# CLOSING STEINBERG SYMBOLS OF THE MAPPING CLASS GROUP: OPEN QUESTIONS

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Abstract. This article studies an algebraic problem which we call Closing the Steinbeg symbol (CS) of the mapping class group of compact Riemann surfaces $\Sigma = \Sigma_g$. For concreteness we focus on the genus two ($g = 2$) case. Formally solving (CS) requires finding finite subsets $I$ of $\Gamma := MCG(\Sigma)$ such that the translates $\sum_{\gamma \in I} \mathscr{B}.\gamma$ of a certain chain sum $\mathscr{B} := \sum_i \alpha_i$ is a nontrivial homology cycle, see 3 for details. Solutions $I$ to (CS) which satisfy additional convexity assumptions (4) lead to our producing candidate equivariant deformation retracts $\mathscr{T}_g \rightsquigarrow \mathscr{X}$ of the Teichmueller space $\mathscr{T}_g$ onto a subvariety $\mathscr{X}$. The construction of these retracts and the subvariety $\mathscr{X}$ is based on the Reduction to Singularity methods of the author's thesis [Mar].

Contents



## 1. MCG and Bieri-Eckmann Duality

Let $\Sigma_g$ be a closed hyperbolic surface genus $g \geq 2$, and let $\Gamma := MCG(\Sigma_g)$ be the mapping class group of orientation preserving diffeomorphisms. According to topologists, we have $MCG(\Sigma) = \pi_0(Diff_+(\Sigma))$ which is the group of orientation preserving automorphisms of $\Sigma$ modulo isotopy. According to algebraists, we have also Dehn-Neilsen-Baer's identification $MCG(\Sigma) = Out(\pi_1(\Sigma))$, where $\pi_1(\Sigma) = \pi_1(\Sigma, pt)$ is Poincaré's pointed fundamental group.

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The group-theoretic (co)homology of $\Gamma$ is defined via the symmetries of proper discontinuous actions $X \times \Gamma \to X$ on $E\Gamma$ models $X$. There is extensive literature on the subject, basic introductions include [Bro82]. The standard $E\Gamma$ model for the mapping class (modular) group of closed orientable Riemann surfaces is Teichmueller's $X = \mathscr{T}_g$, a topological $(6g-6)$-cell $\simeq \mathbb{R}^{6g-6}$, e.g. [HH06]. There are alternative models which are also worth exploring, as will be discussed below (§2).

It is a fundamental observation of Harvey [Har81], Ivanov [Iva15], Harer [Har86], etc., that $\Gamma = MCG(\Sigma_g)$ is a Bieri-Eckmann virtual duality group [BE73], where a key role is played by the action of $\Gamma$ on the simplicial curve complex $\mathscr{C}$ and its reduced singular homology and chain groups. The problem of (CS) is based on homological properties of $\Gamma$, and specifically a representation of Bieri-Eckmann's dualizing module $\mathbb{D}$ with the reduced homology of an excision boundary $\mathbb{D} \simeq \tilde{H}_*(\partial \mathscr{T}[t]; \mathbb{Z})$, where $\mathscr{T}[t]$ is a maximal $\Gamma$-rational excision of Teichmueller space. See section [ss] below.

For the formalists, we present some definitions below.

**Definition 1.** A finitely generated group $\Gamma$ is a duality group of dimension $\nu \geq 0$ with respect to a $\mathbb{Z}\Gamma$-module $\mathbb{D}$, if there exists an element $[B] \in H_\nu(\Gamma; \mathbb{D})$ with the following property: for every $\mathbb{Z}\Gamma$-module $A$, the "cap-product with $[e]$" defines $\mathbb{Z}\Gamma$-module isomorphisms $H^d(\Gamma; A) \approx H_{\nu-d}(\Gamma; A \otimes \mathbb{D})$, $f \mapsto f \cap [e]$.

The basic properties of duality groups are summarized in the following

**Proposition 2** (Bieri-Eckmann duality, [BE73])**.** *Let $\Gamma$ be duality group of dimension $\nu$, with dualizing module $\mathbb{D}$. Then*

*(i) we have $\mathbb{Z}\Gamma$-isomorphism $\mathbb{D} \approx H^\nu(\Gamma; \mathbb{Z}\Gamma) \neq 0$, so $\mathbb{D}$ is a torsion-free additive abelian group;*

*(ii) the homology group $H_\nu(\Gamma; \mathbb{D})$ is infinite cyclic generated by $[e]$ as additive abelian group;*

*(iii) the group $\Gamma$ has cohomological dimension $cd(\Gamma)$ equal to $\nu$.*

*Proof.* The statements are direct consequences of duality. (i) We see $H^\nu(\Gamma; \mathbb{Z}\Gamma) \approx H_0(\Gamma; \mathbf{D}) \approx \mathbf{D}$. (ii) Duality implies $H^0(\Gamma; \underline{\mathbb{Z}})$ is isomorphic to $H_\nu(\Gamma; \underline{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \mathbf{D})$, which in turn is canonically isomorphic to $H_\nu(\Gamma; \mathbf{D})$ since $\underline{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \approx \mathbf{D}$. But $H^0(\Gamma, \underline{\mathbb{Z}})$ is canonically isomorphic to $\underline{\mathbb{Z}}$. (iii) The duality isomorphism implies for every $\mathbb{Z}\Gamma$-module $A$ that $H^*(\Gamma; A)$ is isomorphic to $H_{\nu-*}(\Gamma; A \otimes \mathbf{D})$ which reduces to 0 whenever $\nu - * < 0$. $\square$

Most importantly, for the mapping class group $\Gamma$, the dualizing module $\mathbb{D}$ can be identified with the reduced homology of the simplicial curve complex: that is, we can identify $\mathbb{D}$ up to $\mathbb{Z}\Gamma$-isomorphism as $\mathbb{D} = \tilde{H}_*(\mathscr{C}; \mathbb{Z})$, where of course the reduced homology group inherits the natural structure of $\mathbb{Z}\Gamma$-module from the action of $\Gamma$ on

$\mathscr{C}$. The curve complex has the homotopy-type of a countable bouquet of $(2g-2)$-dimensional spheres, c.f. [Iva15], [Har86]. Thus we deduce that

$$vcd(\Gamma) = 6g - 6 - (2g-2) + 1 = 4g - 5.$$

For example, this implies the Teichmueller space of genus two closed surfaces is (modulo mapping class group) a 6-dimensional manifold, yet having the algebraic topology of a 3-dimensional complex.

## 2. Canonical Riemannian metrics on $\mathscr{T}_g$ and Flat Filling

Teichmueller's theorem that $\mathscr{T}_g$ is a $(6g-6)$-dimensional cell is perhaps misleading because the $\Gamma$-equivariant topology of $\mathscr{T}_g$ is highly nontrivial, and only locally resembling a $(6g-6)$ cell. The construction of $\Gamma$ equivariant metrics $d$ on $\mathscr{T}$ is non canonical and there are many candidates. Popular metrics are Teichmueller's original metric $d_{Teich}$ [HH06], Weil-Peterson's metric $d_{WP}$ [HH06], Thurston's metric [Wol86], or McMullen's [McM00].

For a choice of invariant Riemannian metric $d$ on $\mathscr{T}$ and invariant function $t : \mathscr{C}^0 \to \mathbb{R}_{>0}$, we define an excision $\mathscr{T}[t]$ of $\mathscr{T}$ by excising ("scooping out") convex horoballs centred at various points at infinity. Our method distinguishes the $\Gamma$-rational horoball excisions which enjoy the further property of having $\Gamma$-invariant boundary horospheres. Thus $\Gamma$ acts proper discontinuously on both $\mathscr{T}[t]$ and $\partial\mathscr{T}[t]$. We further emphasize those parameters $t$ which are sufficiently small, in which case the excisions $\mathscr{T}[t]$ have a $\Gamma$-equivariant topological boundary $\partial\mathscr{T}[t]$ with the homotopy-type of $\mathscr{C}$. This important fact implies a canonical isomorphism

$$\mathbb{D} \approx \tilde{H}_*(\partial\mathscr{T}[t]; \mathbb{Z}).$$

For the constructive topologist the $\Gamma$-action on $\mathscr{T}$ is more accesible than the abstract algebraic action on $\mathbb{D}$. The homologically essential spheres of $\mathscr{C}$ can be viewed as spheres at-infinity within the excision $\mathscr{T}[t]$. The $\Gamma$-orbit of these spheres and their singular chain sums generates an important topological $\mathbb{Z}\Gamma$-module called the *Steinberg module*. Following convention we designate the generator of this module a Steinberg symbol $B$, c.f. "modular symbols" in [AR79], [AGM], [Ste07], [Sol], and references therein.

The contractibility of $\mathscr{T}[t]$, $\mathscr{T}$, and the long exact sequence in relative homology implies the natural boundary morphism

$$\delta : C_*(\mathscr{T}[t], \partial\mathscr{T}[t]) \to C_{*-1}(\partial\mathscr{T}[t])$$

is an isomorphism. Here $C_*$ denotes the singular chain groups. However what we require for applications is an inverse operation which is well-defined directly on singular chains, namely

$$FILL := \delta^{-1} : C_{*-1}(\partial\mathscr{T}[t]) \to C_*(\mathscr{T}[t], \partial\mathscr{T}[t])$$

. This inverse operation is a *filling* operation and requires a choice of metric $d'$. For our applications, any metric $d'$ with the following properties would be desirable:

(M1) the metric $d'$ is metrically complete and proper in the interior of $\mathscr{T}$;
(M2) the metric has nonpositive sectional curvature ($\kappa \leq 0$) in $\mathscr{T}$;
(M3) the $(2g-2)$-dimensional spheres generating the Steinberg symbol at infinity admits a unique $d'$-flat filling ($\kappa = 0$) to relative cycles in $(\mathscr{T}[t], \partial\mathscr{T}[t])$.

If we examine the usual metrics, we find the WP metric has properties (M1), (M2). The author does not know if (M3) holds, although [BF06] appears to suggest otherwise. We observe that (M2) has the important consequence that WP-horoballs are geodesically convex in $(\mathscr{T}, d_{WP})$, [Gro91].

With respect to Teichmuller's original metric, we know (a) holds, but (b) fails and Teichmueller's original metric has regions of positive curvature. Moreover recent work of [MR16] shows that "convex hull" constructions are not possible in Teichmueller's metric.

When (M3) holds, the Steinberg symbol $B$ canonically fills to a flat relative cycle $(P, \partial P)$ in $(\mathscr{T}[t], \partial\mathscr{T}[t])$, and we call the flat relative cycles $P = FILL[B]$ "panels". The motivation for the terminology is given in §(3) below. The panels are homologically nontrivial relative cycles in $\mathscr{T}[t]$ modulo the boundary $\partial\mathscr{T}[t]$. The reduction to singularity method of [Mar] does not strictly need flatness in property (M3) – what is essential is rather the *geometric uniqueness* of such fillings. Any metric $d'$ satisfying the properties (M123) on $\mathscr{T}$ would readily lead to the existence of equivariant homotopy reductions of $\mathscr{T}$ onto a candidate spine $\mathscr{X}$ according to [Mar].

The following is very important lemma for our method:

**Lemma 3.** *Let $d'$ be a metric on $\mathscr{T}$ satisfying properties $(M123)$. Let $\mathscr{T}[t]$ be a maximal $\Gamma$-rational excision with sufficiently small parameter $t$. Let $B$ be a Steinberg symbol, and $P = FILL[B]$ the flat-filling. Then $P$ has zero geometric self-intersection in the quotient $\mathscr{T}[t]/\Gamma$.*

*Proof.* □

To motivate Lemma 3, recall the familiar fact that if $\Sigma$ is a closed hyperbolic surface, and $\alpha$ is a closed geodesic on $\Sigma$, then the lifts $\tilde{\alpha}$ of $\alpha$ to the universal covering $\tilde{\Sigma}$ form a $\pi_1(\Sigma)$ orbit in $\tilde{\Sigma}$ where all the translates are disjoint. Likewise Lemma 3 asserts that the $\Gamma$-translates of $P$ are disjoint in the interior of $\mathscr{T}[t]$, and the quotient $\mathscr{T}[t] \to \mathscr{T}[t]/\Gamma$ maps $P$ isometrically onto its image.

The existence of parabolic elements $\gamma$ in $\Gamma$ on the other hand, shows that asymptotically as $t \to 0^+$ the relative cycle $P$ and its translates $P.\gamma$ have intersections "at infinity". However with respect to an equivariant rational parameter $t$, there is no self intersection in the *interior* of $\mathscr{T}[t]$.


FIGURE 1.

## 3. CLOSING STEINBERG

The problem of Closing Steinberg is informally related to stitching a closed football $F$ from a sequence of panels $\{P_i\}_{i\in I}$. The panels $P_i$ are required to have the property that $F = conv\{P_i|\ i \in I\}$ and such that $\sum_{i\in I} \partial P_i = 0$ over $\mathbb{Z}/2\mathbb{Z}$-coefficients. In otherwords, the problem requires finding a sequence of panels $P_i, i \in I$ which assemble to a closed compact convex subset $F$ as defined above. The panels $P = P_i$ of the above footballs are analogous to the flat-filled Steinberg symbols $P = FILL[B]$ and their translates $B.\gamma$, $\gamma \in \Gamma$. Compare Figure 3.

Now we present the formal definition of (CS) as derived from Bieri-Eckmann's homological duality [BE73], [BS73]. When the reader reviews the definition of homology with coefficients in a chain complex [Bro82], then one finds the problem of (CS) amounts to constructing a nontrivial 0-cycle $\xi \in H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma\otimes\mathbb{D})$. Bieri-Eckmann duality says

$$H_0(\Gamma; \underline{\mathbb{Z}_2}\Gamma \otimes \mathbb{D}) \approx H^\nu(\Gamma; \underline{\mathbb{Z}_2}\Gamma) \approx \underline{\mathbb{Z}_2} \otimes_{\mathbb{Z}} \mathbf{D} \neq 0.$$

Thus we deduce the formal existence of nontrivial 0-cycles. However our applications will require nontrivial 0-cycles which satisfy further "convex" assumptions, c.f. (4) below.

The group $\Gamma$ of symmetries flips, rotates, and translates the base cycle $[P]$ throughout the space, and every finite subset $I$ of $\Gamma$ produces a finite chain sum

$$\sum_{\gamma\in I} [P].\gamma,$$

with total chain boundary

$$\partial(\sum_{\gamma\in I}[P].\gamma) = \sum_{\gamma\in I} \partial[P].\gamma.$$

*The basic problem of Closing Steinberg is to produce a finite subset $I \subset \Gamma$ for which the boundary of the nontrivial chain sum $\sum_{\gamma\in I}[P].\gamma$ vanishes in the mod 2 homology group.* The complete definition of Closing Steinberg includes further geometric conditions on the $\Gamma$-translates $F.\Gamma$ of the the closed convex hull $F = conv[P.I]$ of the translates $B.I$. Let $\mathscr{T}[t], \partial\mathscr{T}[t]$ be a $\Gamma$-invariant excision of $\mathscr{T}$. Let $[P]$ be a flat-filled relative cycle representing a nonzero generator of $H_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z})$.

**Definition 4** (**Closing Steinberg**). A finite subset $I$ of $\Gamma$ successfully Closes Steinberg if:

(**i: nontrivial mod** 2) the chain $\xi = \sum_{\gamma\in I} P.\gamma$ is nonvanishing over $\mathbb{Z}/2$ coefficients in the chain group $C_{q+1}(\mathscr{T}[t], \partial\mathscr{T}[t]; \mathbb{Z}/2)$;

(**ii: vanishing boundary mod** 2) the boundary $\partial\xi = \sum_{\gamma\in I} \partial[P].\gamma$ vanishes over $\mathbb{Z}/2$-coefficients in the homology group $[\partial\xi] = 0$ in $H_q(\partial\mathscr{T}[t]; \mathbb{Z})$;

(**iii: well-defined convex hull**) the boundary-chain representing $\partial\xi$ is simultaneously visible from an interior point $x$ in $\mathscr{T}[t]$;

(**iv: well-separated gates**) there exists a finite-index subgroup $\Gamma' < \Gamma$ such that the chain sum $\underline{F} = \sum_{\gamma\in\Gamma'} F.\gamma$ has nonempty *well-separated gates* precisely equal to the principal orbit $\{P.\gamma \mid \gamma \in \Gamma'\}$.

Our definition of Closing Steinberg was inspired by the author's study of [Cre84]. In Cremona's terminology, the problem is to determine a "relation ideal $\mathscr{R}$" and construct a "basic polyhedron $P$ whose transforms fill the space", c.f.[Cre84, pp.290].

The hypotheses (i)–(ii) basically require the chain sum $\xi$ to be nonzero mod 2. The hypotheses (iii)–(iv) are "convex" assumptions, and which need be verified for any nonzero chain. The hypothesis of well-separated gates is related to the following fact: the translates $P, P.\gamma$, for $\gamma \in \Gamma$, are either identical or geometrically disjoint in $\mathscr{T}[t]$ according to Lemma 3. However the translates $P, P.\gamma$ may have nontrivial intersection at infinity in the initial Teichmueller space $\mathscr{T}$. In fact the problem of (CS) is precisly to find such nontrivial intersections at infinity, although again the intersections are disjoint in the interior of $\mathscr{T}[t]$.

## 4. Open Problem: (CS) for Genus 2

To illustrate our ideas, we now study the case of genus two closed Riemann surface. The duality theory of mapping class groups $\Gamma = MCG(\Sigma_g)$ for genus $g = 2$ has been described by [Bro12]. For reference we include the following figure taken from [Bro12, Fig.10], see (4).

The formal problem of (CS) for genus two surfaces has the following symbolic setup. Let $V := \mathbb{Z}/2(\mathscr{C}^0)$ be the abelian topological group consisting of finitely-supported $\mathbb{Z}/2$-valued functions $f : \mathscr{C}^0 \to \mathbb{Z}/2$ on the set of free homotopy classes of simple closed curves on a surface $\Sigma$. Symbolically, we abbreviate such a function $f$ with its support $\alpha + \beta + \ldots$. On the genus two closed surface, consider Broaddus'

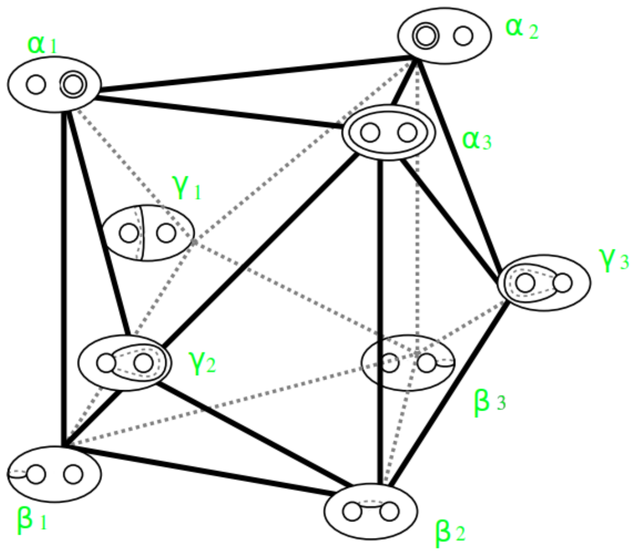


FIGURE 2. Homologically nontrivial 2-sphere in the curve complex $\mathscr{C}$ of genus 2 closed surface. Figure adapted from [Bro12, Fig.10]

set of nine curves $\alpha_i, \beta_j, \gamma_k$, $i, j, k \in \{1, 2, 3\}$, and especially the formal sum

$$\mathscr{B} := \sum_{i,j,k=1}^{3} \alpha_i + \beta_j + \gamma_k.$$

Now finally we make the problem of (CS) totally explicit:

**Definition 5.** A finite subset $I \subset \Gamma$ formally Closes Steinberg for the mapping class group $\Gamma$ of genus two closed surfaces if

$$\sum_{\phi \in I} \sum_{\alpha \in \mathscr{B}} \alpha.\phi = 0, \ mod\ 2, \tag{1}$$

where the zero element 0 on the right hand side is the zero element in $V$, i.e. the constant zero-valued distribution on $\mathscr{C}^0$.

Symbolically the "vanishing mod 2" of the translates $\sum_{\phi \in I} \mathscr{B}.\phi$ says there is an *even* number of coincidences between the translated curves $\alpha.\phi$ where $\phi \in I$, $\alpha \in \mathscr{B}$.

Remark. Obviously the group structure of $\Gamma$ allows us to restrict ourselves to subsets $I$ containing the identity mapping class $Id \in \Gamma$.

To illustrate, let the reader observe that a typical element $\phi \in MCG$ will *totally displace* the curves in $\mathscr{B}$ such that $\mathscr{B} \cap \mathscr{B}.\phi = \emptyset$ for almost every $\phi \in MCG$. On the other hand, if $\phi'$ permutes the curves such that $\mathscr{B}.\phi = \mathscr{B}$, then $I = \{Id, \phi'\}$ would be a solution of (1). However we consider this solution to be trivial in the following sense: the formal sum $[\mathscr{B}] + [\mathscr{B}].\phi' = 2[\mathscr{B}] = 0$ is itself vanishing mod 2. Such trivial solutions are avoided in the case of higher genus closed surfaces as the work of [BBM13] demonstrates, i.e. the identity element is the only mapping class which permutes $\mathscr{B}$. Obviously parabolic type elements $\phi$, i.e. curve stabilizers $\phi \in MCG_\gamma$ satisfy$\mathscr{B} \cap \mathscr{B}.\phi \supset \{\gamma\}$. So naturally one is tempted to find formal solutions to (CS) by choosing a suitable sequence of parabolics.

It is possible that finite subgroups of $\Gamma$ provide convenient solutions $I$ to (CS) in higher genus, as the examples in [Cre84] demonstrate. The structure of finite subgroups in the genus $g = 2$ mapping class group is however too limited. Thus we are forced to search out solutions to (CS) directly. While (1) provides an explicit symbolic statement of (CS), it remains difficult to actually search for such solutions. The problem is to represent the action of the mapping class group on the simplicial curve complex $\mathscr{C}$. The curve complex $\mathscr{C}$ consists of free homotopy classes of simple closed curves $\alpha$ on $\Sigma$. The free homotopy class $[\alpha]$ corresponds to a conjugacy class in $\pi_1$. Therefore the curve complex consists of a special subset of "simple" conjugacy classes $[\alpha]$. The algebraic characterization of those conjugacy classes which represent simple curves appears difficult problem.

Here we find the question:

-*How to symbolically compute the* $\Gamma$*-action* $[\alpha].\phi$ *on* $\mathscr{C}$*?*

To answer the above question would permit a brute force search for formal solutions to (CS). We would represent a neighborhood of the identity element in $\Gamma$ and then randomly search for formal solutions by directly evaluating the chain sums $\sum_\phi [\mathscr{B}].\phi$ and their boundaries $\sum_\phi \partial[\mathscr{B}].\phi$ mod 2. Such a direct approach has been successfully applied to some low dimensional arithmetic groups, e.g. $\Gamma = PGL(\mathbb{Z}^q)$ for $q = 1, 2, 3$, and formal solutions to (CS) have been found for $\Gamma = Sp(\mathbb{Z}^4, \omega)$.

Another possibility arises though, based on Mark C. Bell's curver program `https://curver.readthedocs.io/en/master/index.html`. Our experimentation with curver is however limited by the fact that curver is restricted to surfaces with punctures or marked points, and does not appear immediately applicable to closed surfaces at least from this author's own perspective.

Finally we remark that Professor I.V Nikolaev claims to have established the *linearity* of the mapping class groups of closed surfaces [Nik18], [Nik01], yet we find the articles rather difficult to understand and implement, being based on some results of $AF$ $C^*$-algebras, Perron-Jacobi convergents, Bratelli diagrams, etc.. If indeed a finite-dimensional representation of $MCG$ can be simulated, and a sufficient neighborhood of the identity can be represented in matrix form, then the search for formal solutions $I$ of (CS) can begin.

## 5. A Naive Possibility

The following naive possibility arises. Let $\rho : \pi \to GL_N$ be a linear finite-dimensional representation of $\pi := \pi_1(\Sigma, pt)$ of a pointed closed Riemann surface. Let $N = N_{GL}(\rho)$ be the normalizer of the image of $\rho$ in $GL_N$. Then we find the image $Im(\rho)$ is a normal subgroup of $N$, and we obtain a quotient group $Im(\rho)\backslash N$ with a canonical homomorphism

$$\chi_\rho : Im(\rho)\backslash N \to Out(\pi).$$

**Problem.** *Does there exist a finite-dimensional linear representation $\rho$ of $\pi$ such that the homomorphism $\chi_\rho$ is surjective?*

A solution to Problem 5 would allow an effectively *linear* representation of the elements of $MCG$. The question is whether embedding $\pi$ into a larger matrix group provides a larger class of outer automorphisms to be represented by matrix conjugation.

## 6. Well-Separated Gates: From (CS) To Candidate Spines

Suppose the user succesfully Closes the Steinberg symbol, i.e. finds a finite subset $I$ of $\Gamma$ satisfying conditions 4. Solutions to (CS) allow us to replace $\mathscr{T}$ and rational excisions $\mathscr{T}[t] \subset \mathscr{T}$ with *a chain sum $\underline{F}$ with **well-separated gates***, a term which also appears in item (4)(iv) in the definition of (CS). This is a term introduced in [Mar, pp.13, §5.1], and means $\underline{F} = \sum_{i \in I} F_i$ is a countable chain sum of sets $F_i$ where the intersections $G := F_{ij} := F_i \cap F_j$ have a well-behaved fixed geometry. In our applications the chain summands $F_i$ are convex excisions of the form $F_i = F[t]$ (see [Mar, §5.5]). The idea is next that gated costs (see [Mar, §5.2]) are determined by their restrictions to the summands $F_i$, and also by the gates $G \subset F_i$ which are contained in the given summand. Thus we reduce the study of costs on $\mathscr{T}[t]$ to localized costs defined on the summands $F_i$ and with respect to the gates $G \subset F_i$. This reduces us to the setting of [Mar, §5.9] where we studied various repulsion costs on convex excisions.

If $\underline{F}$ is a chain sum with well-separated gates $\{G\}$, then the singularity locus $\mathscr{Z}$ naturally decomposes as a chain sum $\mathscr{Z} = \sum_i \mathscr{Z} \cap F_i$, and where $\mathscr{Z} \cap F_i$ is the singularity locus of a restricted semicoupling program, with respect to the restricted cost $c|F_i$. Best results are obtained with costs satisfying Properties (D0)–(D4) and we conjecture that the visibility costs satisfy (D0)–(D4) using the notation of [Mar]. Finally using the Reduction to Singularity method of [Mar, Theorems 1.4.1-2], we naturally construct continuous deformation retracts and which even assemble to global continuous retracts $\mathscr{T} \rightsquigarrow \mathscr{Z}$. N.B. constructing the retraction is contingent on the user having an effective computable model of $\mathscr{T}$ available. Solutions to (CS) allow us to localize all computations onto the local chain summands. Generally $\mathscr{Z}$ has large codimension in $\mathscr{T}$ depending on so-called Uniform Halfspace Conditions. Symmetries in the excision boundary (and target measure) on $\partial\mathscr{T}[t]$ increases the maximal codimension of $\mathscr{Z}$ with the possibility of attaining the extreme codimension, even the equivariant spine of $\mathscr{T}$.

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