CLOSING STEINBERG SYMBOLS OF MAPPING CLASS GROUPS

J. H. MARTEL

Abstract. This article introduces an algebraic problem we call Closing the Steinberg symbol (CS) of the mapping class group $\Gamma := Mod(S)$ of compact hyperbolic surfaces $S = S_g$ for $g \geq 2$. Solving (CS) requires finding finite subsets I of Γ which satisfy a subset sum condition, namely that the translates $\sum_{\phi \in I} \phi.\mathscr{B}$ of a certain chain sum $\mathscr{B} := \sum_i \alpha_i$ represent a nontrivial homology cycle (see §5 for details). When formal solutions I to (CS) satisfy additional metric convexity properties relative to a geometric model $X := \mathscr{T}_g$ of Teichmueller space, then we obtain an interesting supply of candidate equivariant deformation retracts $X \rightsquigarrow \mathscr{Z}$ onto subvarieties $\mathscr{Z} \hookrightarrow X$. The construction of these retracts and their properties is based on the author's Reduction-to-Singularity method [Mar] and [Mar22]. Thus we examine the problem of Closing Steinberg symbols as an important tool for constructing small dimensional $E\Gamma$ models and candidate spines.

Contents

1. Mapping Class Group and Bieri-Eckmann Duality

We let $S = S_g$ be a closed hyperbolic surface of genus $g \geq 2$, and let $\Gamma := Mod(S)$ be the mapping class group of S. The topologists define

$$Mod(S) := \pi_0(Diff_+(S))$$

Date: February 21, 2023.

as the group of orientation preserving diffeomorphisms of S modulo isotopy. The algebraists define $Mod(S)$ via Dehn-Nielsen-Baer's theorem

$$Mod(S) = Out(\pi_1(S))$$

where $\pi_1(S) = \pi_1(S, pt)$ is Poincaré's pointed fundamental group, see [FM11].

The group-theoretic (co)homology of Γ is defined via the symmetries of proper discontinuous actions $\Gamma \times X \to X$ on $E\Gamma$ models X. There is extensive literature on this subject, c.f.[Bro82]. The standard $E\Gamma$ model for the mapping class group is Teichmueller's $X = \mathscr{T}_g$, a topological $(6g-6)$-cell $\simeq \mathbb{R}^{6g-6}$, e.g. [Hub06]. We assume here the basic facts that Γ acts proper discontinuously on $\mathscr{T}_g$ with finite covolume. Of course $\mathscr{T}$ is a cell by Teichmuller's theorem and therefore topological trivial. However the Γ-equivariant topology of $\mathscr{T}$ is highly nontrivial.

It is a fundamental observation of Harvey [Har81], Harer [Har86], Ivanov [Iva15], that Γ is a Bieri-Eckmann virtual duality group [BE73]. Strictly speaking we must replace Γ with a finite-index torsion-free subgroup Γ', however there is important role played by the torsion elements in Γ, as we illustrate below.

A key role in our analysis is played by the action of Γ on the simplicial curve complex $\mathscr{C}$ and its reduced singular homology and chain groups. Recall the curve complex $\mathscr{C} = \mathscr{C}(S)$ of the surface is the simplicial complex whose 0-skeleton (vertices) $\mathscr{C}^0$ consists of simple closed curves (modulo isotopy), and where a simplex exists between vertices $a, b, c, \ldots$ if the curves are simultaneously pairwise disjoint on S. For example in genus $g = 2$, the curve complex $\mathscr{C}(S_2)$ is a two-dimensional infinite simplicial complex. Obviously Γ acts on $\mathscr{C}$, c.f. [Bro12]. Now we present the formal definition of homological duality.

Definition 1. A finitely generated group Γ is a duality group of dimension $\nu \geq 0$ with respect to a $\mathbb{Z}\Gamma$-module $\mathbf{D}$, if there exists an element $[B] \in H_\nu(\Gamma; \mathbf{D})$ with the following property: for every $\mathbb{Z}\Gamma$-module A, the "cap-product with $[e]$" defines $\mathbb{Z}\Gamma$-module isomorphisms $H^d(\Gamma; A) \approx H_{\nu-d}(\Gamma; A \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$, $f \mapsto f \cap [e]$.

The basic properties of Bieri-Eckmann's homological duality are summarized in the following theorem from [BE73].

Theorem (Bieri-Eckmann)**.** *Let Γ be duality group of dimension ν, with dualizing module $\boldsymbol{D}$. Then*

(i) we have $\mathbb{Z}\Gamma$-isomorphism $\boldsymbol{D} \approx H^\nu(\Gamma; \mathbb{Z}\Gamma) \neq 0$, so $\boldsymbol{D}$ is a torsion-free additive abelian group;

(ii) the homology group $H_\nu(\Gamma; \boldsymbol{D})$ is infinite cyclic generated by $[e]$ as additive abelian group;

(iii) the group Γ has cohomological dimension $cd(\Gamma)$ equal to ν.

Proof. The statements are direct consequences of Definition 1. (i) We see $H^\nu(\Gamma; \mathbb{Z}\Gamma)$ $\approx H_0(\Gamma; \mathbf{D}) \approx \mathbf{D}$. (ii) Duality implies $H^0(\Gamma; \underline{\mathbb{Z}})$ is isomorphic to $H_\nu(\Gamma; \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$, which in turn is canonically isomorphic to $H_\nu(\Gamma;\mathbf{D})$ since $\underline{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \approx \mathbf{D}$. But $H^0(\Gamma, \underline{\mathbb{Z}})$ is canonically isomorphic to $\underline{\mathbb{Z}}$. (iii) The duality isomorphism implies for every $\mathbb{Z}\Gamma$-module A that $H^*(\Gamma; A)$ is isomorphic to $H_{\nu-*}(\Gamma; A \otimes \mathbf{D})$ which reduces to 0 whenever $\nu - * < 0$. $\square$

Most importantly for the mapping class group Γ, the dualizing module $\mathbf{D}$ can be identified with the reduced homology of the simplicial curve complex. That is, we can identify $\mathbf{D}$ up to $\mathbb{Z}\Gamma$-isomorphism as $\mathbf{D} = \tilde{H}_*(\mathscr{C}; \mathbb{Z})$, where the reduced homology group inherits the natural structure of $\mathbb{Z}\Gamma$-module from the action of Γ on $\mathscr{C}$. The curve complex has the homotopy-type of a countable bouquet of $(2g-2)$-dimensional spheres, c.f. [Iva15], [Har86]. Thus we deduce that

$$vcd(\Gamma) = 6g - 6 - (2g - 2) + 1 = 4g - 5.$$

For example in genus $g = 2$, this implies that $\mathscr{T}_2$ is a 6-dimensional $E\Gamma$ model, and the quotient $\Gamma \backslash \mathscr{T}_2$ has the topology of a 3-dimensional complex. Thus arises the problem of constructing spines or souls for Γ and we seek explicit constructions or realizations of this 3-dimensional singular subvariety for genus $g = 2$. Constructing such subvarieties is the purpose of our (CS) program, as we describe below.

The problem of (CS) is based on homological properties of Γ, and specifically a representation of Bieri-Eckmann's dualizing module $\mathbf{D}$ with the reduced homology of an excision boundary $\mathbf{D} \approx \tilde{H}_*(\partial\mathscr{T}[t]; \mathbb{Z})$, where $\mathscr{T}[t]$ is a maximal Γ-rational excision of Teichmueller space for a sufficiently small parameter t. We elaborate below.

2. Canonical Riemannian metrics on $\mathscr{T}_g$ and Flat Filling

The Teichmueller space $X = Teich(S)$ is not immediately constructible from its definition and Teichmueller's theorem that X is topologically homeomorphic to a $(6g-6)$ cell remains somewhat abstract. The geometrization of X requires a choice of G-equivariant structure, for example an equivariant Riemannian metric. Popular constructions of G-equivariant metrics d on X include Teichmueller's original metric d_{Teich} [Hub06], Weil-Peterson's metric d_{WP} [Hub06], Thurston's metric [Wol86], or McMullen's [McM00]. But none of these constructions appear satisfactorily canonical, although Weil-Peterson's d_{WP} has the most convenient geometric properties thus far.

The choice of invariant Riemannian metric d enters into our constructions by our application of *excisions*. More specifically for a choice of invariant Riemannian metric d on $\mathscr{T}$ and invariant function

$$t : \mathscr{C}^0 \to \mathbb{R}_{>0}$$

we define an excision $\mathscr{T}[t]$ of $\mathscr{T}$ by excising ("scooping out") horoballs centred at various Γ-rational points at-infinity λ and with radius $t(\lambda)$. We focus on these Γ-rational horoballs of $\mathscr{T}$ because they have Γ-invariant boundary horospheres. Thus

Γ acts proper discontinuously on both $\mathscr{T}[t]$ and $\partial\mathscr{T}[t]$, and we obtain the important diagonal action

$$\Gamma \times \mathscr{T}[t] \times \partial\mathscr{T}[t] \to \mathscr{T}[t] \times \partial\mathscr{T}[t].$$

We further emphasize those parameters t which are sufficiently small such that the excisions $\mathscr{T}[t]$ have a Γ-equivariant topological boundary $\partial\mathscr{T}[t]$ with the homotopy-type of $\mathscr{C}$. This important fact implies a canonical equivariant isomorphism

$$\mathbf{D} \approx \tilde{H}_*(\partial\mathscr{T}[t]; \mathbb{Z}). \tag{1}$$

For the constructive topologist the Γ-action on $\mathscr{T}$ is more accesible than the abstract algebraic action on $\mathbf{D}$. The homologically essential spheres of $\mathscr{C}$ can be viewed as spheres at-infinity within the excision $\mathscr{T}[t]$. The Γ-orbit of these spheres and their singular chain sums generates an important topological $\mathbb{Z}\Gamma$-module called the *Steinberg module*. Following convention we designate the generator of this module a Steinberg symbol B. For further references to Steinberg ("modular") symbols, we refer the reader to [Man72], [AR79], [AGM], [Ste07], [Sol] and references therein.

The contractibility of $\mathscr{T}[t]$, $\mathscr{T}$, and the long exact sequence in relative homology implies the natural boundary morphism

$$\delta : C_*(\mathscr{T}[t], \partial\mathscr{T}[t]) \to C_{*-1}(\partial\mathscr{T}[t])$$

is an isomorphism. Here C_* denotes the singular chain groups. However what we require for applications is an inverse operation which is well-defined directly on singular chains, namely

$$FILL := \delta^{-1} : C_{*-1}(\partial\mathscr{T}[t]) \to C_*(\mathscr{T}[t], \partial\mathscr{T}[t]).$$

This inverse operation is a *filling* operation and requires a choice of metric d'. For our applications, any metric d' with the following properties is sufficient:

- (M1) the metric d' is metrically complete and proper in the interior of $\mathscr{T}$;
- (M2) the metric has nonpositive sectional curvature ($\kappa \leq 0$) in $\mathscr{T}$;
- (M3) the $(2g-2)$-dimensional spheres generating the Steinberg symbol at infinity admit unique d'-flat fillings ($\kappa = 0$) to relative cycles in $\mathscr{T}[t]$ mod $\partial\mathscr{T}[t]$.

If we examine the usual metrics, we find the WP metric has properties (M1), (M2). We observe that (M2) has the important consequence that WP-horoballs are geodesically convex in $(\mathscr{T}, d_{WP})$, c.f. [Gro91]. With respect to Teichmuller's original metric, we know (M1) holds, but (M2) fails and Teichmueller's original metric has regions of positive curvature. Moreover recent work of [MR16] shows that convex hull constructions are not possible in Teichmueller's metric. The work of S. Wolpert implies that d_{WP} also satisfies (M3).

3.

Lemma 2 (Wolpert)**.** *Let Q be a geodesic pants decomposition of the hyperbolic surface (S, g). Let $\mathscr{Q}$ be the submanifold of $\mathscr{T}$ passing through (S, g), and such that the Nielsen twist tangents $t(a)$ vanish for every geodesic curve $a \in Q$. Then $\mathscr{Q}$ is a totally flat submanifold of $\mathscr{T}$ having vanishing sectional curvatures with respect to the WP metric d_{WP}.*

Proof. $\square$

Equivalently we find $\mathscr{Q}$ consists of all hyperbolic surfaces such that the dual geodesic pants Q^* intersect Q orthogonally. This is equivalent to saying $\mathscr{Q}$ consists of all surfaces obtained from (S, g) by varying the length parameters ℓ_a $(a \in Q)$ and fixing all twists equal to zero in Fenchel-Nielsen coordinates associated to Q.

4.

Any metric d satisfying the properties (M123) on $\mathscr{T}$ readily leads to the construction of equivariant homotopy reductions of $\mathscr{T}$ onto candidate spines $\mathscr{X}$ according to [Mar]. Specifically when (M3) holds, the Steinberg symbol B canonically fills to a flat relative cycle $(P, \partial P)$ in $(\mathscr{T}[t], \partial\mathscr{T}[t])$. The flat relative cycles $P = FILL[B]$ and their Γ-translates are called "panels". The motivation for the terminology is given in §5 below. The panels are homologically nontrivial relative cycles in $\mathscr{T}[t]$ modulo the boundary $\partial\mathscr{T}[t]$. In practice the condition (M3) could be slightly weakened since the application of our reduction to singularity method [Mar] does not strictly need the Steinberg symbols to admit flat-fillings – what's essential is the geometric uniqueness of the fillings.

The following is important lemma for our method.

Lemma 3. *Let d be a metric on $\mathscr{T}$ satisfying properties (M123). Let $\mathscr{T}[t]$ be a Γ-rational excision with sufficiently small parameter t such that the canonical isomorphism (1) holds. Let B be a Steinberg symbol with flat-filling $P = FILL[B]$. Then P has zero geometric self-intersection in the quotient $\Gamma\backslash\mathscr{T}[t]$, and the quotient projection $\mathscr{T}[t] \to \Gamma\backslash\mathscr{T}[t]$ maps P isometrically onto its image.*

Proof. $\square$

To motivate Lemma 3, recall that if S is a closed hyperbolic surface and α is a closed geodesic on S, then the lifts $\tilde{\alpha}$ of α to the universal covering $\tilde{S}$ form a $\pi_1(S)$ orbit in $\tilde{S}$ where all the translates are disjoint. Likewise Lemma 3 asserts that the Γ-translates of P are *disjoint* in the interior of $\mathscr{T}[t]$.

By contrast the existence of parabolic elements γ in Γ shows that the relative cycle P and its parabolic translates $\gamma.P$ intersect asymptotically "at infinity" when $t \to 0^+$. However there remains no self-intersection in the *interior* of $\mathscr{T}[t]$.

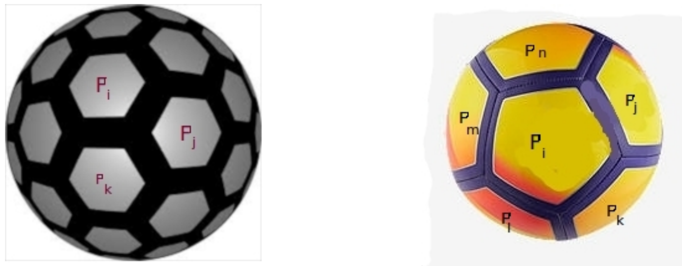

FIGURE 1. Isometric translates of hexagonal and pentagonal panels P_i, P_j, P_k, etc., assemble to closed balls.

5. Closing Steinberg Symbols: Definition and Properties

The problem of Closing Steinberg is informally related to stitching a closed football F from a sequence of panels $\{P_i\}_{i\in I}$. The panels P_i are required to have the property that $F = conv\{P_i \mid i \in I\}$ and such that $\sum_{i\in I} \partial P_i = 0$ over $\mathbb{Z}/2\mathbb{Z}$-coefficients. In otherwords the problem requires finding a sequence of panels P_i $(i \in I)$ which assemble to a closed compact convex subset F as defined above. The panels $P = P_i$ of the above footballs are analogous to the flat-filled Steinberg symbols $P = FILL[B]$ and their translates $\gamma.B$ $(\gamma \in \Gamma)$. Compare Figure 1.

Now we present the formal definition of (CS) as derived from Bieri-Eckmann's homological duality [BE73], [BS73]. The key algebraic construction is the definition of homology with coefficients in a chain complex, [Bro82], where the problem of (CS) amounts to constructing a nontrivial 0-cycle

$$\xi \in H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbf{D}).$$

Here the tensor product $\otimes$ is in the category of $\mathbb{Z}\Gamma$ modules. If Γ is a Bieri-Eckmann duality group, then we find isomorphisms

$$H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbf{D}) \approx H^\nu(\Gamma; \underline{\mathbb{Z}_2}\Gamma) \approx \underline{\mathbb{Z}_2} \otimes_{\mathbb{Z}} \mathbf{D} \neq 0.$$

This fact implies the formal existence of nontrivial 0-cycles.

The group Γ of symmetries flips, rotates, and translates the base cycle $[P]$ throughout the space, and every finite subset I of Γ produces a finite chain sum

$$\sum_{\gamma\in I} \gamma.[P],$$

with total chain boundary

$$\partial(\sum_{\gamma\in I} \gamma.[P]) = \sum_{\gamma\in I} \gamma.\partial[P].$$

The basic problem of Closing Steinberg is to produce a finite subset $I \subset \Gamma$ for which the boundary of the *nontrivial* chain sum $\sum_{\gamma \in I} \gamma.[P]$ vanishes in the mod 2 homology group. The complete definition of Closing Steinberg includes further geometric conditions on the Γ-translates $\Gamma.F$ of the the closed convex hull $F = conv[P.I]$ of the translates $B.I$. Let $\mathscr{T}[t], \partial\mathscr{T}[t]$ be a Γ-invariant excision of $\mathscr{T}$. Let $[P]$ be a flat-filled relative cycle representing a nonzero generator of $H_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z})$.

Definition 4 (**Closing Steinberg**). A finite subset I of Γ successfully Closes Steinberg if:

(i. nontrivial mod 2) the chain $\xi = \sum_{\gamma \in I} \gamma.P$ is nonvanishing over $\mathbb{Z}/2$ coefficients in the chain group $C_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z}/2)$;

(ii. vanishing boundary mod 2) the boundary $\partial\xi = \sum_{\gamma \in I} \gamma.\partial[P]$ vanishes over $\mathbb{Z}/2$-coefficients in the homology group $[\partial\xi] = 0$ in $H_q(\partial\mathscr{T}[t]; \mathbb{Z})$;

(iii. well-defined geometric convex hull) the boundary-chain representing $\partial\xi$ is simultaneously visible from at least one interior point x in $\mathscr{T}[t]$;

(iv. well-separated gates) there exists a finite-index subgroup $\Gamma' < \Gamma$ such that the chain sum $\underline{F} = \sum_{\gamma \in \Gamma'} \gamma.F$ has nonempty *well-separated gates* precisely equal to the principal orbit $\{\gamma.P \mid \gamma \in \Gamma'\}$.

Our definition of Closing Steinberg was inspired by the author's study of [Cre84]. In Cremona's terminology, the problem is to determine a "relation ideal $\mathscr{R}$" and construct a "basic polyhedron P whose transforms fill the space", c.f.[Cre84, pp.290].

The hypotheses (i)–(ii) basically require the chain sum ξ to be nonzero mod 2. The hypotheses (iii)–(iv) are convexity assumptions which need be verified for any nonzero chain. The hypothesis of well-separated gates is related to the following fact: the translates $P, \gamma.P$, for $\gamma \in \Gamma$, are either identical or geometrically disjoint in $\mathscr{T}[t]$ according to Lemma 3. However the translates $P, \gamma.P$ may have nontrivial intersection at infinity in the initial Teichmueller space $\mathscr{T}$. In fact the problem of (CS) is precisly to find such nontrivial intersections at infinity, although again the intersections are disjoint in the interior of $\mathscr{T}[t]$.

Obviously the group structure of Γ allows us to restrict ourselves to subsets I containing the identity mapping class $Id \in \Gamma$. In practice, formal solutions to (CS) can often be found among the torsion elements and finite subgroups of Γ, c.f. [Cre84].

Proposition 5. *Let Γ be a Bieri-Eckmann duality group with dualizing module $\boldsymbol{D}$. Then there exists finite subsets I in Γ for which $\xi = \sum_{\gamma \in I} \gamma.P$ lies in the kernel of ∂_0 over $\mathbb{Z}/2$.*

Proof. The argument is homological. We interpret ξ as a chain sum representing a 0-cycle in $H_0(\Gamma; \mathbb{Z}/2\Gamma \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$. The hypotheses of Closing Steinberg imply ξ is homologically nontrivial cycle. Bieri-Eckmann duality (Proposition 1) implies the kernel $\ker \partial_0$ is naturally isomorphic to the induced $\mathbb{Z}\Gamma$-module $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbf{D}$ which is nonzero. $\square$

To illustrate, let the reader observe that a typical element $\phi \in Mod$ will *totally displace* the curves in $\mathscr{B}$ such that $\mathscr{B} \cap \phi.\mathscr{B} = \emptyset$ for almost every $\phi \in Mod$. On the other hand, if ϕ' permutes the curves such that $\phi.\mathscr{B} = \mathscr{B}$, then $I = \{Id, \phi'\}$ would be a solution of (2). However we consider this solution to be trivial in the following sense: the formal sum $[\mathscr{B}] + \phi'.[\mathscr{B}] = 2[\mathscr{B}] = 0$ is itself vanishing mod 2. Such trivial solutions are avoided in the case of higher genus closed surfaces as the work of [BBM13] demonstrates, i.e. the identity element is the only mapping class which permutes $\mathscr{B}$. Obviously parabolic type elements ϕ, i.e. curve stabilizers $\phi \in Mod_\gamma$ satisfy$\mathscr{B} \cap \phi.\mathscr{B} \supset \{\gamma\}$. So naturally one is tempted to find formal solutions to (CS) by choosing a suitable sequence of parabolics.

Our hypotheses regarding Closing Steinberg have useful consequences, which we summarize in the following theorem.

Theorem 6. *Suppose $I \subset \Gamma$ successfully Closes Steinberg (Definition 4). Define $F := conv[I.P]$. Then*

(i) the Γ-translates $\gamma.F$ ($\gamma \in \Gamma$) form a chain sum

$$\underline{F} := \cdots \gamma.[F] + \gamma'.[F] + \gamma''.[F] + \cdots ,$$

and there exists finite-index subgroup $\Gamma' < \Gamma$ which acts as additive shift-operator on the summands of $\underline{F}$; and

(ii) the support of the chain sum $\underline{F}$ is a simply-connected subset of X, and $\underline{F}$ is a cubical $E\Gamma'$ model.

Proof. We can replace Γ with a finite-index torsion-free subgroup Γ' to ensure Γ' acts freely on X, and therefore the diagonal action is free on $X[t] \times \partial X[t]$. Moreover we can ensure Γ' translates the flat-filled relative cycle $\gamma.[P]$, for $\gamma \in \Gamma'$ freely. Then $\gamma.[P] \neq [P]$ when $\gamma \neq Id$. The definition of Closing Steinberg implies distinct translates F, F' are disjoint unless they intersect in a gate $G' = \gamma'.P$ for some $\gamma' \in \Gamma'$. So $\gamma.F$ equals F only if $\gamma = Id$ is trivial. This proves the summands $\{\gamma.F \mid \gamma \in \Gamma'\}$ of $\underline{F}$ form a principal Γ'-set, and establishes (i). The existence of an interior point $x \in F$ which is simultaneously visible to the translates $P.I$ in $X[t]$ proves $F = conv[P.I]$ is a compact convex set, and homeomorphic to some cube. Thus $\underline{F}$ is a chain sum of cubes, hence a cubical chain sum and therefore (ii). $\square$

The above chain sum $\underline{F}$ provides a convenient global coordinate system on an open domain of X, namely the support of $\underline{F}$. In practice we find the convex hull F more computable and effective than the standard "fundamental domain" of Γ on X, which fundamental domain is noncanonical and noneffective. In the next section we

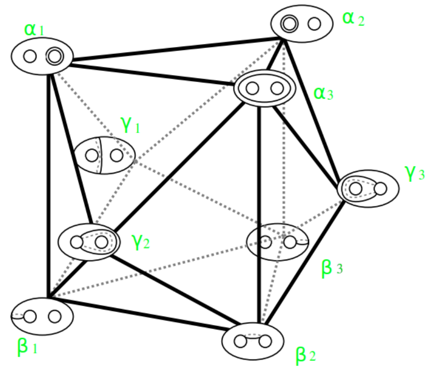

FIGURE 2. Homologically nontrivial 2-sphere in the curve complex $\mathscr{C}$ of genus 2 closed surface. Figure adapted from [Bro12, Fig.10]

illustrate these definitions with some explicit computations in genus $g = 2$, namely $\Gamma = Mod(S_2)$.

6. Closing Steinberg for Genus Two Mapping Class Group

To illustrate our ideas, we now study the case of genus two closed Riemann surface. The duality theory of mapping class groups $\Gamma = Mod(S_g)$ for genus $g = 2$ has been described by [Bro12]. For reference we include the following figure taken from [Bro12, Fig.10], see (2).

The formal problem of (CS) for genus two surfaces has the following symbolic setup. Let $V := \mathbb{Z}/2(\mathscr{C}^0)$ be the abelian topological group consisting of finitely-supported $\mathbb{Z}/2$-valued functions $f : \mathscr{C}^0 \to \mathbb{Z}/2$ on the set $\mathscr{C}^0$ of free homotopy classes of simple closed curves on a surface S. We abbreviate such a function f with its support $\alpha + \beta + \ldots$. On the genus two closed surface, consider Broaddus' set of nine curves $\alpha_i, \beta_j, \gamma_k$ for $i, j, k \in \{1, 2, 3\}$, and the formal sum

$$\mathscr{B} := \sum_{i,j,k=1}^{3} \alpha_i + \beta_j + \gamma_k. \tag{2}$$

Now the problem becomes explicit:

Definition 7. A finite subset $I \subset \Gamma$ formally Closes Steinberg for the mapping class group $\Gamma := Mod(S_2)$ of genus two closed surfaces if

$$\sum_{\phi \in I} \sum_{\alpha \in \mathscr{B}} \phi.\alpha = 0 \ (mod\ 2) \tag{3}$$

where the zero element 0 on the right hand side is the zero element in V, i.e. the constant zero-valued distribution on $\mathscr{C}^0$.

The "vanishing mod 2" of the translates $\sum_{\phi\in I}\phi.\mathscr{B}$ says there is an even number of coincidences between the translated curves $\phi.\alpha$ where $\phi\in I$, $\alpha\in\mathscr{B}$. This can be implemented on python by iterated symmetric differences. For example if $\mathscr{B}$ denotes the *set* of curves defined in (2), then a finite subset $I=\{\phi_1,\dots,\phi_n\}$ is formal solution to (CS) if and only if

$$(\phi_1.\mathscr{B})\Delta\cdots\Delta(\phi_n.\mathscr{B})=\emptyset. \tag{4}$$

We can omit the parentheses since the symmetric difference Δ is an associative operator. Thus we view the equations (2) and (4) as equivalent, although (4) is more useful in practice.

As we indicated earlier, it's interesting to search for formal solutions of (CS) among the torsion elements of Γ. It's convenient to now introduce notation following [NN18]. Let a,b,c,d,e,f be the standard set of Humphries generators for Γ. Then

$$\eta=aecf$$

is an order ten element in Γ. If we define

$$\mu:=\eta^4,$$

then μ is an order five element in Γ. If a,b,c is a geodesic pant decomposition of S, then we define the chain sum

$$B:=[a]+[b]+[c]+\mu.[a]+\mu.[b]+\mu.[c].$$

Lemma 8. *Let $I_0:=\{Id,\mu,\mu^2,\mu^3,\mu^4\}$. Then $\sum_{\phi\in I_0}\phi.B=0$ (mod 2) and I_0 is a formal solution to (CS).*

Proof. The vanishing of the chain sum $\sum_{\phi\in I_0}\phi.B$ is clear. Moreover all the summands $\phi.B$ are distinct for $\phi\in I_0$ and this proves the formal solution is nontrivial. □

We have implemented these calculations using Mark C. Bell's curver program. Using curver the author has developed a jupyter notebook to search for more solutions of (CS) in genus $g=2$. For example, with curver we find that the I_0-translates of B are supported on exactly *ten* curves, where we are counting only the nonseparating curves in Broaddus' Figure 2. Indeed we use curver to compute the conjugacy action of mapping classes on Dehn twists, and using curvers ability to effectively distinguish conjugacy classes. This conjugacy action on Dehn twists is basically equivalent to the action of mapping classes on the curve complex $\mathscr{C}^0$.

In our definition of (CS) we have distinguished between *formal* solutions and the geometric solutions, namely those satisfying the further conditions of 4. Given the formal solution I_0, we proceed in several steps to verify whether the geometric conditions are satisfied. First we need construct the convex hull

$$F:=conv(I_0.B)$$

over these ten curves constituting the I_0 translates of B. Then we need estbalish that the chain sum

$$\underline{F} := \sum_{\phi \in \Gamma} \phi.F$$

has a *well separated gates structure* equal to $\Gamma.B$. The idea of well separated gates is introduced in [Mar, §5.1], and we briefly review the definition. We say the chain sum $\underline{F} = \sum_{i \in I} F_i$ has well separated gates, if the intersections $G := F_{ij} := F_i \cap F_j$ form a principal Γ-set. In other words, the intersections F_{ij} are either empty or isometric to a fixed gate G. The verification of well separated gates is not readily performed on curver. This leads us to following computational problem, where $F = conv(I.B)$ is defined as above.

Problem. *Devise an effective algorithm by which the intersection of the convex hulls $F \cap \phi.F$ can be determined for mapping classes $\phi \in \Gamma$.*

7. Closing Steinberg and Reduction to Singularity

Our goal in this final section is to indicate how solutions to (CS) satisfying Definition 4 lead to constructive equivariant retracts of $\mathscr{T}$ onto subvarieties $\mathscr{Z}$ with large codimensions. In §1 we described how the mapping class group is a Bieri-Eckmann duality group having a very important $\mathbb{Z}\Gamma$ module called the Steinberg module. The topology of this module essentially controls the virtual cohomological dimension of the group and the codimension of equivariant retracts. In fact the spine is essentially Poincare-Lefschetz dual in the sense of intersection theory to the Steinberg symbol, and the max codimension spine will geometrically intersect every flat filled Steinberg symbol in a point.

As we described earlier, we begin with a Γ-rational excision $\mathscr{T}[t]$ of $\mathscr{T}$ with respect to a sufficiently small equivariant parameter t. Thus we obtain a manifold-with-corners $(\mathscr{T}[t], \partial\mathscr{T}[t])$. The rationality implies that Γ acts proper discontinuously on $\mathscr{T}[t] \times \partial\mathscr{T}[t]$, and therefore there exists nontrivial equivariant Borel-Radon measures on $\mathscr{T}[t]$ and $\partial\mathscr{T}[t]$. This means we can study optimal semicoupling programs between source measures σ on $\mathscr{T}[t]$ to target measures τ on $\partial\mathscr{T}[t]$ with respect to continuous costs $c : \mathscr{T}[t] \times \partial\mathscr{T}[t] \to \mathbb{R}_{\geq 0}$. Here the source σ, target τ, and cost c, need to be chosen a priori by the practitioner to begin studying the c-optimal semicouplings from σ to τ. In practice we take σ and τ to be the uniform Lebesgue measures on $\mathscr{T}[t]$ and $\partial\mathscr{T}[t]$ respectively. The Lebesgue measure is defined uniquely up to a multiplicative constant, and our semicoupling hypothesis requires only that $\rho := \int \sigma / \int \tau > 1$ be arbitrarily close to unity, i.e. $\rho \approx 1^+$. The choice of cost c is more interesting, as our thesis finds best results are obtained with *repulsion costs.* The physical idea is to imagine σ and τ as distributions of negative electrical charges, and the semicoupling program looks for an energy-minimizing correlation of some variable amount of source charges from σ to the fixed target charge τ. However the evaluation of such energy costs is not easy, and we use solutions of (CS) to construct more effectively computable costs, namely our so-called gated repulsion costs

$$\underline{c} : (\underline{F} \cap \mathscr{T}[t]) \times (\underline{F} \cap \partial\mathscr{T}[t]) \to \mathbb{R}_{>0} \cup \{+\infty\}. \tag{5}$$

As demonstrated in the author's thesis [Mar] and [Mar22], if $\underline{c}$ is a gated repulsion cost, then we obtain a contravariant functor

$$Z : 2^{\partial\mathscr{T}[t]} \to 2^{\mathscr{T}[t]},\ Y_I \mapsto Z(Y_I) = \cap_{y\in Y_I} \partial^c \psi(y),$$

where $\psi = \psi^{cc}$ is the c-concave Kantorovich potential defined by the maximizers of the dual Kantorovich program. We refer the reader to the above papers for details, but the functor and our topological deformation retract theorems lead to a singularity locus $\mathscr{Z} \hookrightarrow \mathscr{T}[t]$ such that the inclusion is a strong deformation retract. The exact dimension of $\mathscr{Z}$ depends on a local criterion we call Uniform Halfspace Condition, but generally it can be expected that $codim_{\mathscr{T}[t]}\mathscr{Z} \geq 2$. Symmetries in the excision boundary and target measure on $\partial\mathscr{T}[t]$ increases the maximal codimension of $\mathscr{Z}$ with the possibility of attaining the *maximal* codimension, even the equivariant spine of $\mathscr{T}$. The gated structure on $\underline{F}$ and $\underline{c}$ allows us to localize all computations to local chain summands F of $\underline{F}$. If $\underline{F}$ is a chain sum with well-separated gates $\{G\}$, then the singularity locus $\mathscr{Z}$ naturally decomposes as a chain sum $\mathscr{Z} = \sum_i \mathscr{Z} \cap F_i$, and where $\mathscr{Z} \cap F_i$ is the singularity locus of a restricted semicoupling program, with respect to the restricted cost $\underline{c}|F_i$. Using the Reduction to Singularity method of [Mar, Theorems 1.4.1-2], we naturally construct continuous deformation retracts which assemble to global continuous retracts $\mathscr{T} \rightsquigarrow \mathscr{Z}$.

Further details and explicit computations are still required, and this article does not claim to positively construct any spines of the mapping class groups. The author's constructions are obstructed by the inability to positively verify the well separated gates structure of the chain sums arising from solutions to Definition 4. However if Teichmueller space $\mathscr{T}$ was more computable, i.e. if we could explicitly parameterize $\mathscr{T}$ and compute the energies of negative electric charge distributions like in Euclidean space, then we could more readily compute solutions to the optimal transport programs, and more importantly the homotopy types of the singularities.

References

[AGM] A. Ash, P.E. Gunnells, and M. McConnell. "Resolutions of the Steinberg Module for $GL(n)$". In: (). URL: https://www2.bc.edu/avner-ash/Papers/Steinberg-AGM-V-6-23-11-final.pdf (visited on 05/20/2017).

[AR79] A. Ash and L. Rudolph. "The Modular Symbol and Continued Fractions in Higher Dimensions". In: *Inventiones math.* 55 (1979), pp. 241–250. URL: https://eudml.org/doc/186135.

[BE73] R. Bieri and B. Eckmann. "Groups with homological duality generalizing poincare duality". In: *Inventiones. Math.* 20 (1973), pp. 103–124. URL: https://eudml.org/doc/142208 (visited on 04/05/2017).

[BBM13] Joan Birman, Nathan Broaddus, and William Menasco. "Finite rigid sets and homologically non-trivial spheres in the curve complex of a surface". In: (2013). arXiv: 1311.7646 [math.GT].

[BS73] A. Borel and J.-P. Serre. "Corners and arithmetic groups". In: *Comm. Math. Helv* 48 (1973), pp. 436–491. URL: https://eudml.org/doc/139559 (visited on 04/05/2017).

[Bro12] N. Broaddus. "Homology of the curve complex and the Steinberg module of the mapping class group". In: *Duke Math. J.* 161.10 (July 2012), pp. 1943–1969. URL: https://doi.org/10.1215/00127094-1645634.

[Bro82] K.S. Brown. *Cohomology of groups*. Graduate Texts in Mathematics 87. Springer-Verlag, 1982.

[Cre84] J.E. Cremona. "Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields". In: *Compositio Mathematica* 51.3 (1984), pp. 275–324. URL: http://eudml.org/doc/89646.

[FM11] B. Farb and D. Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2011.

[Gro91] M. Gromov. "Sign and geometric meaning of curvature". In: *Rendiconti del Seminario Matematico e Fisico di Milano* 61.1 (1991), pp. 9–123. URL: https://doi.org/10.1007/BF02925201.

[Har86] J.L. Harer. "The virtual cohomological dimension of the mapping class group of an orientable surface." In: *Inventiones mathematicae* 84 (1986), pp. 157–176. URL: http://eudml.org/doc/143338.

[Har81] W. J. Harvey. "Boundary Structure of The Modular Group". In: (1981), pp. 245–252. DOI: https://doi.org/10.1515/9781400881550-019. URL: https://princetonup.degruyter.com/view/book/9781400881550/10.1515/9781400881550-019.xml.

[Hub06] J.H. Hubbard. "Teichmueller Theory and Applications to Geometry, Topology and Dynamics, Volume I: Teichmueller Theory". In: (2006).

[Iva15] Nikolai V. Ivanov. "The virtual cohomology dimension of Teichmueller modular groups: the first results and a road not taken". In: (2015). arXiv: 1510.00956 [math.GT].

[MR16] M.Fortier-Bourque and K. Rafi. "Non-convex balls in the Teichmuller metric". In: (2016). eprint: math/0701398. URL: https://arxiv.org/abs/1606.05170.

[Man72] Y.I. Manin. "Parabolic points and zeta-functions of modular curves". In: *Mathematics of the USSR-Izvestiya* 6.1 (1972), p. 19.

[Mar22] J.H. Martel. "Topology of Singularities of Optimal Semicouplings". In: *arXiv preprint arXiv:2201.12817* (2022).

[Mar] J.H. Martel. "Applications of Optimal Transport to Algebraic Topology: How to Build Spines from Singularity". PhD thesis. University of Toronto. URL: https://github.com/jhmartel/Thesis2019.

[McM00] C.T. McMullen. "The moduli space of Riemann surfaces is Kahler hyperbolic". In: (2000). arXiv: math/0010022 [math.CV].

[NN18] G. Nakamura and T. Nakanishi. "Generation of finite subgroups of the mapping class group of genus 2 surface by Dehn twists". In: *Journal of Pure and Applied Algebra* 222.11 (2018), pp. 3585–3594.

[Sol] L. Solomon. "The Steinberg Character of a Finite Group with BN-pair". In: (), pp. 213–221.

[Ste07] W. Stein. *Modular Forms, a Computational Approach (with an Appendix by P.E. Gunnells)*. Graduate Studies in Mathematics no. 79. Springer-Verlag, 2007.

[Wol86] Scott A. Wolpert. "Thurston's Riemannian metric for Teichmueller space". In: *J. Differential Geom.* 23.2 (1986), pp. 143–174. DOI: 10.4310/jdg/1214440024. URL: https://doi.org/10.4310/jdg/1214440024.

Email address: jhmartel@proton.me