CLOSING STEINBERG SYMBOLS OF THE MAPPING CLASS GROUP

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ABSTRACT. This article introduces an algebraic problem we call Closing the Steinberg symbol (CS) of the mapping class group $\Gamma := Mod(S)$ of compact hyperbolic surfaces $S = S_g$ for $g \geq 2$. Solving (CS) requires finding finite subsets I of Γ which satisfy a subset sum condition, namely that the translates $\sum_{\phi \in I} \phi.\mathscr{B}$ of a certain chain sum $\mathscr{B} := \sum_i \alpha_i$ represent a nontrivial homology cycle (see §3 for details). When formal solutions I to (CS) satisfy additional metric convexity properties relative to a geometric model $X := \mathscr{T}_g$ of Teichmueller space, then we obtain an interesting supply of candidate equivariant deformation retracts $X \rightsquigarrow \mathscr{Z}$ onto subvarieties $\mathscr{Z} \hookrightarrow X$. The construction of these retracts and their properties is based on the author's Reduction-to-Singularity method [Mar] and [JHM22]. Thus we examine the problem of Closing Steinberg symbols as an important tool for constructing small dimensional $E\Gamma$ models and candidate spines.

CONTENTS

1. $Mod(S)$ AND BIERI-ECKMANN DUALITY

We let $S = S_g$ be a closed hyperbolic surface of genus $g \geq 2$, and let $\Gamma := Mod(S)$ be the mapping class group of S. The topologists define

$$Mod(S) := \pi_0(Diff_+(S))$$

Date: August 27, 2022.

as the group of orientation preserving diffeomorphisms of S modulo isotopy. The algebraists define $Mod(S)$ via Dehn-Nielsen-Baer's theorem

$$Mod(S) = Out(\pi_1(S))$$

where $\pi_1(S) = \pi_1(S, pt)$ is Poincaré's pointed fundamental group, see [FM11].

The group-theoretic (co)homology of Γ is defined via the symmetries of proper discontinuous actions $\Gamma \times X \to X$ on $E\Gamma$ models X. There is extensive literature on this subject, c.f.[Bro82]. The standard $E\Gamma$ model for the mapping class group is Teichmueller's $X = \mathscr{T}_g$, a topological $(6g-6)$-cell $\simeq \mathbb{R}^{6g-6}$, e.g. [Hub06]. We assume here the basic facts that Γ acts proper discontinuously on $\mathscr{T}_g$ with finite covolume. Of course $\mathscr{T}$ is a cell by Teichmuller's theorem and therefore topological trivial. However the Γ-equivariant topology of $\mathscr{T}$ is highly nontrivial.

It is a fundamental observation of Harvey [Har81], Harer [Har86], Ivanov [Iva15], that Γ is a Bieri-Eckmann virtual duality group [BE73]. Here a key role is played by the action of Γ on the simplicial curve complex $\mathscr{C}$ and its reduced singular homology and chain groups. Recall the curve complex $\mathscr{C} = \mathscr{C}(S)$ of the surface is the simplicial complex whose 0-skeleton (vertices) $\mathscr{C}^0$ consists of simple closed curves (modulo isotopy), and where a simplex exists between vertices $a, b, c, \ldots$ if the curves are simultaneously pairwise disjoint on S. For example in genus $g = 2$, the curve complex $\mathscr{C}(S_2)$ is a two-dimensional infinite simplicial complex. Obviously Γ acts on $\mathscr{C}$, c.f. [Bro12]. Now we present the formal definition of homological duality.

Definition 1. A finitely generated group Γ is a duality group of dimension $\nu \geq 0$ with respect to a $\mathbb{Z}\Gamma$-module $\mathbf{D}$, if there exists an element $[B] \in H_\nu(\Gamma; \mathbf{D})$ with the following property: for every $\mathbb{Z}\Gamma$-module A, the "cap-product with $[e]$" defines $\mathbb{Z}\Gamma$-module isomorphisms $H^d(\Gamma; A) \approx H_{\nu-d}(\Gamma; A \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$, $f \mapsto f \cap [e]$.

The basic properties of Bieri-Eckmann's homological duality are summarized in the following theorem from [BE73].

Theorem (Bieri-Eckmann)**.** *Let Γ be duality group of dimension ν, with dualizing module $\boldsymbol{D}$. Then*

(i) we have $\mathbb{Z}\Gamma$-isomorphism $\boldsymbol{D} \approx H^\nu(\Gamma; \mathbb{Z}\Gamma) \neq 0$, so $\boldsymbol{D}$ is a torsion-free additive abelian group;

(ii) the homology group $H_\nu(\Gamma; \boldsymbol{D})$ is infinite cyclic generated by $[e]$ as additive abelian group;

(iii) the group Γ has cohomological dimension $cd(\Gamma)$ equal to ν.

Proof. The statements are direct consequences of Definition 1. (i) We see $H^\nu(\Gamma; \mathbb{Z}\Gamma) \approx H_0(\Gamma; \mathbf{D}) \approx \mathbf{D}$. (ii) Duality implies $H^0(\Gamma; \underline{\mathbb{Z}})$ is isomorphic to $H_\nu(\Gamma; \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$, which in turn is canonically isomorphic to $H_\nu(\Gamma; \mathbf{D})$ since $\underline{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \approx \mathbf{D}$. But $H^0(\Gamma, \underline{\mathbb{Z}})$ is canonically isomorphic to $\underline{\mathbb{Z}}$. (iii) The duality isomorphism implies for every $\mathbb{Z}\Gamma$-module A that $H^*(\Gamma; A)$ is isomorphic to $H_{\nu-*}(\Gamma; A\otimes \mathbf{D})$ which reduces to 0 whenever $\nu - * < 0$. $\square$

Most importantly for the mapping class group Γ, the dualizing module $\mathbf{D}$ can be identified with the reduced homology of the simplicial curve complex. That is, we can identify $\mathbf{D}$ up to $\mathbb{Z}\Gamma$-isomorphism as $\mathbf{D} = \tilde{H}_*(\mathscr{C}; \mathbb{Z})$, where the reduced homology group inherits the natural structure of $\mathbb{Z}\Gamma$-module from the action of Γ on $\mathscr{C}$. The curve complex has the homotopy-type of a countable bouquet of $(2g-2)$-dimensional spheres, c.f. [Iva15], [Har86]. Thus we deduce that

$$vcd(\Gamma) = 6g - 6 - (2g - 2) + 1 = 4g - 5.$$

For example in genus $g = 2$, this implies that $\mathscr{T}_2$ is a 6-dimensional $E\Gamma$ model, and the quotient $\Gamma\backslash\mathscr{T}_2$ has the topology of a 3-dimensional complex. Thus arises the problem of constructing spines or souls for Γ and we seek explicit constructions or realizations of this 3-dimensional singular subvariety for genus $g = 2$. Constructing such subvarieties is the purpose of our (CS) program, as we describe below.

The problem of (CS) is based on homological properties of Γ, and specifically a representation of Bieri-Eckmann's dualizing module $\mathbf{D}$ with the reduced homology of an excision boundary $\mathbf{D} \approx \tilde{H}_*(\partial\mathscr{T}[t]; \mathbb{Z})$, where $\mathscr{T}[t]$ is a maximal Γ-rational excision of Teichmueller space for a sufficiently small parameter t. We elaborate below.

2. Canonical Riemannian metrics on $\mathscr{T}_g$ and Flat Filling

In this section we describe geometric properties which are required on $\mathscr{T}$ for our constructions to be effective. The difficulty is that there is no canonical Riemannian metric on $\mathscr{T}$, although there is a canonical symplectic form (c.f. Wolpert, Kerkhoff, et. al). The construction of Γ equivariant metrics d on $\mathscr{T}$ is non canonical and there are many candidates. Popular metrics are Teichmueller's original metric d_{Teich} [Hub06], Weil-Peterson's metric d_{WP} [Hub06], Thurston's metric [Wol86], or McMullen's [McM00].

For a choice of invariant Riemannian metric d on $\mathscr{T}$ and invariant function

$$t : \mathscr{C}^0 \to \mathbb{R}_{>0}$$

we define an excision $\mathscr{T}[t]$ of $\mathscr{T}$ by excising ("scooping out") horoballs centred at various Γ-rational points at-infinity λ and with radius $t(\lambda)$. We focus on these Γ-rational horoballs of $\mathscr{T}$ because they have Γ-invariant boundary horospheres. Thus Γ acts proper discontinuously on both $\mathscr{T}[t]$ and $\partial\mathscr{T}[t]$, and we obtain the important diagonal action

$$\Gamma \times \mathscr{T}[t] \times \partial\mathscr{T}[t] \to \mathscr{T}[t] \times \partial\mathscr{T}[t].$$

We further emphasize those parameters t which are sufficiently small, in which case the excisions $\mathscr{T}[t]$ have a Γ-equivariant topological boundary $\partial\mathscr{T}[t]$ with the homotopy-type of $\mathscr{C}$. This important fact implies a canonical equivariant isomorphism

$$\mathbf{D} \approx \tilde{H}_*(\partial\mathscr{T}[t]; \mathbb{Z}). \tag{1}$$

For the constructive topologist the Γ-action on $\mathscr{T}$ is more accesible than the abstract algebraic action on $\mathbf{D}$. The homologically essential spheres of $\mathscr{C}$ can be viewed as spheres at-infinity within the excision $\mathscr{T}[t]$. The Γ-orbit of these spheres and their singular chain sums generates an important topological $\mathbb{Z}\Gamma$-module called the *Steinberg module*. Following convention we designate the generator of this module a Steinberg symbol B. For further references to Steinberg ("modular") symbols, we refer the reader to [Man72], [AR79], [AGM], [Ste07], [Sol] and references therein.

The contractibility of $\mathscr{T}[t]$, $\mathscr{T}$, and the long exact sequence in relative homology implies the natural boundary morphism

$$\delta : C_*(\mathscr{T}[t], \partial\mathscr{T}[t]) \to C_{*-1}(\partial\mathscr{T}[t])$$

is an isomorphism. Here C_* denotes the singular chain groups. However what we require for applications is an inverse operation which is well-defined directly on singular chains, namely

$$FILL := \delta^{-1} : C_{*-1}(\partial\mathscr{T}[t]) \to C_*(\mathscr{T}[t], \partial\mathscr{T}[t]).$$

This inverse operation is a *filling* operation and requires a choice of metric d'. For our applications, any metric d' with the following properties is sufficient:

- (M1) the metric d' is metrically complete and proper in the interior of $\mathscr{T}$;
- (M2) the metric has nonpositive sectional curvature ($\kappa \leq 0$) in $\mathscr{T}$;
- (M3) the $(2g-2)$-dimensional spheres generating the Steinberg symbol at infinity admit unique d'-flat fillings ($\kappa = 0$) to relative cycles in $\mathscr{T}[t]$ mod $\partial\mathscr{T}[t]$.

If we examine the usual metrics, we find the WP metric has properties (M1), (M2). We observe that (M2) has the important consequence that WP-horoballs are geodesically convex in $(\mathscr{T}, d_{WP})$, c.f. [Gro91]. With respect to Teichmuller's original metric, we know (M1) holds, but (M2) fails and Teichmueller's original metric has regions of positive curvature. Moreover recent work of [MR16] shows that convex hull constructions are not possible in Teichmueller's metric. The work of S. Wolpert implies that d_{WP} also satisfies (M3).

Lemma 2 (Wolpert)**.** *Let Q be a geodesic pants decomposition of the hyperbolic surface (S, g). Let $\mathscr{Q}$ be the submanifold of $\mathscr{T}$ passing through (S, g), and such that the Nielsen twist tangents $t(a)$ vanish for every geodesic curve $a \in Q$. Then $\mathscr{Q}$ is a totally flat submanifold of $\mathscr{T}$ having vanishing sectional curvatures with respect to the WP metric d_{WP}.*

Proof. □

Equivalently we find $\mathscr{Q}$ consists of all hyperbolic surfaces such that the dual geodesic pants Q^* intersect Q orthogonally. This is equivalent to saying $\mathscr{Q}$ consists of all surfaces obtained from (S, g) by varying the length parameters ℓ_a $(a \in Q)$ and fixing all twists equal to zero in Fenchel-Nielsen coordinates associated to Q.

Any metric d satisfying the properties (M123) on $\mathscr{T}$ readily leads to the construction of equivariant homotopy reductions of $\mathscr{T}$ onto candidate spines $\mathscr{X}$ according to [Mar]. Specifically when (M3) holds, the Steinberg symbol B canonically fills to a flat relative cycle $(P, \partial P)$ in $(\mathscr{T}[t], \partial \mathscr{T}[t])$, and these flat relative cycles $P = FILL[B]$ and their Γ-translates are called "panels". The motivation for the terminology is given in §3 below. The panels are homologically nontrivial relative cycles in $\mathscr{T}[t]$ modulo the boundary $\partial\mathscr{T}[t]$. In practice the condition (M3) could be slightly weakened since the application of our reduction to singularity method [Mar] does not strictly need the Steinberg symbols to admit flat-fillings – what's essential is the geometric uniqueness of the fillings.

The following is important lemma for our method.

Lemma 3. *Let d be a metric on $\mathscr{T}$ satisfying properties $(M123)$. Let $\mathscr{T}[t]$ be a Γ-rational excision with sufficiently small parameter t such that the canonical isomorphism* (1) *holds. Let B be a Steinberg symbol with flat-filling $P = FILL[B]$. Then P has zero geometric self-intersection in the quotient $\Gamma\backslash\mathscr{T}[t]$, and the quotient projection $\mathscr{T}[t] \to \Gamma\backslash\mathscr{T}[t]$ maps P isometrically onto its image.*

Proof. □

To motivate Lemma 3, recall that if S is a closed hyperbolic surface and α is a closed geodesic on S, then the lifts $\tilde{\alpha}$ of α to the universal covering $\tilde{S}$ form a $\pi_1(S)$ orbit in $\tilde{S}$ where all the translates are disjoint. Likewise Lemma 3 asserts that the Γ-translates of P are *disjoint* in the interior of $\mathscr{T}[t]$.

By contrast the existence of parabolic elements γ in Γ shows that the relative cycle P and its parabolic translates $\gamma.P$ intersect asymptotically "at infinity" when $t \to 0^+$. However there remains no self-intersection in the *interior* of $\mathscr{T}[t]$.

3. Closing Steinberg Symbols: Definition and Properties

The problem of Closing Steinberg is informally related to stitching a closed football F from a sequence of panels $\{P_i\}_{i\in I}$. The panels P_i are required to have the property that $F = conv\{P_i \mid i \in I\}$ and such that $\sum_{i\in I} \partial P_i = 0$ over $\mathbb{Z}/2\mathbb{Z}$-coefficients. In otherwords the problem requires finding a sequence of panels P_i $(i \in I)$ which assemble to a closed compact convex subset F as defined above. The panels $P = P_i$ of the above footballs are analogous to the flat-filled Steinberg symbols $P = FILL[B]$ and their translates $\gamma.B$ $(\gamma \in \Gamma)$. Compare Figure 1.

Figure 1. Isometric translates of hexagonal and pentagonal panels P_i, P_j, P_k, etc., assemble to closed balls.

Now we present the formal definition of (CS) as derived from Bieri-Eckmann's homological duality [BE73], [BS73]. The key algebraic construction is the definition of homology with coefficients in a chain complex, [Bro82], where the problem of (CS) amounts to constructing a nontrivial 0-cycle

$$\xi \in H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbf{D}).$$

Here the tensor product $\otimes$ is in the category of $\mathbb{Z}\Gamma$ modules. If Γ is a Bieri-Eckmann duality group, then we find isomorphisms

$$H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbf{D}) \approx H^{\nu}(\Gamma; \underline{\mathbb{Z}_2}\Gamma) \approx \underline{\mathbb{Z}_2} \otimes_{\mathbb{Z}} \mathbf{D} \neq 0.$$

This fact implies the formal existence of nontrivial 0-cycles.

The group Γ of symmetries flips, rotates, and translates the base cycle $[P]$ throughout the space, and every finite subset I of Γ produces a finite chain sum

$$\sum_{\gamma \in I} \gamma.[P],$$

with total chain boundary

$$\partial(\sum_{\gamma \in I} \gamma.[P]) = \sum_{\gamma \in I} \gamma.\partial[P].$$

The basic problem of Closing Steinberg is to produce a finite subset $I \subset \Gamma$ for which the boundary of the *nontrivial* chain sum $\sum_{\gamma \in I} \gamma.[P]$ vanishes in the mod 2 homology group. The complete definition of Closing Steinberg includes further geometric conditions on the Γ-translates $\Gamma.F$ of the the closed convex hull $F = conv[P.I]$ of the translates $B.I$. Let $\mathscr{T}[t], \partial\mathscr{T}[t]$ be a Γ-invariant excision of $\mathscr{T}$. Let $[P]$ be a flat-filled relative cycle representing a nonzero generator of $H_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z})$.

Definition 4 (**Closing Steinberg**)**.** A finite subset I of Γ successfully Closes Steinberg if:

(i. nontrivial mod 2**)** the chain $\xi = \sum_{\gamma \in I} \gamma.P$ is nonvanishing over $\mathbb{Z}/2$ coefficients in the chain group $C_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z}/2)$;

(ii. vanishing boundary mod 2**)** the boundary $\partial\xi = \sum_{\gamma \in I} \gamma.\partial[P]$ vanishes over $\mathbb{Z}/2$-coefficients in the homology group $[\partial\xi] = 0$ in $H_q(\partial\mathscr{T}[t]; \mathbb{Z})$;

(iii. well-defined geometric convex hull) the boundary-chain representing $\partial\xi$ is simultaneously visible from at least one interior point x in $\mathscr{T}[t]$;

(iv. well-separated gates) there exists a finite-index subgroup $\Gamma' < \Gamma$ such that the chain sum $\underline{F} = \sum_{\gamma \in \Gamma'} \gamma.F$ has nonempty *well-separated gates* precisely equal to the principal orbit $\{\gamma.P \mid \gamma \in \Gamma'\}$.

Our definition of Closing Steinberg was inspired by the author's study of [Cre84]. In Cremona's terminology, the problem is to determine a "relation ideal $\mathscr{R}$" and construct a "basic polyhedron P whose transforms fill the space", c.f.[Cre84, pp.290].

The hypotheses (i)–(ii) basically require the chain sum ξ to be nonzero mod 2. The hypotheses (iii)–(iv) are convexity assumptions which need be verified for any nonzero chain. The hypothesis of well-separated gates is related to the following fact: the translates $P, \gamma.P$, for $\gamma \in \Gamma$, are either identical or geometrically disjoint in $\mathscr{T}[t]$ according to Lemma 3. However the translates $P, \gamma.P$ may have nontrivial intersection at infinity in the initial Teichmueller space $\mathscr{T}$. In fact the problem of (CS) is precisly to find such nontrivial intersections at infinity, although again the intersections are disjoint in the interior of $\mathscr{T}[t]$.

Obviously the group structure of Γ allows us to restrict ourselves to subsets I containing the identity mapping class $Id \in \Gamma$. In practice, formal solutions to (CS) can often be found among the torsion elements and finite subgroups of Γ, c.f. [Cre84].

Proposition 5. *Let Γ be a Bieri-Eckmann duality group with dualizing module $\boldsymbol{D}$. Then there exists finite subsets I in Γ for which $\xi = \sum_{\gamma \in I} \gamma.P$ lies in the kernel of ∂_0 over $\mathbb{Z}/2$.*

Proof. The argument is homological. We interpret ξ as a chain sum representing a 0-cycle in $H_0(\Gamma; \mathbb{Z}/2\Gamma \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$. The hypotheses of Closing Steinberg imply ξ is homologically nontrivial cycle. Bieri-Eckmann duality (Proposition 1) implies the kernel $\ker \partial_0$ is naturally isomorphic to the induced $\mathbb{Z}\Gamma$-module $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbf{D}$ which is nonzero. $\square$

To illustrate, let the reader observe that a typical element $\phi \in Mod$ will *totally displace* the curves in $\mathscr{B}$ such that $\mathscr{B} \cap \phi.\mathscr{B} = \emptyset$ for almost every $\phi \in Mod$. On the other hand, if ϕ' permutes the curves such that $\phi.\mathscr{B} = \mathscr{B}$, then $I = \{Id, \phi'\}$ would be a solution of (2). However we consider this solution to be trivial in the following sense: the formal sum $[\mathscr{B}] + \phi'.[\mathscr{B}] = 2[\mathscr{B}] = 0$ is itself vanishing mod 2. Such trivial solutions are avoided in the case of higher genus closed surfaces as the work of [BBM13] demonstrates, i.e. the identity element is the only mapping class which permutes $\mathscr{B}$. Obviously parabolic type elements ϕ, i.e. curve stabilizers $\phi \in Mod_\gamma$ satisfy$\mathscr{B} \cap \phi.\mathscr{B} \supset \{\gamma\}$. So naturally one is tempted to find formal solutions to (CS) by choosing a suitable sequence of parabolics.

Our hypotheses regarding Closing Steinberg have useful consequences, which we summarize in the following theorem.

Theorem 6. *Suppose $I \subset \Gamma$ successfully Closes Steinberg (Definition 4). Define $F := conv[I.P]$. Then*

(i) the Γ-translates $\gamma.F$ $(\gamma \in \Gamma)$ form a chain sum

$$\underline{F} := \cdots \gamma.[F] + \gamma'.[F] + \gamma''.[F] + \cdots ,$$

and there exists finite-index subgroup $\Gamma' < \Gamma$ which acts as additive shift-operator on the summands of $\underline{F}$; and

(ii) the support of the chain sum $\underline{F}$ is a simply-connected subset of X, and $\underline{F}$ is a cubical $E\Gamma'$ model.

Proof. We can replace Γ with a finite-index torsion-free subgroup Γ' to ensure Γ' acts freely on X, and therefore the diagonal action is free on $X[t] \times \partial X[t]$. Moreover we can ensure Γ' translates the flat-filled relative cycle $\gamma.[P]$, for $\gamma \in \Gamma'$ freely. Then $\gamma.[P] \neq [P]$ when $\gamma \neq Id$. The definition of Closing Steinberg implies distinct translates F, F' are disjoint unless they intersect in a gate $G' = \gamma'.P$ for some $\gamma' \in \Gamma'$. So $\gamma.F$ equals F only if $\gamma = Id$ is trivial. This proves the summands $\{\gamma.F \mid \gamma \in \Gamma'\}$ of $\underline{F}$ form a principal Γ'-set, and establishes (i). The existence of an interior point $x \in F$ which is simultaneously visible to the translates $P.I$ in $X[t]$ proves $F = conv[P.I]$ is a compact convex set, and homeomorphic to some cube. Thus $\underline{F}$ is a chain sum of cubes, hence a cubical chain sum and therefore (ii). □

4. (CS) for Genus Two Mapping Class Group ($g = 2$)

To illustrate our ideas, we now study the case of genus two closed Riemann surface. The duality theory of mapping class groups $\Gamma = Mod(S_g)$ for genus $g = 2$ has been described by [Bro12]. For reference we include the following figure taken from [Bro12, Fig.10], see (4).

The formal problem of (CS) for genus two surfaces has the following symbolic setup. Let $V := \mathbb{Z}/2(\mathscr{C}^0)$ be the abelian topological group consisting of finitely-supported $\mathbb{Z}/2$-valued functions $f : \mathscr{C}^0 \to \mathbb{Z}/2$ on the set $\mathscr{C}^0$ of free homotopy classes of simple closed curves on a surface S. We abbreviate such a function f with its support $\alpha + \beta + \ldots$. On the genus two closed surface, consider Broaddus' set of nine curves $\alpha_i, \beta_j, \gamma_k$ for $i, j, k \in \{1, 2, 3\}$, and the formal sum

$$\mathscr{B} := \sum_{i,j,k=1}^{3} \alpha_i + \beta_j + \gamma_k. \tag{2}$$

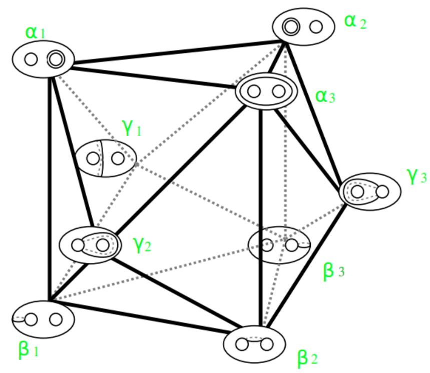

FIGURE 2. Homologically nontrivial 2-sphere in the curve complex $\mathscr{C}$ of genus 2 closed surface. Figure adapted from [Bro12, Fig.10]

Now finally we make the problem of (CS) totally explicit:

Definition 7. A finite subset $I \subset \Gamma$ formally Closes Steinberg for the mapping class group Γ of genus two closed surfaces if

$$\sum_{\phi \in I} \sum_{\alpha \in \mathscr{B}} \phi.\alpha = 0, \ mod\ 2 \tag{3}$$

where the zero element 0 on the right hand side is the zero element in V, i.e. the constant zero-valued distribution on $\mathscr{C}^0$.

Symbolically the "vanishing mod 2" of the translates $\sum_{\phi \in I} \phi.\mathscr{B}$ says there is an even number of coincidences between the translated curves $\phi.\alpha$ where $\phi \in I$, $\alpha \in \mathscr{B}$. This can be implemented on python by iterated symmetric differences. For example if $\mathscr{B}$ denotes the *set* of curves defined in (2), then a finite subset $I = \{\phi_1, \ldots, \phi_n\}$ is formal solution to (CS) if and only if

$$\phi_1.\mathscr{B} \Delta \cdots \Delta \phi_n.\mathscr{B} = \emptyset.$$

We can omit the parentheses since the symmetric difference Δ is an associative operator.

Now we introduce some standard notation following [NN18]. Let

$$\eta = aecf$$

be the order ten element in $Mod(S_2)$ and define

$$\mu := \eta^4.$$

Then μ is an order five element in Mod. If a, b, c is a geodesic pant decomposition, then we define the chain sum

$$B := [a] + [b] + [c] + \mu.[a] + \mu.[b] + \mu.[c].$$

Lemma 8. *Let $I_0 := \{Id, \mu, \mu^2, \mu^3, \mu^4\}$. Then $\sum_{\phi \in I_0} \phi.B = 0$ (mod 2) and I_0 is a formal solution to (CS).*

Proof. The vanishing of the chain sum $\sum_{\phi \in I_0} \phi.B$ is clear. Moreover all the summands $\phi.B$ are distinct for $\phi \in I_0$ and this proves the formal solution is nontrivial. $\square$

By computation using Mark C. Bell's curver [ref] we have found that the I_0-translates of B is supported on ten curves. Given the formal solution I_0, we proceed in several steps. First we construct the convex hull

$$F := conv(I_0.B)$$

over these ten curves constituting the I_0 translates of B. Then we need estbalish that the chain sum

$$\underline{F} := \sum_{\phi \in \Gamma} \gamma.F$$

has a *well-separated gates structure* equal to $\Gamma.B$. The idea of well-separated gates is introduced in [Mar, pp.13, §5.1], and means $\underline{F} = \sum_{i \in I} F_i$ is a countable chain sum of sets F_i where the intersections $G := F_{ij} := F_i \cap F_j$ form a principal Γ-set.

5.

If $\underline{F}$ is a chain sum with well-separated gates $\{G\}$, then the singularity locus $\mathscr{Z}$ naturally decomposes as a chain sum $\mathscr{Z} = \sum_i \mathscr{Z} \cap F_i$, and where $\mathscr{Z} \cap F_i$ is the singularity locus of a restricted semicoupling program, with respect to the restricted cost $c|F_i$. Best results are obtained with costs satisfying Properties (D0)–(D4) and we conjecture that the visibility costs satisfy (D0)–(D4) using the notation of [Mar]. Finally using the Reduction to Singularity method of [Mar, Theorems 1.4.1-2], we naturally construct continuous deformation retracts and which even assemble to global continuous retracts $\mathscr{T} \rightsquigarrow \mathscr{Z}$.

N.B. Constructing the retraction is contingent on the user having an effective computable model of $\mathscr{T}$ available. Solutions to (CS) allow us to localize all computations onto the local chain summands. Generally $\mathscr{Z}$ has large codimension in $\mathscr{T}$ depending on so-called Uniform Halfspace Conditions. Symmetries in the excision boundary (and target measure) on $\partial\mathscr{T}[t]$ increases the maximal codimension of $\mathscr{Z}$ with the possibility of attaining the extreme codimension, even the equivariant spine of $\mathscr{T}$.

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