A Spine for Teichmueller Space of Closed Hyperbolic Surfaces (Draft)

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We claim a minimal dimension spine W of Teichmueller space $Teich(S)$ consists of surfaces which are filled by their shortest essential nonseparating geodesics. The key element in our geometric retraction is the construction of nonconstant harmonic functions on S depending uniquely and canonically on short C-systoles. This allows explicit Teichmueller type deformation of the hyperbolic metric to increase all the C-systoles simultaneously, and increase the homological rank of the systoles. [- J.H. Martel]

1 C-Systoles, Homological Rank, and Filling.

We let (S, g) be a closed hyperbolic surface with constant Gauss curvature $K = -1$. We begin with a series of definitions and notation.

Definition: A collection C of curves is *essential* if the curves are homotopically nontrivial. The collection is *nonseparating* if the curves are nonzero $[\alpha] \neq 0$ in $H_1(S, \mathbf{Z})$ for $\alpha \in C$.

Definition: Let $C = C(g)$ be the set of geodesic nonseparating essential curves on (S, g).

Definition: Let C' be a collection of essential nonseparating geodesics. The complexity of C' is defined as the rank of the homological image of C', that is

$$\xi(C') := \dim \operatorname{span}(H_1(C')).$$

We observe $\xi = \xi(C')$ is an integer taking every integral value between $1 \leq \xi \leq 2g$ where g is the genus of the surface S.

Definition: The C-systole of (S, g) consists of the shortest curves in C relative to g-length. We denote the C-systoles of a given metric g by $C' = C'(g)$.

Definition: A collection of curves C_0 fills the surface S if the complement $S - C_0$ is a disjoint union of topological disks.

Lemma: A subset $C_0 \subset C(S, g)$ is filling if and only if $\xi(C_0) = 2g$. *Proof:*

2 Belt Tightening Lemma

The following Belt Tightening Lemma is our main observation:

Belt Tightening Lemma: Let (S, g) be hyperbolic surface with C-systoles C'.

There exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that:

- the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$;
- the curve lengths $\ell(\gamma, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\gamma \in C'$.

The proof depends on the construction of an explicit harmonic function on the surface-with-corners $S - C'$. This construction is provided in Lemma [ref].

Proof: Let $\phi = d\hat{u}$ be the unique harmonic one-form constructed in Lemma [ref]. Let $q_t := (e^t\phi + ie^{-t}\phi^*)^2$ for $t \geq 0$. Define g_t to be the resultant hyperbolic structure $(q_t)\#g$. [Incomplete]

Claim: The lengths of the curves in C' are provided by [formula] and the lengths are simultaneously increasing with respect to the deformation. QED.

3 Harmonic Functions on Surfaces-With-Corners

On a given Riemann surface (S, g) we let $*$ ("star") denote the Hodge star operator. The Cauchy-Riemann theorem says harmonic one-forms on Riemann surfaces are precisely the real parts of analytic complex differentials. In Lemma [ref] we construct a holomorphic one-form $\phi + i\phi^*$ which is canonically defined by the C-systoles of a Riemann surface S. The complex square $q := (\phi + i\phi^*)^2$ defines a holomorphic quadratic differential on the surface. The transverse invariant foliations of this quadratic differential are the real and imaginary parts of $\sqrt{q}$, and indeed defined by ϕ and ϕ^*. The retraction constructed in Lemma [ref] depends on repeated application of a flow $t \mapsto e^{-t}\phi + ie^t\phi^*$ constructed from the harmonic one-form ϕ.

In general the systoles $C' = C'(g)$ define a link in the surface. This means C' has an almost everywhere uniquely defined normal vector except at the geometric intersections of certain curves $\gamma \cap \gamma'$. But these intersections are finite. Therefore we consider the normal derivative n as defined everywhere except some isolated finite number of points on C in S.

Lemma: (Harmonic Extensions) Let (S, g) be a hyperbolic surface with C-systoles C'. There exists harmonic functions u on S satisfying the measurable Neumann boundary condition

$$\frac{\partial u}{\partial n} = 1 \quad \text{on the curves} \quad C'.$$

The harmonic functions are unique up to additive constant.

Proof. The curves C' are geodesics in the surface S, and $C' \hookrightarrow S$ is a link. We cut the surface along this link to obtain the hyperbolic surface with geodesic boundary $S - C'$. The hypothesis of constant unit normal derivative becomes a Neumann boundary condition on $\partial(S - C')$. Poisson's fundamental theorem [insert ref] says there exists a unique harmonic extension $\hat{u}$ onto $S - C'$ with prescribed normal derivative on the boundary.

This extension $\hat{u}$ is nonconstant since $n \cdot \nabla u = 1 \neq 0$ by hypothesis and is unique up to additive constant. Therefore the harmonic one-form $\phi := d\hat{u}$ is uniquely defined on S.

Remark: The main idea behind the harmonic one-form ϕ is to define the quadratic holomorphic differential $q = (\phi + i\phi^*)^2$ whose measured foliations are precisely given by ϕ, ϕ^*. The key point is that Teichmueller type deformations `$t q__t:=(e^t  + e^{-t}  ^*) $` for $t \geq 0$ are well-defined analytic deformations on Teichmueller space. In otherwords $\{ \ q_t \mid t \geq 0 \ \}$ is an explicit variation of hyperbolic structure, and thre is a sense of pushforward $g_t := q_t\#g$.

Remark: We want to relate $\phi = d\hat{u}$ to the hypothesis that $\xi(C') < 2g$. In otherwords, we need a comment on cohomology and Hodge theorem to say that ϕ is nonzero when $\xi < 2g$. This could be established by period arguments, since by construction it's evident that $\int_\gamma \phi \neq 0$ (?) [Error Careful]

Lemma: If C' does not fill S, then the harmonic extension $\bar{u}$ constructed in Lemma [ref] is nonconstant on S.

Proof. [Incomplete]

4 Well-Rounded Retract of Teichmueller Space

Here is our proposal for constructing well-rounded retracts of $Teich(S)$.

Remark: We observe that if C' does *not* fill S, then the harmonic extensions $\hat{u}$ are everywhere nonconstant by Lemma [ref]. [Error?]. This implies the Teichmueller deformations q_t are nontrivial for $t \geq 0$.

Definition: For every index $1 \leq j \leq 2g$, let W_j be the subvariety of $Teich(S)$ consisting of hyperbolic metrics whose C-systoles satisfy $\xi(C) \geq j$.

Theorem: *The Teichmueller space $Teich$ continuously and equivariantly retracts onto $W = W_{2g}$. Moreover W has codimension $2g-1$ in $Teich$ and therefore is a minimal dimension spine of $Teich$.*

Proof: The retract $Teich \to W$ is defined as a composition of retracts $W_1 \to W_2 \to \cdots \to W_{2g}$. The general retract $W_j \to W_{j+1}$ is defined as follows:

Let (S, g) be a hyperbolic surface in W_j with $\xi(C(g)) = j < 2g$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma which simultaneously expands the lengths of C'.

Claim: There exists a minimal stopping time $\tau = \tau(g)$ which depends continuously on g such that $g_\tau \in W_{j+1}$. Equivalently τ is the unique minimal time such that a new *independant* C-systole appears and which *strictly increases the complexity* ξ. Analytically τ is defined as the least time t such that

$$\xi(S_0(g_t)) > \xi(S_0(g)).$$

Claim: The one-parameter deformation defines a continuously well-posed global retraction $g \mapsto g_\tau$ from W_j to W_{j+1}.

Claim: The subvariety W_{j+1} is a codimension one subvariety of W_j. Therefore W_{2g} is a codimension $2g - 1$ subvariety of $Teich$. This is the minimal possible dimension according to Bieri-Eckmann homological duality. QED.

Remark: Geometric minimality requires a further homological duality argument a la [Souto-Pettet]. [Insert details]