AN ELEMENTARY PROOF THAT BLUM'S MEDIAL AXIS TRANSFORM IS A HOMOTOPY-ISOMORPHISM

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ABSTRACT. This article presents an elementary proof that the inclusion of Blum's medial axis transform $M(A) \hookrightarrow A$ is a homotopy-isomorphism for all open bounded subsets $A \subset \mathbb{R}^N$.

Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ defined by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x,y)\} \geq 2\}. \tag{1}$$

Blum [Blu67] conjectured that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism. This implies $M(A)$ contains all the topology of A, is connected whenever A is, and formalizes the idea that the medial axis is a *complete shape descriptor*. A formal proof of Blum's conjecture for bounded open subsets of $\mathbb{R}^N$ is established in [Lie04]. The present article describes a short proof of the above homotopy-isomorphism.

Notation: a "ball" in this article designates some Euclidean open ball contained in $\mathbb{R}^N$. The open ball centered at x with radius $r > 0$ is denoted $B_r(x)$. The directed geodesic segment between a pair of points x, y is denoted $[x, y]$.

Definition 1 (Max-Radius)**.** For $x \in A$, let $r(x)$ be the maximal radius of those balls $B_r(x)$ centred at x and contained in A. Thus $r(x) := \sup\{ r > 0 \mid B_r(x) \subset A\}$.

Lemma 2 (Max-Radius is Continuous)**.** *If A is open bounded set, then the max-radius function $r : A \to \mathbb{R}_{>0}$ defined in (1) is continuous.*

Proof. The max-radius $r(x)$ is numerically equal to the distance-to-boundary $x \mapsto dist(x, \partial A)$, which is 1-Lipschitz by the triangle inequality and continuous since ∂A is compact. Thus $x, x' \in A$ satisfy $|r(x) - r(x')| \leq dist(x, x')$. ☐

Lemma 3 (Unique Max-Balls)**.** *Let A be bounded open set. For every $x \in A$ there exists a unique maximal ball $M = M_x$ satisfying $B_{r(x)}(x) \subset M \subset A$.*

Proof. The definition of the maximal ball M_x has the following variational definition. For $y_0 \in \partial A$ define the subset

$$I(y_0) := \{ x \in A \mid d(x, y_0) \leq d(x,y) \text{ for all } y \in \partial A\}.$$

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In the euclidean geometry it's clear $I(y_0)$ is a convex compact subset (possibly empty). For $x \in I(y_0)$ the max-centre map $m(x)$ is constant and given by the maximizers of the program

$$\max_{x \in I(y_0)} d(x, y_0).$$

It's clear that the maximum of the convex function $x \mapsto d(x, y_0)$ is attained on the boundary of the domain $I(y_0)$ and is unique. This unique maximizer corresponds to the max-centre $m(x)$. □

N.B. Given $x \in A$, there need not exist a unique maximal ball containing x in A, but this maximal ball M is unique specified when we further require that M contains $B_{r(x)}(x)$.

Definition 4 (Max-Centre Map)**.** For every $x \in A$, let $m(x)$ be the centre of the unique maximum ball M_x.

Lemma 5 (Max-Centre Map is Continuous)**.** *Let A be bounded open subset. Then the max-centre map $m : A \to A$ is continuous.*

Proof. The proof of (3) gives a variational definition of m. The max radius $r = r(x)$ varies continuously with x, and also the intersection $B_{r(x)}(x) \cap \partial A = \{y_0\}$. Therefore the domain $I(y_0)$ varies continuously with x. Likewise $\{m\} = argmax_{x' \in I(y_0)} d(x', y_0)$ varies continuously with y_0, and this implies the continuity of $m = m(x)$. □

Lemma 6. *Let A be bounded open subset. The image of the max-centre map $m(A)$ coincides set-theoretically with Blum's medial axis $M(A)$.*

Proof. The set-theoretic equality $M(A) = m(A)$ is obvious once we observe that every maximal ball M_x intersects the boundary ∂A in at least two distinct points. This proves $m(A) \subset M(A)$. The reverse inclusion is obvious from the definition of $M(A)$. □

Lemma 7. *For every $x \in A$, the Euclidean distance $dist(x, m(x))$ varies continuously with x.*

Proof. Corollary of Lemma (5). □

For every $t \in [0, 1]$, let $[x, m(x)]_t$ be the unique point on the segment which is distance exactly $t \cdot dist(x, m(x))$ from x. Now we define the deformation retract from A to $M(A)$.

Theorem 8. *For $x \in A$, $t \in [0, 1]$, the function $h(x, t) := [x, m(x)]_t$ defines a continuous strong deformation retract $h : A \times [0, 1] \to A$ from A onto $M(A)$.*

Proof. We need demonstrate:

(i) that h is continuous;

(ii) that $h(m,t) = m$ for all $m \in M(A)$ and $t \in [0,1]$;
(iii) that $h(x,1) \in M(A)$ for all $x \in A$.

Lemma (7) implies the continuity of h, and this proves (i). If $x \in M(A)$, then $M_x = B_{r(x)}(x)$ and $x = m(x)$, and therefore $h(x,t) = x$ for all t, and this proves (ii). Finally we clearly see $m(x) \in M(A)$, and this proves (iii). ☐

Corollary 9. *The inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism.*

Proof. Immediate consequence of the continuity of h in (8). ☐

References

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