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Abstract.

Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ defined by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

Blum [Blu67] conjectured that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism. This implies $M(A)$ contains all the topology of A, is connected whenever A is, and formalizes Blum's idea that the medial axis is a *complete shape descriptor*. A formal proof of Blum's conjecture for bounded open subsets of $\mathbb{R}^N$ is established in [Lie04].

The present article establishes a strong deformation retract version of Blum's theorem. We define a closed subset $N = N(A) \supset M(A)$ and prove that the inclusion $N(A) \hookrightarrow A$ is a strong deformation retract.

Theorem 1. *If A is bounded open subset satisfying [assumptions], then $N(A) \hookrightarrow A$ is a homotopy-isomorphism, and the max-centre map defines a strong deformation retract $A \rightsquigarrow N(A)$.*

Our proof is independant of Lieutier's [Lie04], and is elementary, depending on our definition of *max-centre map* m (see Def. 10).

Notation: a "ball" in this article designates some Euclidean open ball contained in $\mathbb{R}^N$. The open ball centered at x with radius $r > 0$ is denoted $B_r(x)$. The directed geodesic segment between a pair of points x, y is denoted $[x, y]$.

1. Max-Radii and Max-Centres

Definition 2 (Max-Radius)**.** For $x \in A$, let $r(x) = r_A(x)$ be the maximal radius of those balls $B_r(x)$ centred at x and contained in A. Thus $r(x) := \sup\{\, r > 0 \mid B_r(x) \subset A\}$.

Lemma 3 (Max-Radius is Continuous)**.** *If A is open bounded set, then the max-radius function $r : A \to \mathbb{R}_{>0}$ defined in (2) is Lipschitz continuous.*

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Proof. The max-radius $r(x)$ is numerically equal to the distance-to-boundary $x \mapsto dist(x, \partial A)$, which is 1-Lipschitz by the triangle inequality and continuous since ∂A is compact. Thus $x, x' \in A$ satisfy $|r(x) - r(x')| \leq dist(x, x')$. $\square$

For $y_0 \in \partial A$ define the subset

$$I(y_0) := \{\ x \in A \mid d(x, y_0) \leq d(x, y) \text{ for all } \ y \in \partial A\}.$$

Equivalently one has

$$I(y_0) = \cap_{y \in \partial A} \{x \in A \mid d(x, y) - d(x, y_0) \geq 0\}.$$

This shows $I(y_0)$ is a closed subset and the intersection of A with a convex set, namely the intersection of closed halfspaces containing y_0 in the interior.

Lemma 4. *If A is a bounded open subset with C^1-boundary, then $y \mapsto I(y)$ is continuous in the Gromov-Hausdorff topology.*

Proof. If the boundary ∂A has continuously defined tangent plane at $y_0 \in \partial A$, then $I(y_0)$ is a one-dimensional arc normal to the boundary at y_0, and in this case a compact one-dimensional convex segment. $\square$

If the boundary ∂A does not admit a tangent plane at y_0, then $I(y_0)$ becomes larger and is an upper semicontinuous function of y_0. The set $I(y_0)$ is generally a nonconvex subset of A wherever A is nonconvex. However $I(y_0)$ is the intersection of A with a convex set – thus if $x_0, x_1 \in I(y_0)$ and $x_{1/2} := \frac{1}{2}(x_0 + x_1) \in A$, then $x_{1/2} \in I(y_0)$.

Lemma 5 (Unique Max-Balls)**.** *Let A be bounded open set. For every $x \in A$ there exists a unique maximal ball $M = M_x$ satisfying $B_{r(x)}(x) \subset M \subset A$.*

Proof. $\square$

N.B. Given $x \in A$, there need not exist a unique maximal ball containing x in A, but this maximal ball M is uniquely specified when we further require that M contains $B_{r(x)}(x)$.

Definition 6 (Free points)**.** A point $x_0 \in A$ is called *free* if there exists a sufficiently small open neighborhood U of x_0 in A such that M_x is strictly larger than $B_{r(x)}(x)$ for all $x \in U$, i.e. $M_x \supsetneq B_{r(x)}(x)$.

If a point x_0 is *not-free*, then the definition implies x_0 is a limit point of $M(A)$. This observation implies the following.

Lemma 7 (Non-free points are closed)**.** *The set of non-free points $N = N(A)$ of A is a closed topological subset of A.*

Proof. A point $x \in N$ is non-free iff x is an accumulation point of $M(A)$. If $\{x_k\}_k$ is a Cauchy sequence in N, then a diagonalization argument implies the limit point x_∞ is an accumulation point of $M(A)$, hence non-free in A. □

So A is free at x_0 if *all* max-radius balls $B_{r(x)}(x)$ for x sufficiently near x_0 can be strictly increased. Our definition of free and not-free points is actually equivalent to the standard loci of differentiability and nondifferentiability. Recall the max-radius function $r(x)$ is differentiable almost-everywhere on A, and $N = N(A)$ equals the domain of nondifferentiability of $r(x)$. Since differentiability is a local condition, we find $dom(Dr)$ is an open subset of full measure in A since r is Lipschitz 3.

Lemma 8. *The set of free points $A - N$ coincides with the locus of differentiability of $r(x)$, namely*

$$A - N = dom(Dr).$$

Proof. □

Now we use the definition of free points to modify the definition of max balls.

Definition 9. For $x \in A$, define M'_x to be the *minimal* ball M' which contains $B_{r(x)}(x)$ and contained in A (i.e. $B \subset M' \subset A$), and is centred over N (i.e. M' is a ball with a non-free centre).

In general we have $B_{r(x)}(x) \subset M'_x \subset M_x \subset A$. It's possible that $B_{r(x)}(x) = M'_x \neq M_x$. Following the argument of 5 we see M'_x is uniquely defined. Now we define the modified max-centre map.

Definition 10 (Max-Centre Map)**.** For $x \in A$, define $m(x)$ to be the centre of M'_x.

Thus we obtain a map $m : A \to A$, where $m(x)$ is centre of the minimum ball containing $B_{r(x)}(x)$ and centred over N. The definition is informally equivalent to defining $m(x)$ as the centre of the "maximal" ball containing $B_{r(x)}(x)$ with free centre (i.e. centred over $dom(Dr)$). However this maximum is not always strictly attained. It's more convenient to define M'_x as the minimal ball with *non-free* centre, in which case the minimum *is* attained. Thus the image of the max-centre map consists of non-free points

$$m(A) = N(A). \tag{2}$$

Our modified max-centre map involves M' and centres over N in order to address Saùl Rodrigues Martin's "jellybean" example provided in []. This example demonstrates the need to define $m(x)$ based on local data at x, and not just set-theoretically as a function of the point x. This strategy is useful to obtain the important *continuity* of m [ref].

Lemma 11 (Max-Centre Map is Continuous)**.** *If A is a bounded open subset, then the max-centre map $m : A \to A$ (Def. 10) is continuous.*

Proof. □

2. Constructing the Deformation Retract

Lemma 12. *For every $x \in A$, the Euclidean distance $dist(x, m(x))$ varies continuously with x.*

Proof. Corollary of Lemma (11). □

For every $t \in [0, 1]$, let $[x, m(x)]_t$ be the unique point on the segment which is distance exactly $t \cdot dist(x, m(x))$ from x. Now we define the deformation retract from A to $N(A)$.

Theorem 13. *For $x \in A$, $t \in [0, 1]$, the function $h(x, t) := [x, m(x)]_t$ defines a continuous strong deformation retract $h : A \times [0, 1] \to A$ from A onto $N(A)$.*

Proof. We need demonstrate:

(i) that h is continuous;
(ii) that $h(m, t) = m$ for all $m \in N(A)$ and $t \in [0, 1]$;
(iii) that $h(x, 1) \in N(A)$ for all $x \in A$.

Lemma (12) implies the continuity of h, and this proves (i). If $x \in N(A)$, then $M_x = B_{r(x)}(x)$ and $x = m(x)$, and therefore $h(x, t) = x$ for all t, and this proves (ii). Finally we clearly see $m(x) \in N(A)$, and this proves (iii). □

Corollary 14. *The inclusion $N(A) \hookrightarrow A$ is a homotopy-isomorphism.*

Proof. Immediate consequence of the continuity of h in (13). □

References

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