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Abstract.

Contents

1.

Let $\mathscr{T}$ be the Teichmueller space of marked hyperbolic structures on a closed compact surface of genus $g \geq 2$. Let $S \in \mathscr{T}$ be a reference hyperbolic surface. Every surface S' (rel S) can be represented uniquely by a holomorphic quadratic differential $q = q_{S'}$ on S, namely the Hopf differential $q = dw$ where w is the unique harmonic map transporting the hyperbolic structure from S to S'. [ref: Wolf, Hopf]. Thus we choose an explicit parameterization $(\mathscr{T}, S) \approx Q(S)$, and we study surfaces via this correspondance S' rel $S \mapsto q = q_{S'} \in Q(S)$. Obviously S (rel S) is represented by the zero differential $q = 0$ in $Q(S)$.

Lemma 1. *The assignment (S' rel S)$\mapsto q$ defines continuous map $(\mathscr{T}, S) \to Q(S)$.*

Proof. □

Now we define the so-called *spectral cover* of S relative to $q \in Q(S)$. The following references are useful: [ref: Bridgeland-Smith, Roman Contreras, Thurston]. The spectral cover $\hat{S}$ relative to q is the unique double cover of S which is ramified over the odd multiplicity zeros and poles of q. Let $p : \hat{S} \to S$ be the covering map, and

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let τ be the deck transformation $\tau : \hat{S} \to \hat{S}$ satisfying $\tau \circ \tau = Id$. In the double cover $\hat{S}$ the quadratic $\hat{q}$ splits as the square of an abelian differential ω.

The basic idea in this article is to study *periods* of ω as length-type functions on $\mathscr{T}$. Thus for curves $\hat{a}$ on $\hat{S}$, we replace the study of length functions $\ell_a(S)$ with the period integrals $\int_{\hat{a}} \omega$ on $\hat{S}$. The distinction is that $\ell_a(S)$ is not a homotopy-invariant of the curve a, and the length needs be computed as a minimum over all isotopies of a on S. Here $\hat{a}$ will be a homology class in the "hat" homology group $\hat{H}(q) := H_1(\hat{S})^-$. The periods are topological and invariant with respect to isotopy. The involution τ acts on the homology group $H_1(\hat{S})$ and we find a splitting into $(+1)$ and (-1) eigenspaces $H_1(\hat{S}) = H^+ \oplus H^-$. We find $H^+ \approx H^1(S)$ has real dimension $2g$ when $\partial S = \emptyset$, and H^- has real dimension $6g-6$ when ω is holomorphic. The identity $\tau(\omega) = -\omega$ implies that the periods $\int_a \omega$ vanish on the τ-invariant subgroup H^+. Thus the study of periods on $\hat{S}$ is restricted to the τ anti-invariant cycles.

Now we define a type of transfer map from $t_* : H_1(S) \to \hat{H}$. Let a be a homology cycle in $H_1(S)$. If a is a nonzero homology class in S, then a has two oriented lifts a_1, a_2 in $\hat{S}$. We define $\pm t_*(a) := \pm(a_1 - a_2)$ as the transfer homomorphism $t_* : H_1(S) \to \hat{H}$. The definition of t_* implies $t_*(a)$ is τ anti-invariant. In terms of the involution we have $\tau(a_1) = a_2$ and the cycle $t_*(a)$ can be represented $t_*(a) = a_1 - \tau(a_1)$ where $a_1 = \tilde{a}$ is a lift of a. It's clear that $t_*(a)$ projects to a trivial homology class on S, where $p_*(t_*(a)) = p_*(a_1) - p_*(a_2) = a - a = 0$. Moreover we find $t_*(a) = 0$ if and only if $a_1 = \tau a_1$ in $\hat{(H)}$. If the lifts are non homologous in $\hat{S}$ then $t_*(a) \neq 0$ in $\hat{H}$. The definition of $tr_*(a)$ is ambiguous up to sign, but we obtain well-defined periods $\int_{tr_*(a)} \omega = \int_{a_1 - a_2} \omega$ via the identity $\int_{a_1} \omega = -\int_{a_2} \omega$.

Lemma 2. *For given curve $\hat{a}$, the period $\int_{\hat{a}} \omega$ varies continuously with S' rel S.*

Lemma 3. *For given curve $\hat{a}$, the derivative of the composition $S' \mapsto q \mapsto \int_a \omega$ is equal to [??].*

Everything can be visualized if we begin with a geodesic pants decomposition of the basepoint S. Then we have two pairs of hyperbolic pants. We imagine on these two dimensional pair of paints all quadratic differentials with two zeros of degree three on each pant. This makes a total of four zeros of degree three.

2. Does there exist length-length coordinates on Teichmueller space?

3.

Lemma 4 (Max Noether [ref: Linear Dependance Relations). *] If S is hyperbolic surface of genus $g \geq 2$, then $Q(S) = A^{\otimes 2}$.*

4.

5.

Another problem: we have our candidate solution $I = \{\mu^i \mid i = 0, 1, 2, 3, 4\}$ to Closing the Steinberg symbol in genus $g = 2$.

2.1 Need the convex hull of $P = conv(\xi)$ to be three-dimensionsal in the model of $Teich$.
2.2 Need construct $F = conv(I.\xi)$ the convex hull of the I-translates of ξ.
2.3 Need the chain sum $\underline{F} = \sum_{\gamma \in \Gamma} \gamma.F$ to have well-separated gates structure $\{G\}$ equal to Γ-translates of P.

The condition 2.1 is tricky, for Broaddus' two-sphere (see Fig.10 in [ref]) is a deceiving image. Indeed while the surface of Broaddus' sphere B is two-dimensional, there is no canonical filling of this sphere to a three-dimensional ball in $Teich$. For generic metrics d' on $Teich$, we would expect the generic convex hull of six points to be five dimensional, not three!

6.

Given the pant decomposition and its dual we obtain a filling collection of curves on the surface. We say a hyperbolic surface is orthogonal relative to the filling collection if the geodesic representatives of the filling curves intersect orthogonally if they intersect at all. This is obviously the image of the Fenchel-Nielsen length parameters with zero twisting parameter relative to a pant decomposition. There is ambiguity in which pant decomposition describes the image of orthogonal surfaces for a given collection of curves C, and this ambiguity is similar to the Weyl groups in the Bruhat-Tits theory of linear algebraic groups. We let $P = P(C)$ be the subset of $\mathscr{T}$ consisting of all surfaces which are orthogonal relative to the filling collection C. We view P as a "panel" in $\mathscr{T}$. Let $I := \{Id, \mu, \mu^2, \mu^3, \mu^4$ be the formal solution to (CS) obtained in [section]. The orbits $\mu^k.P$ of the panel are specially interesting for our applications.

For a given hyperbolic surface (S, g) we need to assume the existence of well-defined projection maps $proj_P : \mathscr{T} \to P$ from a given point to the C-orthogonal surfaces $P = P(C)$. We require the projection to be equivariant with respect to the action of $Mod(S)$ on the surface and the curve collection C.

onto the panels of a given domain.

where I is

Consider the orbit of the panel $\mathscr{P}.\phi$ with respect to the mapping class group $\phi \in Mod(S)$. The orbit is basically disjoint in $\mathscr{T}$, except for parabolic elements which fix the vertices.

7.

Let S be closed hyperbolic surface with genus $g \geq 2$. Let $Teich(S)$ be the Teichmueller space of S. It's well known that $Teich(S)$ is diffeomorphic to a (6g-6)-dimensional cell, where a coordinatization is given by the Fenchel-Nielsen length-twist coordinates $\{(\ell_a, \tau_a)\}_{a \in P}$ associated to a pant decomposition P of S.

Question: can we replace the twist parameters with length parameters of other curves, and thereby replace the "length-twist" coordinates with "length-length" coordinates on $Teich(S)$?

Related question: are any formulas known which express the twist differentials $d\tau_a$ with linear combinations of length differentials $d\ell_{a'}$ in Wolpert's formula for the Weil-Petersson symplectic form $\omega_{WP} = \sum_{a \in P} d\ell_a \wedge d\tau_a$? I.e. can we express ω_{WP} using only differentials of length functions?

The same question: if we are given a hyperbolic surface S' with known length parameters relative to a pant decomposition, then what metric properties on the surface S' do we use to identify the twist parameters $\{\tau_a\}_a$?

Remark. This question risks being a [duplicate.][1] However we find the answer to the above question unsatisfactory, as indicated by our comments below.

In genus $g = 2$ we obtain the following almost canonical collection of six simple closed curves on the surface S. The value of the lengths of the green curves do not distinguish between left and right Nielsen twists along the red curves. However the derivatives of the lengths of the green curves *do* distinguish between left and right Nielsen twists along the red curves. This is similar to how the derivatives of strictly convex functions $f : \mathbb{R}^3 \to \mathbb{R}$ are injective where $Df(x_1) = Df(x_2)$ if and only if $x_1 = x_2$. Here we are assuming that the lengths of the green curves are basically convex functions in the Neilsen twist parameters in the red curves.

[![The lengths of the red curves plus their angles of intersection coordinatize Teichmueller space. Wolpert's work shows the differentials of the lengths of the green curves are parameterized by the angles of geodesic intersections between the red and green curves.][2]][2]

Answer: Consider genus $g = 2$. Let $t(a)$, $t(b)$, $t(c)$ be the Nielsen tangent vectors in Teich defined by the red geodesic pant decomposition $\{a, b, c\}$. Let $\{a', b', c'\}$ be the "dual pant". Then I propose that the functions

$$\ell_a, \ell_b, \ell_c$$

together with the cosines of the angles of intersection

$$d\ell_{b'}(t(a)), d\ell_{c'}(t(b)), d\ell_{a'}(t(c)))$$

are globally well defined coordinates on Teichmueller space, i.e. an injective continuous map $\mathscr{T} \to \mathbb{R}^6$. Notice the collection is of cardinality $6g - 6 = 6$ for genus $g = 2$. So they are not (length, length) coordinates, but (length, "d"length) coordinates.

Here I'm assuming all of Wolpert's work, especially pp.252 in Wolpert's 1983 paper referenced by Alex Nolte's answer.

Likewise if we use Wolpert's Reciprocity formula $d\ell_a(t(b)) = -d\ell_b(t(a))$, then we obtain another "reciprocal" global coordinate system on Teich. The idea is that the angles of geodesic intersection are effective parameters of the Nielsen twist parameter along the pant cuffs.

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