A NEW SPINE FOR TEICHMUELLER SPACE OF HYPERBOLIC SURFACES

J.H. MARTEL

Abstract. We provide a self-contained construction of a minimal dimension equivariant spine for Teichmueller space of closed hyperbolic surfaces. The spine consists of closed hyperbolic surfaces which are filled by their shortest nonseparating essential curves. This spine is not identical with Thurston's original proposal. Our spine gives simple proof of the homological formula $dimTeich(S) - vcdMCG(S) = 2g - 1 = dimH_1(S) - 1$.

1. Introduction

Let S be a closed compact connected surface. Let $T = Teich(S)$ be the Teichmueller space of hyperbolic metrics g on S. In this article we prove the following

Main Theorem. *A minimal dimension equivariant spine W of Teichmueller space T consists of hypberbolic surfaces which are filled by their shortest essential nonseparating curves.*

The spine W is distinct from W. Thurston's original construction [7], the key difference being our emphasis on essential *nonseparating* curves. The complete proof of our Main Theorem requires constructing a strong deformation retract of T onto W which is invariant with respect to mapping class group, and prove that W has codimension equal to $2g-1$. Our construction in fact clarifies the numerical coincidence between

$$\dim T - vcd(MCG(S)) = 2g - 1$$

and $\dim H_1(S) - 1$. Compare e.g. [2]. To construct continuous retractions requires canonical tangent vectors, and this is addressed in Lemma 2 which constructs canonical harmonic one forms ϕ on the surface adapted to short nonseparating curves. Our main observation is Belt Tightening Lemma 4 in §4 which says if short nonseparating curves are non filling, then we can *simultaneously increase* their lengths

Date: 2024-02-02.

by flowing along a Teichmueller deformation in the "direction of ϕ". We iterate the construction of ϕ and the Teichmueller deformations to obtain a sequence of continuous retracts $W_j \to W_{j+1}$ whose composition yields the desired deformation retract of Teichmueller space onto W.

2. C-Systoles, Homological Rank, and Filling

We begin with some definitions and notation. Let (S, g) be a closed hyperbolic surface with constant Gauss curvature $\kappa = -1$. A collection of curves is *essential* if the curves are homotopically nontrivial. The collection is *nonseparating* if the curves are nonzero in $H_1(S, \mathbf{Z})$. Let $C = C(g)$ denote the set of all geodesic nonseparating essential curves on (S, g). A collection of curves C_0 *fills* the surface S if the complement $S - C_0$ is a disjoint union of topological disks. We define the C-systoles of (S, g) to be the shortest curves in C relative to g-length. We emphasize that C consists of essential *nonseparating* geodesics. The C-systoles of a given metric g are denoted by $C' = C'(g)$. The *complexity* of a hyperbolic metric g is defined as the dimension of the homological image generated by the C-systoles $C'(g)$. That is

$$\xi(g) := \dim \mathrm{span}(H_1(C'(g))).$$

We observe $\xi = \xi(g)$ is an integer taking every integral value between $1 \leq \xi \leq 2\mathrm{genus}(S)$.

Lemma 1. *A subset $C_0 \subset C$ fills S if and only if $H_1(C_0) = H_1(S)$.*

Proof. By definition C_0 consists of essential nonseparating curves, and their homological image generates a subspace H' in H_1. By classification of surfaces, the complement $S - C'$ is a disjoint union of surfaces with geodesic corners and topological types $S_{g,b}$ for various values of $g \geq 0$, $b \geq 1$. Here g should not be confused with the hyperbolic metric. If C' does not fill S, then by definition there exists some connected component $S_{g,b}$ with $g + b \geq 2$. We have two cases:

(Case 1) If $g > 0$, then it's evident there exists a nonseparating interior homology curve on $S_{g,b}$ which injects into $H_1(S)$.

(Case 2) If $g = 0$ then our assumptions imply the component has topological type $S_{0,b}$ with $b \geq 2$. This implies there exists nontrivial relative one cycles in $S_{0,b}$ modulo $\partial S_{0,b}$. This nonzero relative one cycle extends nonuniquely to some nonseparating curve $\hat{\beta}$ in S. We claim the nonunique homology cycle $[\hat{\beta}]$ is linearly independant from H'. This follows from Poincare-Lefschetz duality. $\square$

3. Canonical Harmonic One Forms

The construction of equivariant retracts of Teichmueller space requires defining "canonical flow directions". This is subtle and crucial aspect of global continuous retracts, and leads to controversy especially with respect to Thurston's preprint [7], [1]. In the following lemma we apply a simple variational idea to define canonical harmonic one forms depending on C'. These harmonic one forms are crucial in defining the specific "Teichmueller flow" retracting onto the spine.

Lemma 2. *If the C-systoles C' do not fill the hyperbolic surface, then there exists a canonical harmonic one form ϕ such that:*

(i) the kernel $\ker \phi^*$ is parallel to α for all $\alpha \in C'$; and

(ii) the periods satisfy $\int_\alpha \phi = 1$ for all $\alpha \in C'$; and

(iii) ϕ is the minimal energy harmonic one form satisfying (i) and (ii).

Proof of Lemma 2. Consider the homological image $H' := H_1(C')$ in $H_1(S)$. The annihilator $Ann(H')$ of H' in $H^1(S)$ consists of one forms ψ which satisfy $\int_\alpha \psi = 0$ for all $\alpha \in H'$. If C' does not fill S, then $H' \neq H^1(S)$ and $Ann(H')$ is a nonzero subspace of $H^1(S)$. Now consider the closed convex subset K' of all one forms ψ which satisfy $\int_\alpha \psi = 1$ for all $\alpha \in H'$. From the definitions it's clear this subset is nonempty and disjoint from 0 if and only if C' is not filling. Next we observe there exists a uniquely defined L^2 shortest one form ψ_0 in K'. By Hodge's theorem there exists a unique harmonic one form ϕ representing ψ_0. Clearly ϕ satisfies (ii) and it remains to prove that ϕ satisfies (i). We observe that ϕ is by construction orthogonal to $Ann(H')$, i.e. $\iint_S \phi^* \wedge \beta = 0$ for all $\beta \in Ann(H')$. This implies $\phi^* \in Ann(H')$ and $\int_\alpha \phi^* = 0$ for all $\alpha \in H'$. We conclude our argument with Claim (i) that the integrand $\phi^*(\alpha') = 0$ vanishes for all $\alpha \in C'$. We claim (i) follows from Dirichlet's Hydrodynamic First Existence Theorem [3]. Indeed (i) asserts that the harmonic one-form ϕ further satisfies the abelian condition $\phi^* = 0$ on C'. □

Let $j = 1, 2, ...$ be an integer. We define W_j to be the subset of $T := Teich(S)$ consisting of hyperbolic metrics g such that $\xi(g) \geq j$. The subset W_j is a closed subset of T. In fact we claim W_j has dimension equal to $j - 1$, see section 5 below.

Lemma 3. *The canonical harmonic one form $\phi = \phi_0$ constructed in Lemma 2 varies continuously with respect to the hyperbolic metric g on S if and only if ξ is constant along the variation.*

Proof. The proof is trivial given the hypotheses. The C-systoles $C'(g)$ of the hyperbolic metric g is upper semicontinuous with respect to variations in the metric, i.e. if g_k is a sequence of hyperbolic metrics with limit $\lim_k g_k = g_\infty$, then $C'(g_\infty)$ contains the Gromov-Hausdorff limit of $C'(g_k)$. In simple terms, this means short vectors are preserved and there is possibility of new short vectors appearing in the limit. This implies that $Ann(H')$ and K' are lower semicontinuous with respect to g. Therefore $K'(g_\infty)$ is always contained in the Gromov-Hausdorff limit of $K'(g_k)$. All of these subsets are Gromov-Hausdorff continuous when the spans $H' = H'(g)$ vary continuously with respect to g and this occurs iff the dimension ξ is constant. □

When the limit is upper discontinuous, then the limit metric g_∞ admits a new homologically independant short curve. The affine subspace $K'(g_\infty)$ is then a strictly proper subset of $\mathrm{GH}-\lim_k K'(g_k)$. Therefore it's evident that the least g_∞-energy one form $\phi_0(g_\infty)$ is likely not equal to the Gromov-Hausdorff limit of the least energy $\mathrm{GH}-\lim_k \phi_0(g_k)$ one forms.

4. Belt Tightening Lemma

Recall that $C' = C'(g)$ consists of the shortest essential nonseparating geodesics on the hyperbolic surface (S, g) and $\xi(g)$ is the dimension of the homological image of C'. The following lemma is our main observation.

Lemma 4 (Belt Tightening)**.** *Let (S, g) be hyperbolic surface with C-systoles C'. If C' does not fill the surface, then there exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that*

(i) the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$, and

(ii) the curve lengths $\ell(\gamma, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\gamma \in C'$.

Belt Tightening is proved by constructing the metrics g_t as solutions of an initial value problem of an ODE. In otherwords we define infinitesimal deformations of the hyperbolic metric. Teichmueller theory identifies infinitesimal variations dg of hyperbolic metrics g with the real parts of holomorphic quadratic differentials. See [6, Ch. 17, §60-63], [4].

Therefore if dg is a variation of hyperbolic metrics, then $dg = Re(q)$ where q is a holomorphic quadratic differential and $Re(q)$ is its real part. The harmonic one forms $\phi = \phi_0$ constructed in Lemma 2 provide holomorphic quadratic differentials on S by setting $q := (\phi + i\phi^*)^{\otimes 2}$, which implies $Re(q) = \phi\phi - \phi^*\phi^*$. Our retractions depend on defining the complex squareroot $q^{1/2}$ which has $Re(q^{1/2}) = \sqrt{\phi\phi - \phi^*\phi^*}$, and this provides a canonical direction along which we infitesimally deform the metric $g + dg$.

Consider the initial value problem defined by:

$$g' = g^{1/2}.Re(q^{1/2}) = g^{1/2}.(\phi\phi - \phi^*\phi^*)^{1/2}, \quad g(0) = g. \tag{1}$$

Standard results from ODEs imply that solutions "integral curves" to (1) exist uniquely and vary continuously with respect to initial conditions only if ξ is everywhere constant. It's convenient to define a solution g_t of (1) as *regular* if ξ is constant along g_t.

Lemma 5. *If g_t is a regular solution of (1) and γ is a curve in S, then the length $\ell(\alpha, g_t)/\ell(\alpha, g)$ satisfies $\frac{d}{dt}\ell(\gamma, g_t) = \frac{1}{2}\int_\gamma \sqrt{\phi(\gamma')^2 - \phi^*(\gamma')^2}$ for every curve γ on S.*

Proof. The proof is a direct computation. We compute length according to standard definition $\ell(\gamma, g_t) = \int \sqrt{g_t(\gamma', \gamma')}ds$. We differentiate under the integral and use chain rule to obtain

$$\frac{d}{dt}\ell(\gamma, g_t) = \frac{1}{2}\int \frac{1}{\sqrt{g_t(\gamma', \gamma')}}\frac{d}{dt}(g_t(\gamma', \gamma'))ds$$

which equals $\frac{1}{2}\int \sqrt{\phi(\gamma')^2 - \phi^*(\gamma')^2}ds$ according to the definition of the IVP. $\square$

The integral formula in Lemma 5 is valid for general curve γ. If $\gamma = \alpha$ belongs to $C'(g_t)$ for every t, then the harmonic one form $\phi = \phi(g)$ constructed in Lemma 2 satisfies

$$\frac{d}{dt}\ell(\alpha, g_t) = \frac{1}{2}\int \sqrt{\phi(\alpha')^2}ds = \frac{1}{2}\int_\alpha \phi = \frac{1}{2}.$$

From Lemma 3 we know ϕ, ϕ^* vary continuously with respect to g whenever $\xi(g)$ is constant. It's clear that ξ is *locally constant* in a sufficiently small neighborhood of g. It's also clear that ξ can increase "discontinuously" along convergent sequences $g_k \to g_\infty$ and this occurs when new homologically independant short curves appear in the limit $C'(g_\infty) > \lim_k C'(g_k)$. In section 5 we define strong deformation retracts $W_{j+1} \to W_j$ by Belt Tightening 4 along the C' curves.

5. Retract of Teichmueller Space onto $W = W_{2g}$

In this section we construct the well-rounded retract of $T = Teich(S)$. Given a hyperbolic metric g, if the C-systoles $C'(g)$ do not fill S, then Lemma 2 defines a canonical harmonic one form $\phi = \phi(g)$ such that $ker\phi^*$ is parallel to α and $\int_\alpha \phi = 1$ for every $\alpha \in C'(g)$. The Belt Tightening Lemma 4 shows that deforming the metric g_t in the ϕ direction simultaneously increases the lengths of $\alpha \in C'(g_t)$ when $\xi(g_t)$ is constant. .

Definition: For every index $1 \le j \le 2g$, let W_j be the subvariety of T consisting of hyperbolic metrics whose C-systoles satisfy $\xi(C) \ge j$.

Theorem 6. *For every index $1 \le j \le 2g-1$, there exists a continuous equivariant deformation retract $W_j \to W_{j+1}$. Moreover W_{j+1} has codimension one in W_j.*

Proof. The general retract $W_j \to W_{j+1}$ is defined as follows. Let (S, g) be a hyperbolic surface in W_j with $\xi(C(g)) = j < 2g$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma 4 which simultaneously increase the lengths of C'. The theorem follows from the following Claims (i), (ii), (iii):

Claim (i) There exists a minimal stopping time $\tau = \tau(g)$ which depends continuously on g such that $g_\tau \in W_{j+1}$. Equivalently τ is the unique minimal time such that a new homologically independant C-systole appears and which strictly increases the complexity ξ.

Claim (ii) The one-parameter deformation g_t defines a continuously well-posed global retraction $g \mapsto g_\tau$ from W_j to W_{j+1}.

Claim (iii) The subvariety W_{j+1} is a codimension one subvariety of W_j.

These claims are established below. □

Proof of Main Theorem. The retract $T \to W$ is defined as the composition of retracts $W_1 \to W_2 \to \cdots \to W_{2g}$ constructed in Theorem 6. It follows that W_{2g} is a codimension $2g-1$ subvariety of T, and this is the minimal possible dimension according to Bieri-Eckmann homological duality. □

Remark. Geometric minimality requires a further homological duality argument a la Souto-Pettet [5].

References

[1] Maxime Fortier Bourque. "The dimension of Thurston's spine". In: *arXiv preprint arXiv:2211.08923* (2022).

[2] Nathan Broaddus. "Homology of the curve complex and the Steinberg module of the mapping class group". In: (2012).

[3] Harvey Cohn. *Conformal Mapping on Riemann Surfaces*. Dover, New York, 1967.

[4] Athanase Papadopoulos. *Handbook of Teichmüller theory*. Vol. V. European Mathematical Society, 2007.

[5] Alexandra Pettet and Juan Souto. "Minimality of the well-rounded retract". In: *Geometry & Topology* 12.3 (2008), pp. 1543–1556.

[6] Oswald Teichmuller. "Extremale quasikonforme Abbildungen und quadratische Differentialen". In: *Abh. Preuss. Akad. Wiss.* 22 (1939), pp. 3–197.

[7] William Thurston. "A spine for Teichmüller space". In: *preprint* (1986).