A NEW SPINE FOR TEICHMUELLER SPACE OF CLOSED HYPERBOLIC SURFACES

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Abstract. We provide a self-contained construction of a minimal dimension equivariant spine for Teichmueller space of hyperbolic surfaces. The spine consists of hyperbolic surfaces which are filled by their shortest nonseparating essential curves. The spine is not identical with Thurston's original proposal.

1. Introduction

Let S be a closed compact connected hyperbolic surface. In this article we prove the following

Main Theorem. *A minimal dimension equivariant spine W of Teichmueller space $Teich(S)$ consists of surfaces which are filled by their shortest essential nonseparating geodesics.*

The spine W is distinct from W. Thurston's original construction [3], the key difference being our emphasis on essential *nonseparating* curves. The complete proof of our Main Theorem requires constructing a continuous mapping class group equivariant strong deformation retract of $Teich(S)$ onto W, and proving that W has codimension equal to $2g-1$ in $Teich(S)$. To construct continuous retractions requires canonical tangent vectors, and this is addressed in Lemma **??** which constructs harmonic one forms ϕ on the surface adapted to short nonseparating curves. Our observation is the Belt Tightening Lemma 3 in §4 which says: whenever short nonseparating curves are non filling, we can *simultaneously increase* their lengths by flowing along a Teichmueller deformation in the "direction of ϕ". We iterate the construction of ϕ and the Teichmueller deformations to obtain a sequence of continuous retracts $W_j \to W_{j+1}$ whose composition yields the desired deformation retract of Teichmueller space onto W.

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2. C-Systoles, Homological Rank, and Filling

We begin with some definitions and notation. Let (S, g) be a closed hyperbolic surface with constant Gauss curvature $\kappa = -1$. A collection of curves is *essential* if the curves are homotopically nontrivial. The collection is *nonseparating* if the curves are nonzero in $H_1(S, \mathbf{Z})$. Let $C = C(g)$ denote the set of all geodesic nonseparating essential curves on (S, g). Let C_0 be an arbitrary subset of C. The complexity of C_0 is defined as the rank of the homological image of C_0, namely

$$\xi(C_0) := \dim \operatorname{span}(H_1(C_0)).$$

We observe $\xi = \xi(C_0)$ is an integer taking every integral value between $1 \leq \xi \leq 2g$ where g is the genus of the surface S. A collection of curves C_0 *fills the surface* S if the complement $S - C_0$ is a disjoint union of topological disks. We define the C-systoles of (S, g) to be the shortest curves in C relative to g-length. We emphasize that C consists of essential *nonseparating* geodesics. The C-systoles of a given metric g are denoted by $C' = C'(g)$.

These definitions imply the following

Lemma 1. *A subset $C_0 \subset C$ fills S if and only if $H_1(C_0) = H_1(S)$.*

Proof. By definition C_0 consists of essential nonseparating curves, and their homological image generates a subspace H' in H_1. By classification of surfaces, the complement $S - C'$ is a disjoint union of surfaces with geodesic corners and topological types $S_{g,b}$ for various values of $g \geq 0$, $b \geq 1$. If C' does not fill S, then by definition there exists some connected component $S_{g,b}$ with $g + b \geq 2$. We have two cases:

(Case 1) If $g > 0$, then it's evident there exists a nonseparating interior homology curve on $S_{g,b}$ which injects into $H_1(S)$.

(Case 2) If $g = 0$ then our assumptions imply the component has topological type $S_{0,b}$ with $b \geq 2$. This implies there exists nontrivial relative one cycles in $S_{0,b}$ modulo $\partial S_{0,b}$. This nonzero relative one cycle extends nonuniquely to some nonseparating curve $\hat{\beta}$ in S. We claim the nonunique homology cycle $[\hat{\beta}]$ is linearly independant from H'. This follows from Poincare-Lefschetz duality. $\square$

3. Canonical Harmonic One Forms Adapted to C-Systoles

The construction of equivariant retracts of Teichmueller space requires defining "canonical flow directions". This is subtle and crucial aspect of global continuous retracts, and leads to controversy especially with respect to Thurston's preprint [3]. In the following lemma we apply a simple variational idea to canonically define harmonic one forms along which we will "Teichmueller flow".

Lemma 2. *If the C-systoles C' do not fill the hyperbolic surface, then there exists a canonical harmonic one form ϕ such that:*

(i) the kernel $\ker \phi^*$ *is parallel to α for all $\alpha \in C'$; and*

(ii) the harmonic form ϕ satisfies $|\phi| = 1$ almost everywhere along the curves $\alpha \in C'$. Equivalently if the curves α are parameterized by arclength, then $|\phi(\alpha')| = 1$ almost eveywhere along C'.

Proof of Lemma. Consider the homological image $H' := H_1(C')$ in $H_1(S)$. The annihilator of H' in the cohomology group $H^1(S)$ consists of one forms ψ which satisfy $\int_\alpha \psi = 0$ for all $\alpha \in H'$. If C' does not fill S, then $H' \neq H^1(S)$ and $Ann(H')$ is a nonzero subspace of $H^1(S)$. Now consider the closed convex subset K' of all one forms ϕ which satisfy $\int_\alpha \phi \geq 1$ for all $\alpha \in H'$. This subset is nonempty if C' is not filling. We observe that there exists a uniquely defined L^2 shortest one form ϕ_0 in K'. This shortest one form is also orthogonal to $Ann(H')$, i.e. we have $\iint_S \alpha \wedge \beta^* = 0$ for all $\beta \in Ann(H')$. By Hodge's theorem ϕ_0 is uniquely represented as a harmonic one form. This shortest harmonic one form $\phi = \phi_0$ is the desired canonical one form. □

4. Belt Tightening Lemma

Recall that $C' = C'(g)$ consists of the shortest essential nonseparating geodesics on the hyperbolic surface (S, g). The following lemma is our main observation.

Lemma 3 (Belt Tightening)**.** *Let (S, g) be hyperbolic surface with C-systoles C'. If C' does not fill the surface, then there exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that*

(i) the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$, and

(ii) the curve lengths $\ell(\gamma, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\gamma \in C'$.

The harmonic one form $\phi = \phi_0$ constructed in Lemma 2 allows us to define holomorphic quadratic differentials $q := (\phi + i\phi^*)^2$ on S.

Moreover for every real parameter $t \geq 0$ we define $q_t := (e^t \phi + i\, e^{-t} \phi^*)^2$. By construction q_t is a holomorphic quadratic differential for every $t \geq 0$ and $q_0 = q$. Teichmueller's Theorem [1] says holomorphic quadratic forms q induce deformations of hyperbolic structures on S via $g' := g + Re(q)$. If $q = (\phi + i\phi^*)^2$, then we find

$$g' = g + (\phi\phi - \phi^*\phi^*).$$

The idea behind the following Belt Tightening Lemma is to explicitly deform the hyperbolic metric via $g_t := g + tq_t$ for $t \geq 0$, and then directly compute the variation in the g_t-lengths of the curves α in C'.

Lemma 4. *For $t \geq 0$ let $g_t := g + tq_t$ be the symmetric quadratic form constructed above. For every curve α we have ratio $\ell(\alpha, g)/\ell(\alpha, g_t)$ satisfying [insert].*

Proof. □

5. Continuous Well Rounded Retracts of Teichmueller Space

In this section we construct the well-rounded retract of $Teich(S)$. If the systoles C' do not fill S, then Lemma **??** proves there exists a canonical harmonic one form ϕ such that $ker\phi^*$ is parallel to α and ϕ is a.e. uniform for every $\alpha \in C'$. By Belt Tightening Lemma 3 we can simultaneously increase the lengths in the direction of ϕ. This leads us to defining our well rounded retract.

Definition: For every index $1 \leq j \leq 2g$, let W_j be the subvariety of $Teich(S)$ consisting of hyperbolic metrics whose C-systoles satisfy $\xi(C) \geq j$.

Theorem 5. *For every index $1 \leq j \leq 2g - 1$, there exists a continuous equivariant deformation retract $W_j \to W_{j+1}$. Moreover W_{j+1} has codimension one in W_j.*

Proof. The general retract $W_j \to W_{j+1}$ is defined as follows. Let (S, g) be a hyperbolic surface in W_j with $\xi(C(g)) = j < 2g$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma 3 which simultaneously increase the lengths of C'. The result follows from the following Claims (i), (ii), (iii).

Claim (i): There exists a minimal stopping time $\tau = \tau(g)$ which depends continuously on g such that $g_\tau \in W_{j+1}$.

Equivalently τ is the unique minimal time such that a new homologically independant C-systole appears and which strictly increases the complexity ξ. Analytically τ is defined as the least time t such that $\xi(S_0(g_t)) > \xi(S_0(g))$.

Claim (ii): The one-parameter deformation defines a continuously well-posed global retraction $g \mapsto g_\tau$ from W_j to W_{j+1}.

Claim (iii): The subvariety W_{j+1} is a codimension one subvariety of W_j.

These claims are established below. $\square$

Proof of Main Theorem. The retract $Teich \to W$ is defined as the composition of retracts $W_1 \to W_2 \to \cdots \to W_{2g}$ constructed in Theorem 5. It follows that W_{2g} is a codimension $2g-1$ subvariety of $Teich(S)$, and this is the minimal possible dimension according to Bieri-Eckmann homological duality. $\square$

Remark: Geometric minimality requires a further homological duality argument a la Souto-Pettet [2].

References

[1] Athanase Papadopoulos. *Handbook of Teichmüller theory*. Vol. V. European Mathematical Society, 2007.
[2] Alexandra Pettet and Juan Souto. "Minimality of the well-rounded retract". In: *Geometry & Topology* 12.3 (2008), pp. 1543–1556.
[3] William Thurston. "A spine for Teichmüller space". In: *preprint* (1986).