CLOSING STEINBERG SYMBOLS OF THE MAPPING CLASS GROUP: OPEN QUESTIONS

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This article studies an algebraic problem which we call Closing the Steinbeg symbol (CS) of the mapping class group of closed hyperbolic surfaces, and with special attention to genus two ($g = 2$). In simple terms this requires finding finite subsets I of $\Gamma := MCG(\Sigma_g)$ such that the translates $\sum_{\gamma \in I} \mathscr{B}.\gamma$ of a certain chain sum $\mathscr{B} := \sum_i \alpha_i$ is a nontrivial homology cycle, see [ss] for details. Solutions I to (CS) which satisfy additional convexity assumptions [ref] lead to our producing candidate equivariant deformation retracts $\mathscr{T}_g \rightsquigarrow \mathscr{X}$ of the Teichmueller space $\mathscr{T}_g$ onto a subvariety $\mathscr{X}$. The subvariety $\mathscr{X} = \mathscr{X}_\pi$ is a closed subset of the singularity locus of an optimal semicoupling $\pi = \pi(c, \sigma, \tau)$ from a source measure π to a target measure τ, and where everything is with respect to a cost c. The bridge from solutions to (CS) to deformation retracts $\mathscr{T}_g \rightsquigarrow \mathscr{X}$ is based on the "reduction to singularity" methods of the author's thesis [Mar], where we show the homotopy type of $\mathscr{X}$ is determined by an analytic Uniform Halfspace Condition (UHS), requiring an "averaged interaction" vector potential denoted $\eta_{avg}(x, y_0, y_1)$ to be uniformly bounded away from zero $\mathbf{0}$. Verifying (UHS) is a computational problem, and is beyond the scope of this present article. The present article concludes with a question to [ref] concerning further computational problems in genus $g = 2$.

1. MCG and Bieri-Eckmann Duality

Let Σ_g be a closed hyperbolic surface genus $g \geq 2$, and let $\Gamma := MCG(\Sigma_g)$ be the so-called mapping class group. According to topologists, we have $MCG(\Sigma) = \pi_0(Diff_+(\Sigma))$ which is the group of orientation preserving automorphisms of Σ modulo isotopy. According to algebraists, we have also Dehn-Neilsen-Baer's identification $MCG(\Sigma) = Out(\pi_1(\Sigma))$, where $\pi_1(\Sigma) = \pi_1(\Sigma, pt)$ is Poincaré's pointed fundamental group.

The group-theoretic (co)homology of Γ is defined via the symmetries of proper discontinuous actions $X \times \Gamma \to X$ on $E\Gamma$ models X, c.f. [Bro82]. The standard $E\Gamma$ model for the mapping class (modular) group of Riemann surfaces is Teichmueller's $X = \mathscr{T}_g$, a topological $(6g-6)$-cell $\simeq \mathbb{R}^{6g-6}$, e.g. [HH06].

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It is a fundamental observation of Harvey [Har81], Ivanov [Iva15], Harer [Har86], etc., that $\Gamma = MCG(\Sigma_g)$ is a Bieri-Eckmann virtual duality group [BE73], and that a key role is played by the action of Γ on the simplicial curve complex $\mathscr{C}$ and its reduced singular homology groups.

The problem of (CS) is based on homological properties of Γ, and specifically a representation of Bieri-Eckmann's dualizing module $\mathbb{D}$ with the reduced homology of an excision boundary $\mathbb{D} \simeq \tilde{H}_*(\partial\mathscr{T}[t];\mathbb{Z})$, where $\mathscr{T}[t]$ is a maximal Γ-rational excision of Teichmueller space. See section [ss] below.

Definition 1. A finitely generated group Γ is a duality group of dimension $\nu \geq 0$ with respect to a $\mathbb{Z}\Gamma$-module $\mathbb{D}$, if there exists an element $[B] \in H_\nu(\Gamma;\mathbb{D})$ with the following property: for every $\mathbb{Z}\Gamma$-module A, the "cap-product with $[e]$" defines $\mathbb{Z}\Gamma$-module isomorphisms $H^d(\Gamma;A) \approx H_{\nu-d}(\Gamma;A\otimes\mathbb{D})$, $f \mapsto f \cap [e]$.

The basic properties of duality groups are summarized in the following

Proposition 2 (Bieri-Eckmann duality, [BE73])**.** *Let Γ be duality group of dimension ν, with dualizing module $\mathbb{D}$. Then*

(i) we have $\mathbb{Z}\Gamma$-isomorphism $\mathbb{D} \approx H^\nu(\Gamma;\mathbb{Z}\Gamma) \neq 0$, so $\mathbb{D}$ is a torsion-free additive abelian group;

(ii) the homology group $H_\nu(\Gamma;\mathbb{D})$ is infinite cyclic generated by $[e]$ as additive abelian group;

(iii) the group Γ has cohomological dimension $cd(\Gamma)$ equal to ν.

Proof. The statements are direct consequences of duality. (i) We see $H^\nu(\Gamma;\mathbb{Z}\Gamma) \approx H_0(\Gamma;\mathbf{D}) \approx \mathbf{D}$. (ii) Duality implies $H^0(\Gamma;\underline{\mathbb{Z}})$ is isomorphic to $H_\nu(\Gamma;\underline{\mathbb{Z}}\otimes_{\mathbb{Z}\Gamma}\mathbf{D})$, which in turn is canonically isomorphic to $H_\nu(\Gamma;\mathbf{D})$ since $\underline{\mathbb{Z}}\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}\Gamma \approx \mathbf{D}$. But $H^0(\Gamma,\underline{\mathbb{Z}})$ is canonically isomorphic to $\underline{\mathbb{Z}}$. (iii) The duality isomorphism implies for every $\mathbb{Z}\Gamma$-module A that $H^*(\Gamma;A)$ is isomorphic to $H_{\nu-*}(\Gamma;A\otimes\mathbf{D})$ which reduces to 0 whenever $\nu - * < 0$. □

Most importantly, for the mapping class group Γ, the dualizing module $\mathbb{D}$ can be identified with the reduced homology of the simplicial curve complex: that is, we can identify $\mathbb{D}$ up to $\mathbb{Z}\Gamma$-isomorphism as $\mathbb{D} = \tilde{H}_*(\mathscr{C};\mathbb{Z})$, where of course the reduced homology group inherits the natural structure of $\mathbb{Z}\Gamma$-module from the action of Γ on $\mathscr{C}$.

The curve complex has the homotopy-type of a countable bouquet of $(2g-2)$-dimensional spheres, c.f. [Iva15], [Har86]. Thus we deduce that

$$vcd(\Gamma) = 6g - 6 - (2g-2) + 1 = 4g - 5.$$

For example, this implies the Teichmueller space of genus two closed surfaces is (modulo mapping class group) a 6-dimensional manifold, yet having the algebraic topology of a 3-dimensional complex.

2. Canonical Riemannian metrics on $\mathscr{T}_g$

Teichmueller's theorem that $\mathscr{T}_g$ is a $(6g-6)$-dimensional cell is perhaps misleading, because the Γ-equivariant topology of $\mathscr{T}_g$ is not trivial, and only locally resembles a $(6g-6)$ cell. Now metrics on $\mathscr{T}$ of course need be Γ-equivariant, but such metric structures on $\mathscr{T}$ appear (from the author's perspective) to be noncanonical and a question of taste. Popular metrics are Teichmueller's original metric d_{Teich} [HH06], Weil-Peterson's metric d_{WP} [HH06], Thurston's metric [Wol86], or McMullen [McM00].

For a choice of invariant Riemannian metric d on $\mathscr{T}$ and invariant function $t : \mathscr{C}^0 \to \mathbb{R}_{>0}$, we define an excision $\mathscr{T}[t]$ of $\mathscr{T}$ by excising ("scooping out") convex horoballs centred at various points at infinity. Our method distinguishes the Γ-rational horoball excisions which have the further property of having Γ-invariant boundary horospheres. We further emphasize those parameters t which are sufficiently small, in which case the excisions $\mathscr{T}[t]$ below have a Γ-equivariant topological boundary $\partial\mathscr{T}[t]$ with the homotopy-type of $\mathscr{C}$. This important fact implies a canonical isomorphism

$$\mathbb{D} \approx \tilde{H}_*(\partial\mathscr{T}[t];\mathbb{Z}).$$

For the constructive topologist, the Γ-action on $\mathscr{T}$ is more accesible than an abstract algebraic action on $\mathbb{D}$. The homologically essential spheres of $\mathscr{C}$ can be viewed as spheres at-infinity within the excision $\mathscr{T}[t]$. The Γ-orbit of these spheres, and their singular chain sums, generates an important topological $\mathbb{Z}\Gamma$-module called the Steinberg module. Following convention we designate the generator of this module a Steinberg symbol B, c.f. "modular symbols" in [AR79], [AGM], [Ste07], [Sol], and references therein.

The contractibility of $\mathscr{T}[t]$, $\mathscr{T}$, and the long exact sequence in relative homology implies the natural boundary morphism [formula] is an isomorphism. However what we require for applications is a well-defined inverse operation which is defined directly on singular chains. The inverse operation is a filling operation, and requiring a choice of metric d'. For our applications, any metric d' with the following properties would be desirable:

- (M1) the metric d' is metrically complete and proper in the interior of $\mathscr{T}$;
- (M2) the metric has nonpositive sectional curvature ($\kappa \leq 0$);
- (M3) the $(2g-2)$-dimensional spheres generating the Steinberg symbol at infinity admits a unique d'-flat filling ($\kappa = 0$) to relative cycles in $(\mathscr{T}[t], \partial\mathscr{T}[t])$.

The WP metric has properties (M1), (M2). The author does not know if (M3) holds, although [BF06] appears to suggest otherwise. We observe that (M2) has the important consequence that WP-horoballs are geodesically convex in $(\mathscr{T}, d_{WP})$, [Gro91].

With respect to Teichmuller's original metric, we know (a) holds, but (b) fails and Teichmueller's original metric has regions of positive curvature. Moreover recent work of [MR16] shows that "convex hull" constructions are not possible in Teichmueller's metric.

The reduction to singularity method of [Mar] does not strictly need flatness in property (M3) – what is essential is rather the *geometric uniqueness* of such fillings. However our opinion is that the canonical metric on $\mathscr{T}$ should satisfy all three conditions (M123) because such a metric would immediately homotopy-reduce to the maximal spine $\mathscr{X}$ of $\mathscr{T}$ according to [Mar]. When (M3) holds, the Steinberg symbol B canonically fills to a flat relative cycle $(P, \partial P)$ in $(\mathscr{T}[t], \partial\mathscr{T}[t])$, and we call the flat relative cycles $P = FILL[B]$ "panels". The motivation for the terminology is given in §(3) below. The panels are homologically nontrivial relative cycles in $\mathscr{T}[t]$ modulo the boundary $\partial\mathscr{T}[t]$. The following is very important lemma for our method:

Lemma 3. *Let d' be a metric on $\mathscr{T}$ satisfying properties $(M123)$. Let $\mathscr{T}[t]$ be a maximal Γ-rational excision with sufficiently small parameter t. Let B be a Steinberg symbol, and $P = FILL[B]$ the flat-filling. Then P has zero geometric self-intersection in the quotient $\mathscr{T}[t]/\Gamma$.*

Proof. □

The existence of parabolics on the other hand, shows that asymptotically as $t \to 0^+$ the relative cycles P have nontrivial intersection at infinity. However, with respect to an equivariant rational parameter t, there is no self intersection in the interior of the space; only asymptotically when all the horoballs collapse to points is there self-intersection.

3. Closing Steinberg

The problem of Closing Steinberg is informally related to stitching a closed football F from a sequence of panels $\{P_i\}_{i\in I}$. The panels P_i are required to have the property that $F = conv\{P_i|\ i \in I\}$ and such that $\sum_{i\in I} \partial P_i = 0$ over $\mathbb{Z}/2\mathbb{Z}$-coefficients. In otherwords, the problem amounts to finding a sequence of panels $P_i, i \in I$ which "triangulate" (in a convex sense) a compact convex subset F as defined above. The panels $P = P_i$ of the above footballs are comparable to the flat-filled Steinberg symbols $P = FILL[B]$ and their translates $B.\gamma$, $\gamma \in \Gamma$. Compare Figure 3.

Now we present the formal definition of (CS) as derived from Bieri-Eckmann's homological duality [BE73], [BS73]. When the reader reviews the definition of homology with coefficients in a chain complex [Bro82], then one finds the problem of (CS) amounts to constructing a nontrivial 0-cycle $\xi \in H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbb{D})$. Bieri-Eckmann duality says

$$H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbb{D}) \approx H^\nu(\Gamma; \underline{\mathbb{Z}_2}\Gamma) \approx \underline{\mathbb{Z}_2} \otimes_{\mathbb{Z}} \mathbf{D} \neq 0.$$

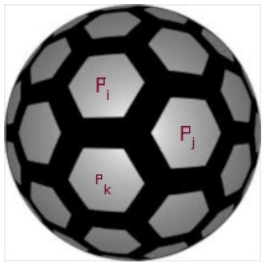

FIGURE 1.

Thus we deduce the formal existence of nontrivial 0-cycles. However our applications will require nontrivial 0-cycles which satisfy further “convex” assumptions, c.f. (4) below.

The group Γ of symmetries flips, rotates, and translates the base cycle $[P]$ throughout the space, and every finite subset I of Γ produces a finite chain sum

$$\sum_{\gamma \in I} [P].\gamma,$$

with total chain boundary

$$\partial(\sum_{\gamma \in I} [P].\gamma) = \sum_{\gamma \in I} \partial[P].\gamma.$$

The basic problem of Closing Steinberg is to produce a finite subset $I \subset \Gamma$ for which the boundary of the nontrivial chain sum $\sum_{\gamma \in I}[P].\gamma$ vanishes in the mod 2 homology group.

The complete definition of Closing Steinberg includes further geometric conditions on the Γ-translates $F.\Gamma$ of the the closed convex hull $F = conv[P.I]$ of the translates $B.I$. Let $\mathscr{T}[t], \partial\mathscr{T}[t]$ be a Γ-invariant excision of $\mathscr{T}$. Let $[P]$ be a flat-filled relative cycle representing a nonzero generator of $H_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z})$.

Definition 4 (**Closing Steinberg**)**.** A finite subset I of Γ successfully Closes Steinberg if:

(i)[**nontrivial mod** 2] the chain $\xi = \sum_{\gamma \in I} P.\gamma$ is nonvanishing over $\mathbb{Z}/2$ coefficients in the chain group $C_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z}/2)$;

(ii)[**vanishing boundary mod** 2] the boundary $\partial\xi = \sum_{\gamma \in I} \partial[P].\gamma$ vanishes over $\mathbb{Z}/2$-coefficients in the homology group $[\partial\xi] = 0$ in $H_q(\partial\mathscr{T}[t]; \mathbb{Z})$;

(iii)[**well-defined convex hull**] the boundary-chain representing $\partial\xi$ is simultaneously visible from an interior point x in $\mathscr{T}[t]$;

(iv)[**well-separated gates**] there exists a finite-index subgroup $\Gamma' < \Gamma$ such that the chain sum $\underline{F} = \sum_{\gamma \in \Gamma'} F.\gamma$ has nonempty well-separated gates structure precisely equal to the principal orbit $\{P.\gamma \mid \gamma \in \Gamma'\}$.

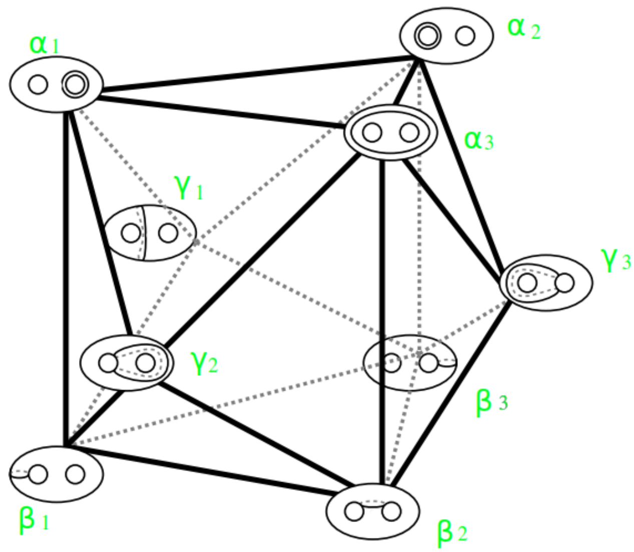

FIGURE 2. Homologically nontrivial 2-sphere in the curve complex $\mathscr{C}$ of genus 2 closed surface.

Our definition of Closing Steinberg was inspired by the author's study of [Cre84]. In Cremona's terminology, the problem is to determine a "relation ideal $\mathscr{R}$" and construct a "basic polyhedron P whose transforms fill the space", c.f.[Cre84, pp.290].

The hypothesis of well-separated gates is related to the following fact: the translates $P, P.\gamma$, for $\gamma \in \Gamma$, are either identical or geometrically disjoint in $\mathscr{T}[t]$ (recall Lemma (3)). However the translates $P, P.\gamma$ may have nontrivial intersection at infinity in the initial Teichmueller space $\mathscr{T}$. In fact the problem of (CS) is precisly to find such nontrivial intersections at infinity, although the intersections are disjoint in $\mathscr{T}[t]$.

The hypotheses (i)–(ii) basically require the chain sum ξ to be nonzero mod 2. The hypotheses (iii)–(iv) are "convex" assumptions, and which need be verified for any nonzero chain.

4. Example: Genus Two Closed Surfaces

Now we study the above definitions in the case of genus two. The duality theory of mapping class groups $\Gamma = MCG(\Sigma_g)$ for genus $g = 2$ has been described by [Bro12]. For reference we include the following figure taken from [Bro12, Fig.10], see (4).

The formal problem of (CS) for genus two surfaces reduces to the following symbolic setup. Let $V := \mathbb{Z}/2(\mathscr{C}^0)$ be the abelian topological group consisting of finitely-supported $\mathbb{Z}/2$-valued functions $f : \mathscr{C}^0 \to \mathbb{Z}/2$ on the set of free homotopy classes of simple closed curves on a surface Σ. Symbolically, we abbreviate such a function f with its support $\alpha + \beta + \ldots$. On the genus two closed surface, consider Broaddus' set of nine curves $\alpha_i, \beta_j, \gamma_k$, $i, j, k \in \{1, 2, 3\}$, and especially the formal sum

$$\mathscr{B} := \sum_{i,j,k=1}^{3} \alpha_i + \beta_j + \gamma_k.$$

Now finally we make the problem of (CS) totally explicit:

Definition 5. A finite subset $I \subset \Gamma$ formally Closes Steinberg for the mapping class group Γ of genus two closed surfaces if

$$\sum_{\phi \in I} \sum_{\alpha \in \mathscr{B}} \alpha.\phi = 0, \ mod\ 2,$$

where the zero element 0 on the right hand side is the zero element in V, i.e. the constant zero-valued distribution on $\mathscr{C}^0$.

Symbolically the "vanishing mod 2" of the translates $\sum_{\phi \in I} \mathscr{B}.\phi$ says there is an *even* number of coincidences between the translated curves $\alpha.\phi$ where $\phi \in I$, $\alpha \in \mathscr{B}$.

Remark. Obviously the group structure of Γ allows us to restrict ourselves to subsets I containing the identity mapping class $Id \in \Gamma$.

To illustrate, let the reader observe that a typical element $\phi \in MCG$ will totally displace the curves in $\mathscr{B}$, i.e. $\mathscr{B} \cap \mathscr{B}.\phi = \emptyset$ for almost every $\phi \in MCG$. On the other hand, if ϕ' permutes the curves such that $\mathscr{B}.\phi = \mathscr{B}$, then $I = \{Id, \phi'\}$ would be a solution of [eqref]. However we consider this solution to be trivial in the following sense: the formal sum $[\mathscr{B}] + [\mathscr{B}].\phi' = 2[\mathscr{B}] = 0$ is itself vanishing mod 2. Such trivial solutions are avoided in the case of higher genus closed surfaces as the work of [BBM13] demonstrates, i.e. the identity element is the only mapping class which permutes $\mathscr{B}$. Obviously parabolic type elements ϕ, i.e. curve stabilizers $\phi \in MCG_\gamma$ satisfy$\mathscr{B} \cap \mathscr{B}.\phi \supset \{\gamma\}$. So naturally one is tempted to find formal solutions to (CS) by choosing a suitable sequence of parabolics.

It is possible that finite subgroups of Γ provide convenient solutions I to (CS) in higher genus. However the structure of finite subgroups in genus $g = 2$ appears too limited.

While we have an explicit symbolic statement of (CS), it is difficult to computationally search for such solutions. By comparison, we have used WOLFRAM to determine solutions of (CS) in various arithmetic groups, e.g. $Sp(\mathbb{Z}^4, \omega)$. The the present article provides some motivation for finding *linear* representations of the mapping class group – for how do we compute except by matrices? There are at least two difficulties:

(a) how to represent mapping class elements in WOLFRAM?

(b) how to compute the action of mapping class elements on the curve complex in WOLFRAM?

However, the author has not fully experimented with Mark Bell's curver https://curver.readthedocs.io/en/master/index.html. Can curver solve (CS)?

5. From (CS) To Spines

Suppose the user succesfully Closes the Steinberg symbol, i.e. finds a finite subset I of Γ satisfying conditions 4. In this section we indicate how candidate equivariant spines $\mathscr{X}$ of $\mathscr{T}$ are produced from such solutions I. The complete development of these ideas is provided in our PhD thesis [Mar]. Briefly, solutions to (CS) allow us to replace $\mathscr{T}$ and it's rational excisions $\mathscr{T}[t]$ with a chain sum $\underline{F}$ with *well-separated gates*. This is a term introduced in [Mar], and means $\underline{F} = \sum_{i \in I} F_i$ is a countable chain sum of sets F_i where the intersections $F_{ij} := F_i \cap F_j$ have a well-behaved fixed geometry.

In our applications the chain summands F_i are convex excisions of the form $F_i = F[t]$ (see [Mar][]). The idea is next that gated costs (see [Mar][Ch.5]) are determined by their restrictions to the summands F_i, and also by the gates $G \subset F_i$ which are contained in the given summand. Thus we reduce the study of costs on $\mathscr{T}[t]$ to localized costs defined on the summands F_i and with respect to the gates $G \subset F_i$. This reduces us to the setting of [Mar][Ch.6] where we studied various repulsion costs on convex excisions.

If $\underline{F}$ is a chain sum with well-separated gates $\{G\}$, then the singularity locus $\mathscr{X}$ naturally decomposes as a chain sum $\mathscr{X} = \sum_i \mathscr{X} \cap F_i$, and where $\mathscr{X} \cap F_i$ is the singularity locus of a restricted semicoupling program, with respect to the restricted cost $c|F_i$, c.f. [Mar][Ch.5]. Best results are obtained with costs satisfying Properties (D0)–(D4) and we conjecture that the visibility costs satisfy (D0)–(D4) using the notation of [Mar]. Finally using the reduction to singularity methods of [Mar], we naturally construct continuous deformation retracts and which even assemble to global continuous retracts $\mathscr{T} \rightsquigarrow \mathscr{X}$. Generally $\mathscr{X}$ has large codimension in $\mathscr{T}$ (depending on so-called Uniform Halfspace Conditions). Symmetries in the excision boundary (and target measure) on $\partial\mathscr{T}[t]$ increases the maximal codimension of $\mathscr{X}$ with the possibility of attaining the extreme codimension, even the equivariant spine of $\mathscr{T}$. Here there is much to say, but we go no further in this article.

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