A NEW SPINE FOR TEICHMUELLER SPACE OF HYPERBOLIC SURFACES

J.H. MARTEL

ABSTRACT. We provide a self-contained construction of a minimal dimension equivariant spine for Teichmueller space of closed hyperbolic surfaces. The spine consists of closed hyperbolic surfaces which are filled by their shortest nonseparating essential curves. This spine is not identical with Thurston's original proposal.

1. INTRODUCTION

Let S be a closed compact connected hyperbolic surface. In this article we prove the following

Main Theorem. *A minimal dimension equivariant spine W of Teichmueller space $Teich(S)$ consists of surfaces which are filled by their shortest essential nonseparating geodesics.*

The spine W is distinct from W. Thurston's original construction [4], the key difference being our emphasis on essential *nonseparating* curves. The complete proof of our Main Theorem requires constructing a continuous mapping class group equivariant strong deformation retract of $Teich(S)$ onto W, and proving that W has codimension equal to $2g-1$ in $Teich(S)$. To construct continuous retractions requires canonical tangent vectors, and this is addressed in Lemma **??** which constructs harmonic one forms ϕ on the surface adapted to short nonseparating curves. Our observation is the Belt Tightening Lemma 4 in §4 which says: whenever short nonseparating curves are non filling, we can *simultaneously increase* their lengths by flowing along a Teichmueller deformation in the "direction of ϕ". We iterate the construction of ϕ and the Teichmueller deformations to obtain a sequence of continuous retracts $W_j \to W_{j+1}$ whose composition yields the desired deformation retract of Teichmueller space onto W.

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2. C-Systoles, Homological Rank, and Filling

We begin with some definitions and notation. Let (S, g) be a closed hyperbolic surface with constant Gauss curvature $\kappa = -1$. A collection of curves is *essential* if the curves are homotopically nontrivial. The collection is *nonseparating* if the curves are nonzero in $H_1(S, \mathbf{Z})$. Let $C = C(g)$ denote the set of all geodesic nonseparating essential curves on (S, g). Let C_0 be an arbitrary subset of C. The complexity of C_0 is defined as the rank of the homological image of C_0, namely

$$\xi(C_0) := \dim \operatorname{span}(H_1(C_0)).$$

We observe $\xi = \xi(C_0)$ is an integer taking every integral value between $1 \leq \xi \leq 2g$ where g is the genus of the surface S. A collection of curves C_0 *fills the surface* S if the complement $S - C_0$ is a disjoint union of topological disks. We define the C-systoles of (S, g) to be the shortest curves in C relative to g-length. We emphasize that C consists of essential *nonseparating* geodesics. The C-systoles of a given metric g are denoted by $C' = C'(g)$.

These definitions imply the following

Lemma 1. *A subset $C_0 \subset C$ fills S if and only if $H_1(C_0) = H_1(S)$.*

Proof. By definition C_0 consists of essential nonseparating curves, and their homological image generates a subspace H' in H_1. By classification of surfaces, the complement $S - C'$ is a disjoint union of surfaces with geodesic corners and topological types $S_{g,b}$ for various values of $g \geq 0$, $b \geq 1$. If C' does not fill S, then by definition there exists some connected component $S_{g,b}$ with $g + b \geq 2$. We have two cases:

(Case 1) If $g > 0$, then it's evident there exists a nonseparating interior homology curve on $S_{g,b}$ which injects into $H_1(S)$.

(Case 2) If $g = 0$ then our assumptions imply the component has topological type $S_{0,b}$ with $b \geq 2$. This implies there exists nontrivial relative one cycles in $S_{0,b}$ modulo $\partial S_{0,b}$. This nonzero relative one cycle extends nonuniquely to some nonseparating curve $\hat{\beta}$ in S. We claim the nonunique homology cycle $[\hat{\beta}]$ is linearly independant from H'. This follows from Poincare-Lefschetz duality. $\square$

3. Canonical Harmonic One Forms Adapted to C-Systoles

The construction of equivariant retracts of Teichmueller space requires defining "canonical flow directions". This is subtle and crucial aspect of global continuous retracts, and leads to controversy especially with respect to Thurston's preprint [4]. In the following lemma we apply a simple variational idea to define canonical harmonic one forms depending on C'. These harmonic one forms are crucial in defining the specific "Teichmueller flow" retracting onto the spine.

Lemma 2. *If the C-systoles C' do not fill the hyperbolic surface, then there exists a canonical harmonic one form ϕ such that:*

(i) the kernel $\ker \phi^*$ is parallel to α for all $\alpha \in C'$; and

(ii) ϕ satisfies $\int_\alpha \phi \geq 1$ for all $\alpha \in C'$; and

(iii) ϕ is the minimal energy harmonic one form satisfying (i) and (ii).

Proof of Lemma 2. Consider the homological image $H' := H_1(C')$ in $H_1(S)$. The annihilator $Ann(H')$ of H' in $H^1(S)$ consists of one forms ψ which satisfy $\int_\alpha \psi = 0$ for all $\alpha \in H'$. If C' does not fill S, then $H' \neq H^1(S)$ and $Ann(H')$ is a nonzero subspace of $H^1(S)$. Now consider the closed convex subset K' of all one forms ϕ which satisfy $\int_\alpha \phi \geq 1$ for all $\alpha \in H'$. From the definitions it's clear this subset is nonempty if and only if C' is not filling. Now we observe that there exists a uniquely defined L^2 shortest one form ϕ_0 in K'. This shortest one form is orthogonal to $Ann(H')$, i.e. we have $\iint_S \alpha \wedge \beta^* = 0$ for all $\beta \in Ann(H')$. By Hodge's theorem ϕ_0 is uniquely represented as a harmonic one form. This shortest harmonic one form $\phi = \phi_0$ is the desired canonical one form. ☐

For our applications we remark that the canonical one form $\phi = \phi_0$ constructed in Lemma 2 varies continuously with respect to variations in the hyperbolic metric g when the dimension of the homological span $H'(g) := H_1(C'(g))$ is constant.

Lemma 3. *The canonical harmonic one form $\phi = \phi_0$ constructed in Lemma 2 varies continuously with respect to the hyperbolic metric g on S if and only if ξ is constant along the variation.*

Proof. The proof is trivial. The C-systoles $C'(g)$ of the hyperbolic metric g is upper semicontinuous with respect to variations in the metric, i.e. if g_k is a sequence of hyperbolic metrics with limit $\lim_k g_k = g_\infty$, then $C'(g_\infty)$ contains the Gromov-Hausdorff limit of $C'(g_k)$. In simple terms, this means short vectors are preserved and there is possibility of *new* short vectors appearing in the limit. This implies that $Ann(H')$ and K' are *lower* semicontinuous with respect to g. Therefore $K'(g_\infty)$ is always contained in the Gromov-Hausdorff limit of $K'(g_k)$. All of these subsets are Gromov-Hausdorff continuous when the spans $H' = H'(g)$ vary continuously with respect to g, and this iff the dimensions are constant. $\square$

4. Belt Tightening Lemma

Recall that $C' = C'(g)$ consists of the shortest essential nonseparating geodesics on the hyperbolic surface (S, g). The following lemma is our main observation.

Lemma 4 (Belt Tightening)**.** *Let (S, g) be hyperbolic surface with C-systoles C'. If C' does not fill the surface, then there exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that*

(i) the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$, and

(ii) the curve lengths $\ell(\gamma, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\gamma \in C'$.

Belt Tightening is proved by constructing the metrics g_t as solutions of an initial value problem of an ODE. In otherwords we define infinitesimal deformations of the hyperbolic metric. Teichmueller theory identifies infinitesimal variations of hyperbolic metrics with the real parts of holomorphic quadratic differentials. See [3, Ch. 17, §60-63], [1]. Thus if g_t is a variation of hyperbolic metrics, then $g_t' = Re(q)$ where q is a holomorphic quadratic differential and $Re(-)$ denotes the real part. The harmonic one forms $\phi = \phi_0$ constructed in Lemma 2 define holomorphic quadratic differentials on S by setting $q := (\phi + i\phi^*)^2$. Consequently $Re(q) = \phi\phi - \phi^*\phi^*$ and therefore infinitesimal deformations dg of the hyperbolic metric g in the "ϕ-direction" are approximately equal to $dg = (\phi\phi - \phi^*\phi^*)$ plus higher order terms.

Recall that $\xi(g)$ is the dimension of the homological image of the C systoles of g. For hyperbolic metric g, let $\phi = \phi(g)$ be the canonical harmonic one-form constructed in Lemma 2. Consider the initial value problem defined by:

$$g' = \phi\phi - \phi^*\phi^*, \ g(0) = g. \tag{1}$$

From Lemma 3 we know ϕ, ϕ^* vary continuously with respect to g whenever ξ is constant. It's clear that ξ is *locally constant* in a sufficiently small neighborhood of g. It's also clear that ξ can increase along convergent sequences, and this occurs precisely when "new" homologically independant short curves appear in the limit. Standard results from ODEs imply that solutions to the ((1)) exist uniquely and vary continuously with respect to initial conditions only if ξ is everywhere constant. It's convenient to define a solution g_t of ((1)) as *regular* if ξ is constant along g_t.

Lemma 5. *If g_t is a regular solution of ((1)) and α is a curve in S, then $\ell(\alpha, g_t)/\ell(\alpha, g)$ satisfies [formula] for every curve α.*

Proof. □

5. Retract of Teichmueller Space onto $W = W_{2g}$

In this section we construct the well-rounded retract of $Teich(S)$. If the systoles C' do not fill S, then Lemma **??** proves there exists a canonical harmonic one form ϕ such that $ker\phi^*$ is parallel to α and ϕ is a.e. uniform for every $\alpha \in C'$. By Belt Tightening Lemma 4 we can simultaneously increase the lengths in the direction of ϕ. This leads us to defining our well rounded retract.

Definition: For every index $1 \leq j \leq 2g$, let W_j be the subvariety of $Teich(S)$ consisting of hyperbolic metrics whose C-systoles satisfy $\xi(C) \geq j$.

Theorem 6. *For every index $1 \leq j \leq 2g-1$, there exists a continuous equivariant deformation retract $W_j \to W_{j+1}$. Moreover W_{j+1} has codimension one in W_j.*

Proof. The general retract $W_j \to W_{j+1}$ is defined as follows. Let (S, g) be a hyperbolic surface in W_j with $\xi(C(g)) = j < 2g$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma 4 which simultaneously increase the lengths of C'. The result follows from the following Claims (i), (ii), (iii).

Claim (i): There exists a minimal stopping time $\tau = \tau(g)$ which depends continuously on g such that $g_\tau \in W_{j+1}$.

Equivalently τ is the unique minimal time such that a new homologically independant C-systole appears and which strictly increases the complexity ξ. Analytically τ is defined as the least time t such that $\xi(S_0(g_t)) > \xi(S_0(g))$.

Claim (ii): The one-parameter deformation defines a continuously well-posed global retraction $g \mapsto g_\tau$ from W_j to W_{j+1}.

Claim (iii): The subvariety W_{j+1} is a codimension one subvariety of W_j.

These claims are established below. $\square$

Proof of Main Theorem. The retract $Teich \to W$ is defined as the composition of retracts $W_1 \to W_2 \to \cdots \to W_{2g}$ constructed in Theorem 6. It follows that W_{2g} is a codimension $2g-1$ subvariety of $Teich(S)$, and this is the minimal possible dimension according to Bieri-Eckmann homological duality. $\square$

Remark: Geometric minimality requires a further homological duality argument a la Souto-Pettet [2].

References

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