A NEW SPINE FOR TEICHMUELLER SPACE OF HYPERBOLIC SURFACES

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ABSTRACT. We provide a self-contained construction of a minimal dimension equivariant spine W for Teichmueller space T of closed hyperbolic surfaces (S, g). The spine W consists of closed hyperbolic surfaces which are homologically filled by their shortest nonseparating essential curves. This spine is not identical with Thurston's original proposal. Our construction clarifies the numerical coincidence between $dimT(S) - vcdMCG(S) = 2g - 1$ and $dimH_1(S) - 1$. Comments welcome at jhmlabs [at] gmail [dot] com.

1. INTRODUCTION

Let S be a closed compact connected surface. Let $T = Teich(S)$ be the Teichmueller space of hyperbolic metrics g on S. In this article we prove the following

Main Theorem. *A minimal dimension equivariant spine W of Teichmueller space T consists of hyperbolic surfaces which are homologically filled by their shortest essential nonseparating curves.*

The spine W is distinct from W. Thurston's original construction [7], the key difference being our emphasis on essential nonseparating curves and their homological filling properties. The complete proof of our Main Theorem requires constructing a strong deformation retract of T onto W which is invariant with respect to mapping class group, and prove that W has codimension equal to $2g-1$. Our construction clarifies the numerical coincidence between $\dim T - vcd(MCG(S)) = 2g-1$ and $\dim H_1(S) - 1$. Compare [2], [3]. To construct continuous retractions requires canonical tangent vectors, and this is addressed in Lemma 1 which constructs canonical harmonic one forms ϕ on the surface adapted to short nonseparating curves. Our main observation is Belt Tightening Lemma 2 in §4 which says if short nonseparating curves

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are not homologically filling, then we can *simultaneously increase* their lengths by flowing along a Teichmueller deformation in the "direction of ϕ". We iterate the construction of ϕ and the Teichmueller deformations to obtain a sequence of continuous retracts $W_j \to W_{j+1}$ whose composition yields the desired deformation retract of Teichmueller space onto W.

2. Homological Filling and C-systoles

Let (S, g) be a closed hyperbolic surface with constant Gauss curvature $\kappa = -1$. A collection of curves is *essential* if the curves are homotopically nontrivial. The collection is *nonseparating* if the curves are nonzero in $H_1(S) := H_1(S, \mathbf{Z})$. Let $C = C(g)$ denote the set of all geodesic nonseparating essential curves on (S, g). A subset of geodesic nonseparating curves $C' \subset C$ with the property that the inclusion $i : C' \to S$ induces a homological isomorphism $H_1(i) : H_1(C') \to H_1(S)$ is called *homologically filling* in S.

We define the *C-systoles* of (S, g) to be the shortest curves in C relative to the hyperbolic metric g. We emphasize that C consists of essential *nonseparating* geodesics. The C-systoles of a given metric g are denoted by $C' = C'(g)$. The *complexity* of a hyperbolic metric g is defined as the dimension of the homological image generated by the C-systoles $C'(g)$. That is

$$\xi(g) := \dim \operatorname{span}(H_1(C'(g))).$$

We observe $\xi = \xi(g)$ is an integer taking every integral value between $1 \leq \xi \leq 2\mathrm{gen}(S)$. By definition $\xi(g) = 2\mathrm{gen}(S)$ iff $C_0 := C'(g)$ homologically fills S.

3. Canonical Harmonic One Forms

The construction of equivariant retracts of Teichmueller space requires defining canonical flow directions. This is subtle and crucial aspect of global continuous retracts, and leads to controversy especially with respect to Thurston's preprint [7], [1]. In the following lemma we apply a simple variational idea to define canonical harmonic one forms depending on C'. These harmonic one forms are key to defining the specific Teichmueller flow retracting onto the new spine W. Recall the standard Riemannian Dirichlet energy

$$E(\phi) := \frac{1}{2} \iint_S ||\phi||^2 dS = \frac{1}{2} \iint_S \phi \wedge \phi^* dS. \tag{1}$$

Lemma 1 (Canonical Harmonic One Form)**.** *If the C-systoles $C' := C'(g)$ do not homologically fill S, then there exists nonzero harmonic one form ϕ, unique up to sign, which minimizes Dirichlet energy* (1) *such that*

(a) the abelian condition $\phi^*|_{C'} = 0$ is satisfied along the geodesics C', i.e. $\phi^*(\alpha'(s)) = 0$ identically;

(b) the local circulation of ϕ along the geodesics $\alpha \in C'$ is a.e. constant $|\phi(\alpha'(s))| = 1$ with respect to arclength s.

Proof of Lemma 1. Suppose the geodesics $\alpha \in C'$ are parameterized by arclength $\alpha = \alpha(s)$. Consider the Lagrangian

$$L : H^1(S, \mathbf{R}) \times \mathbf{R}^{C'} \times \mathbf{R}^{C'} \to \mathbf{R}$$

defined by

$$L(\phi, u, v) = E(\phi) - u \cdot (\int [\phi(\alpha'(s)) - 1]^2 ds) - v \cdot (\int \phi^*(\alpha'(s))^2 ds).$$

We claim L has unique (up to sign) energy minimizing critical point $\nabla_{\phi,u,v} L(\phi_0, u_0, v_0) = 0$ and ϕ_0 is the desired harmonic one-form.

□

4. Belt Tightening Lemma

Recall that $C' = C'(g)$ consists of the shortest essential nonseparating geodesics on the hyperbolic surface (S, g) and $\xi(g)$ is the dimension of the homological image of C'. The following lemma is our main observation.

Lemma 2 (Belt Tightening)**.** *Let (S, g) be hyperbolic surface with C-systoles C'. If C' does not homologically fill S, then there exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that*

(i) the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$, and

(ii) the curve lengths $\ell(\gamma, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\gamma \in C'$.

Belt Tightening is proved by constructing the metrics g_t as solutions of the initial value problem (2). The initial value problem depends on the homological image generated by the C-systoles $C'(g)$, which image generates the choice of harmonic one form ϕ. Therefore the IVP is locally constant for $g \in W_j$. Teichmueller theory identifies infinitesimal variations dg of hyperbolic metrics g with the real parts of holomorphic quadratic differentials $dg = Re(q)$. See [6, Ch. 17, §60-63], [4]. The harmonic one forms $\phi = \phi_0$ constructed in Lemma 1 provide holomorphic quadratic differentials on S by setting $q := (\phi + i\phi^*)^{\otimes 2}$, with $Re(q) = \phi\phi - \phi^*\phi^*$. This provides the canonical direction $\phi\phi - \phi^*\phi^*$ along which we infitesimally deform the hyperbolic metric g. Here is the formal definition.

Let ϕ be a harmonic one form on S. Consider the initial value problem on $Teich(S)$ defined by

$$g' = Re(q) = (\phi\phi - \phi^*\phi^*), \quad g(0) = g. \tag{2}$$

Standard results from ODEs imply that integral curves of (2) exist uniquely and vary continuously with respect to initial conditions g and parameters ϕ.

Lemma 3. *If g_t is an integral curve of (2) and $\gamma = \gamma(s)$ is a geodesic curve in S parameterized by arclength s, then the length $\ell(\alpha, g_t)$ satisfies*

$$\frac{d}{dt}\ell(\gamma, g_t) = \frac{1}{2}\int \phi(\gamma')^2 - \phi^*(\gamma')^2 ds.$$

If $\gamma = \alpha$ belongs to $C'(g_t)$ for every t, then

$$\frac{d}{dt}\ell(\alpha, g_t) = \frac{1}{2}\int \phi(\alpha')^2 ds = \frac{1}{2}\ell(\alpha, g_t).$$

Proof. We compute length according to standard definition $\ell(\gamma, g_t) = \int \sqrt{g_t(\gamma', \gamma')}ds$. We differentiate under the integral and use chain rule to obtain

$$\frac{d}{dt}\ell(\gamma, g_t) = \frac{1}{2}\int \frac{1}{\sqrt{g_t(\gamma', \gamma')}}\frac{d}{dt}(g_t(\gamma', \gamma'))ds$$

which equals $\frac{1}{2}\int [\phi(\gamma')^2 - \phi^*(\gamma')^2]ds$ according to the definition of the IVP and the hypothesis that $g_t(\gamma', \gamma') = 1$, i.e. $\gamma'(s)$ is unit length.

If $\gamma = \alpha$ belongs to $C'(g_t)$ for every t, then the harmonic one form $\phi = \phi(g)$ constructed in Lemma 1 has constant circulation a.e. on $\alpha \in C'$, therefore $\int \phi^2(\alpha'(s))ds = \int 1 ds = \ell(\alpha, g_t)$ as claimed. □

Now we complete the proof of Belt Tightening.

Proof of Belt Tightening 2. We define the deformation g_t as the integral curve of IVP (2). Lemma 3 proves the curves are simultaneously increasing throughout the flow when ϕ is the canonical harmonic one form constructed in Lemma 1.

□

Lemma 4. *The canonical harmonic one form $\phi = \phi_0$ constructed in Lemma 1 varies continuously with respect to the hyperbolic metric g on S if and only if ξ is constant along the variation.*

Proof. The proof is trivial given the hypotheses. The C-systoles $C'(g)$ of the hyperbolic metric g is upper semicontinuous with respect to variations in the metric, i.e. if g_k is a sequence of hyperbolic metrics with limit $\lim_k g_k = g_\infty$, then $C'(g_\infty)$ contains the Gromov-Hausdorff limit of $C'(g_k)$. In simple terms, this means short vectors are preserved and there is possibility of new short vectors appearing in the limit. Therefore the dimension possibly increases. This implies that $Ann(H')$ and K' are lower semicontinuous with respect to g. Therefore $K'(g_\infty)$ is always contained in the Gromov-Hausdorff limit of $K'(g_k)$. All of these subsets are Gromov-Hausdorff continuous when the spans $H' = H'(g)$ vary continuously with respect to g and this occurs iff the dimension ξ is constant. □

When the limit is upper discontinuous, then the limit metric g_∞ admits a new homologically independant short curve. The affine subspace $K'(g_\infty)$ is then a strictly proper subset of GH-limit $GH - \lim_k K'(g_k)$, and the minimal g_∞-energy one form $\phi_0(g_\infty)$ is generally not equal to the Gromov-Hausdorff limit of the least energy $\mathrm{GH} - \lim_k \phi_0(g_k)$ one forms.

5. Retract of Teichmueller Space onto $W = W_{2g}$

In this section we construct the well-rounded retract of $T = Teich(S)$. Given a hyperbolic metric g, if the C-systoles $C'(g)$ do not homologically fill S, then Lemma 1 defines a canonical harmonic one form ϕ such that Belt Tightening the metric g in the ϕ direction simultaneously increases the lengths of the geodesics $\alpha \in C'(g)$. This suggests the following well rounded retract of T:

Definition: For every index $1 \le j \le 2g$, let W_j be the subvariety of T consisting of hyperbolic metrics whose C-systoles satisfy $\xi(C) \ge j$.

Theorem 5. *For every index $1 \leq j \leq 2g-1$, there exists a continuous equivariant deformation retract $W_j \to W_{j+1}$. Moreover W_{j+1} has codimension one in W_j.*

The retraction $W_j \to W_{j+1}$ is defined as follows. Let (S,g) be a hyperbolic surface in W_j with $\xi(C(g)) = j < 2g$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma 2 which simultaneously increases the lengths of C'.

Lemma 6. *For given $g \in W_j$, let g_t be the flow defined by the IVP (2) with respect to $\phi = \phi_{g_t}$.*

(a) There exists a finite minimal stopping time $\tau = \tau(g) \in \mathbf{R}_{\geq 0}$ such that $g_\tau \in W_{j+1}$.

(b) The stopping time τ is a continuous function on W_j.

Proof. The variation of length formula (3) shows that curves β which are Poincare dual to $\alpha \in C$ will be strictly decreasing in length. This implies $\tau(g)$ is finite. Moreover we claim the variation of length is non-increasing on geodesic curves β which are homologically independant of the image of $H_1(C')$ in $H_1(S)$.

□

The deformation retract of W_j onto W_{j+1} is formally defined by

$$h : W_j \times [0,1] \to W_j, \qquad h(g,t) = g_{t\tau(g)} \tag{3}$$

where g_t is the integral curve of the IVP (2). Evidently $h(g,t) = g$ for all $g \in W_{j+1} \subset W_j$ and $t \in [0,1]$.

Claim. The deformation retract h defined in (3) is continuous.

\end{proof}

Proof of Main Theorem. The retract $T \to W$ is defined as the composition of retracts $W_1 \to W_2 \to \cdots \to W_{2g}$ constructed in Theorem 5. In Lemm [ref] we proved every retract $W_j \to W_{j+1}$ is codimension one. This follows from, e.g. the continuity of the stopping time τ.

It follows that W_{2g} is a codimension $2g-1$ subvariety of T, and this is the minimal possible dimension according to Bieri-Eckmann homological duality. □

The equations for W_{2g} are, up to change of variabels, of the form $\ell(g, \alpha) = \ell(g, \beta)$ for every pair of curves $\alpha, \beta \in C'(g)$ in the C-systoles.

Claim. The dimension of the affine span of the gradients $\nabla \ell(g, \alpha)$, $\alpha \in C'(g)$, is equal to $\xi(g)$-1.

[Idea: We are separating via the normal homological dimensions]

Remark. Our argument can be applied almost verbtaim to provide an alternative construction of the well rounded retract of flat n-dimensional tori. Geometric minimality requires a further homological duality argument a la Souto-Pettet [5].

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