A Spine for Teichmueller Space of Closed Hyperbolic Surfaces (Draft)

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We claim a minimal dimension spine W of Teichmueller space $Teich(S)$ consists of surfaces which are filled by their shortest essential nonseparating geodesics. [-J.H. Martel]

1 Minimal Geodesic Subsurface Lemma

We assume (S, g) is a compact hyperbolic surface with constant Gauss curvature $K = -1$.

Definition: A collection C of curves is π_1-essential if the curves are homotopically nontrivial. The collection is nonseparating, or H_1-essential, if the curves are nonzero $[\alpha] \neq 0$ in $H_1(S, \mathbf{Z})$ for $\alpha \in C$.

Definition: Let C be a collection of homotopy essential geodesics, and let S_0 be a geodesic subsurface of S containing C. The complexity of C is defined as the rank of the homological image of C, that is

$$\xi(C) := \dim \operatorname{span}(H_1(C)).$$

We see $\xi = \xi(C)$ is an integer taking every integer value between $1 \leq \xi \leq 2g$ where g is the genus of the surface S.

Lemma: A collection of homotopy essential nonseparating geodesics C has $\xi(C) = 2g$ if and only if C is filling, i.e. iff $S - C$ is a disjoint union of topological disks.

Remark: If S_0 is a proper nonempty geodesic subsurface of S, then S_0 is a surface with geodesic boundary ∂S_0. In otherwords there are no topological embeddings of $S_{p,0} \hookrightarrow S_{q,0}$ when $p \leq q$. Every proper geodesic subsurface has *some* boundary.

Remark: The definition of ξ extends to subsurfaces $S_0 \hookrightarrow S$ with

$$\xi(S_0) := \dim \operatorname{image}(H_1(S_0)).$$

We begin with a Lemma/Definition which constructs a canonical geodesic subsurface S_0 containing C.

Lemma 1 (Unique Minimal Complexity Geodesic Subsurfaces): Let C be a collection of essential nonseparating geodesic curves on (S, g). There exists a unique geodesic subsurface $S_0 = S_0(C)$ such that - S_0 contains C in its interior; and - the boundary ∂S_0 is homotopically essential in S; and - the subsurface has minimal complexity $\xi(S_0) = \xi(C)$.

Equivalently C fills a unique essential geodesic subsurface S_0 in S.

Proof of Lemma 1: If S_0 is a geodesic subsurface containing C and satisfying $\xi(S_0) = \xi(C)$, then S_0 strong deformation retracts onto C. Therefore if S_0, S_1 are two such subsurfaces with $\xi(S_0) = \xi(S_1) = \xi(C)$, then S_0 and S_1 are homotopic relative to C. It follows that there exists a unique geodesic subsurface in the homotopy class rel C by energy minimizing argument [incomplete].

Remark: The geodesic subsurface S_0 constructed in Lemma 1 is a degenerate subsurface when the curves of C are pairwise disjoint, i.e. the subsurface is basically the one-dimensional geodesic link $C \subset S$. The subsurface S_0 is nondegenerate when there exists a nontrivial intersection amongst some elements of C in which case the genus of S_0 is nonzero.

2 Belt Tightening Lemma

The purpose of constructing the canonical geodesic subsurface S_0 in Lemma 1 is to contract the boundary ∂S_0 and thereby deform the hyperbolic structure. The following Belt Tightening Lemma is our main observation:

Belt Tightening Lemma: Let S_0 be a proper subsurface of S with geodesic boundary ∂S_0. Let C be a collection of essential nonseparating geodesics disjoint from ∂S_0. Suppose $\xi(S_0) = \xi(C)$.

There exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that:

- the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$;
- the boundary lengths $\ell(\gamma, g_t)$ are decreasing for all $t \geq 0$ and all $\gamma \in \partial S_0$;
- the curve lengths $\ell(\alpha, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\alpha \in C$.

Proof: [Thurston, Minimal Stretch Maps preprint]

3 Well-Rounded Retract of Teichmueller Space

Here is our proposal for constructing well-rounded retracts of $Teich(S)$.

Definition: Let $C = C(g)$ be the set of geodesic nonseparating π_1-essential curves on (S, g).

Definition: The C-systole of (S, g) consists of the shortest curves in C relative to g length. We denote the C-systoles of a given metric g by $C'(g)$.

Definition: A collection of curves C_0 fills the surface S if the complement $S - C_0$ is a disjoint union of topological disks.

Notation: For metric g, let $S_0 := S_0(C'(g))$ be the minimal geodesic subsurface constructed in Lemma 1 and containing the C-systoles.

Remark: Evidently $\xi(S_0) \leq \xi(S)$ with equality if and only if $C'(g)$ fills S. Otherwise if C' does *not* fill, then $S_0(C')$ has nontrivial geodesic boundary which we can contract by Belt Tightening Lemma. Iterating this process defines the retract.

Definition: For every index $1 \leq j \leq 2g$, let W_j be the subvariety of $Teich(S)$ consisting of hyperbolic metrics whose C-systoles satisfy $\xi(C) \geq j$.

Theorem: *The Teichmueller space $Teich$ continuously and equivariantly retracts onto $W = W_{2g}$. Moreover W has codimension $2g-1$ in $Teich$ and therefore is a minimal dimension spine of $Teich$.*

Proof: The retract $Teich \to W$ is defined as a composition of retracts

$$W_1 \to W_2 \to \cdots \to W_{2g}.$$

The general retract $W_j \to W_{j+1}$ is defined as follows:

Let (S, g) be a hyperbolic surface in W_j with $\xi(C(g)) = j < 2g$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma which *contracts* the geodesic boundary $\partial S_0(g)$ and simultaneously expands the lengths of C in $S_0(g)$.

Claim: There exists a minimal stopping time $\tau = \tau(g)$ which depends continuously on g such that $g_\tau \in W_{j+1}$. Equivalently τ is the unique minimal time such that a new *independant* C-systole appears and which *strictly increases the complexity* ξ of the supporting minimal subsurface $S_0(g_\tau))$. Analytically τ is defined as the least time t such that

$$\xi(S_0(g_t)) > \xi(S_0(g)).$$

Claim: The one-parameter deformation defines a continuously well-posed global retraction $g \mapsto g_\tau$ from W_j to W_{j+1}.

Claim: The subvariety W_{j+1} is a codimension one subvariety of W_j. Therefore W_{2g} is a codimension $2g - 1$ subvariety of $Teich$. This is the minimal possible dimension according to Bieri-Eckmann homological duality. QED.

Remark: Geometric minimality requires a further homological duality argument a la [Souto-Pettet]. [Insert details]