5, p. 504], [3, p. 19]. It is interesting problem to reinterpret Hatcher's construction according to Dold-Thom.

## 3. Implicit Sweepouts via Explicit Optimal Transport

We discuss the problem of generating topological sweepouts using optimal transport data. The idea was suggested by the techniques and methods of McCann-Pass [7]. Given a source $(X, \sigma)$, a target $(Y, \tau)$, and a cost $c : X \times Y \to \mathbf{R} \cup \{+\infty\}$, then Monge-Kantorovich duality generates a $c$-concave potential $\psi : Y \to \mathbf{R} \cup \{-\infty\}$ satisfying the dual max program. The $c$-subdifferential $\partial^c \psi$ defines closed subsets

$$Z(y) := \partial^c \psi \hookrightarrow X.$$

The subsets $y \mapsto Z(y)$ defines a $Y$-parameter family of subsets $Z(y) \hookrightarrow X$, for $y \in Y$.

When the cost $c$ satisfies certain assumptions, denoted (A0123) below, we find solutions to Kantorovich's dual program are unique and sufficiently regular that $y \mapsto Z(y)$, $Y \mapsto 2^X$ defines a topological sweepout of the source $X$ in the sense of Almgren, Guth [3, 4]. The assumptions (A0123) imply $Z(y)$ is a closed subvariety of $X$ and $Z(y_1) \cap Z(y_2) =: Z(y_1, y_2)$ is not necessarily empty. Therefore the sweepouts obtained via optimal transport are more general than fibres of continuous or Lipschitz maps, which fibres are necessarily disjoint.

In the dual Kantorovich program, the key definition is the $c$-subdifferential $\partial^c \psi(y) \hookrightarrow X$ for $y \in Y$. Using these $c$-subdifferential we define Kantorovich's contravariant functor $Z : 2^Y \to 2^X$ by

$$Z(Y_I) := \cap_{y \in Y_I} \partial^c \psi(y)$$

for every closed subset $Y_I \hookrightarrow Y$. We view $Z$ as defining a $Y$-parameter family of cells $\partial^c\psi(y)$ in $X$. A priori these cells need not be disjoint in $X$. There are standard assumptions on the cost $c$ such that almost every cell $Z(y)$ is an $(n-k)$-dimensional submanifold of $X$, and otherwise singular subvarieties, as discussed below.

Our applications require cost functions $c : X \times Y \to \mathbf{R}$ which satisfy the following assumptions:

**(A0)** The cost $c$ is twice-continuously differentiable throughout its domain, jointly in the source and target variables $(x, y)$, and nonnegative, and proper. Thus $c(x, y) \geq 0$ and all proper closed sublevels are compact.

**(A1)** For every $y \in Y$, we assume $x \mapsto c(x, y)$ is nonconstant on every open subset of $X$.

**(A2)** The cost satisfies (Twist) condition with respect to the source variable throughout $dom(c)$: for every $x \in X$ the rule $y \mapsto \nabla_x c(x, y)$ defines an injective mapping $dom(c_x) \to T_x X$.

The (Twist) condition (A2) is motivated by Kantorovich duality, and guarantees the uniqueness a.e. of $c$-optimal transports and gives important equation describing the fibres of the optimal transport mapping (see $X_1(y)$ defined below).

The assumptions (A012) imply every $c$-concave function $\psi : Y \to \mathbf{R} \cup \{+\infty\}$ is Lipschitz and uniformly semiconcave, and differentiable $\tau$-almost everywhere on $Y$. The active domain $A$ is naturally defined as $dom(\phi)$ with closure equal to $dom(D\phi)$. Therefore $dom(\nabla_x \phi)$ is a full measure subset of the active domain $A \hookrightarrow X$. Furthermore $D^2\psi$ is defined $\tau$-almost everywhere on $dom(D\psi)$.

Following McCann-Pass [7] we define the following two subsets:

$$X_1(y) := \{x \in X \mid \nabla_y c(x, y) = \nabla_y \psi(y)\},$$

$$X_2(y) := \{x \in X_1(y) \mid \nabla^2_{yy} c(x, y) - \nabla^2_{yy}\psi(y) \geq 0\}.$$

Clearly

$$\partial^c\psi(y) \subset X_2(y) \subset X_1(y).$$

The definition of $c$-concavity implies $x \in \partial^c\psi(y)$ if and only if equality is attained in $-\phi(x) + \psi(y) \leq c(x, y)$ where $\phi = \psi^c$. But $x \in X_1$ if $y \in dom(D\psi)$, and $x \in X_2(y)$ if $y \in dom(D^2\psi)$. Moreover $X_2$ is a closed subset of $X_1$. If $D^2_{xy}c(x, y)$ is nonsingular for every $x \in X_1(y)$, then the Inverse Function Theorem implies $X_1(y)$ is a smooth submanifold of $X$.

The key inequality $-\phi(x) + \psi(y) \leq c(x, y)$ has equality if and only if $x \in \partial^c\psi(y)$ and if and only if $y \in \partial^c\phi(x)$. If $\psi$ is differentiable at $y \in Y$, then we find $D_y c(x, y) - D_y\psi(y) = 0$ for almost every $x \in \partial^c\psi(y)$.

A key observation of McCann-Pass [7] is that if $D^2_{xy}c$ is nondegenerate, then we can apply Inverse Function Theorem and deduce $X_1(y)$ is a closed codimension $(n-k)$ submanifold of $X$ for every $y \in dom(D\psi)$. If we want $X_1(y)$ to be codimension $Y$ singular subvariety, then $D^2_{xy}c$ might degenerate, but must degenerate regularly to give a singular subvariety with constant dimension.

The optimal transport data thus provides a $Y$-parameter family of closed subsets $Z(y)$ on $X$. We seek $Y$-parameter families $Z(y)$ which define sweepouts of $X$ continuous in Almgren's flat chain topology. In general we cannot expect $Z(y)$ to be smooth for every set theoretic $y \in Y$. The classical Sard's theorem applied to the $c$-subdifferential implies the subsets $Z(y)$ are smooth submanifolds of $X$ for a.e. $y \in Y$. For applications to sweepouts, we require that $Z(y)$ represent topological cycles in $X$, i.e. have empty topological boundary. In otherwords we want the fibres $Z(y)$ to be *closed* subvarieties in $X$ where every point is an interior point. This requires a further assumption depending on the $c$-convex potential, namely

**(A3)** If $\phi$ is a $c$-convex potential on $X$, then for every $y \in Y$ we assume the gradient $\nabla_x(c(x, y) + \phi(x))$ is uniformly bounded away from zero for all $x \in \partial^c\psi(y)$.

Assumption (A3) is a strong form of Clarke's nonsmooth implicit function theorem, c.f. [10]. If the above gradient is bounded away from zero uniformly for all $x \in Z(y)$, then Clarke's theorem implies every point in $Z(y)$ is an *interior point*. The idea is the gradient controls the radius of open neighbourhoods and degenerates to zero iff $x$ approaches a "boundary point" in $\partial^c\psi(y)$.

The assumptions (A012) restrict the cost $c$. However (A3) is an assumption on the $c$-convex potential $\psi$, and implicitly restricts the cost $c$ *and* the measures $\sigma, \tau$.

If $c, \sigma, \tau, \psi, Z$ are defined as above and satisfy the Assumptions (A0123), then the following lemma proves that $y \mapsto Z(y)$ defines a sweepout of $X$ according to the conventional terminology [3].

**Lemma 8.** *Let $c, \sigma, \tau, \psi$ be defined as above and satisfying Assumptions (A0123). Suppose $\sigma$ is fully supported on $X$ and $\int_X \sigma = \int_Y \tau$. Then* *(a) The cells $Z(y) \hookrightarrow X$ are closed cycles on $X$, i.e. $Z(y)$ has vanishing boundary for every $y \in Y$.*

*(b) The cycles $Z(y)$, $y \in Y$, assemble to the fundamental class of $X$.*

*(c) The $Y$-parameter sweepout $y \mapsto Z(y)$ of $X$ is continuous in Almgren's flat chain topology.*

*Proof.* □

**Lemma 9** (Almgren Continuity)**.** *Under the above conditions, if $y_0, y_1 \in Y$ are sufficiently close, then the $(n-k)$ cycles $Z(y_0)$ and $Z(y_1)$ bound a Lipschitz $(n-k+1)$ chain $C$ of arbitrarily small area with $\partial C = Z(y_1) - Z(y_0)$.*

*Proof.* The hypotheses imply $\psi$ and $\phi := \psi^c$ are locally Lipschitz. Therefore the source potential $\phi$ has gradient which is bounded locally everywhere, and this keeps the areas of bounding chains small.

□

Without the hypothesis that $\phi, \psi$ are locally Lipschitz, it is difficult to control the convergence of cells and $\lim_{y \to y_1} Z(y) \neq Z(y_1)$ in general.

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