A Spine for Teichmueller Space of Closed Hyperbolic Surfaces (Draft)

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We claim the minimal spine W of Teichmueller space $Teich(S)$ consists of surfaces which are filled by the shortest geodesics which are simultaneously nonzero in $H_1(S)$ and nonzero in $\pi_1(S)$.

1 Two Lemmas: Minimal Geodesic Subsurfaces and Belt Tightening

We assume (S, g) is a compact hyperbolic surface with constant Gauss curvature $K = -1$.

Definition: A collection C of curves is π_1-essential if the curves are homotopically nontrivial. The collection is nonseparating, or H_1-essential, if the curves are nonzero $[\alpha] \neq 0$ in $H_1(S, \mathbf{Z})$ for $\alpha \in C$.

Definition: A subsurface S_0 of S is a surface S_0 with an isometric embedding $S_0 \hookrightarrow S$. The restriction of the metric g to $S_0 \times S_0$ is again hyperbolic, where we identify S_0 with it's image in S.

Remark: If S_0 is a proper nonempty geodesic subsurface of S, then S_0 is a surface with geodesic boundary ∂S_0.

We begin with a simple Lemma/Definition which constructs the minimal subsurface S_0 containing C.

Lemma 1: Let C be a collection of π_1-essential simple closed nonseparating geodesic curves on (S, g). Let $S_0 = S_0(C, g)$ be the intersection of all geodesic subsurfaces of S that contain C and whose boundaries ∂S_0 are disjoint from C. If C does not fill S, then S_0 is a proper subsurface of S with nonempty geodesic boundary ∂S_0.

The following Belt Tightening Lemma is our main observation.

Belt Tightening Lemma: Let S_0 be a proper subsurface of S with geodesic boundary ∂S_0. Let C be a collection of simple geodesics disjoint from ∂S_0. Then there exists a one-parameter deformation $\{g_t\}$ in $Teich(S)$ such that:

- the metric g_t is hyperbolic for all $t \geq 0$ and $g_0 = g$;
- the boundary lengths $\ell(\gamma, g_t)$ are decreasing for all $t \geq 0$ and all $\gamma \in \partial S_0$;
- the curve lengths $\ell(\alpha, g_t)$ are simultaneously increasing for all $t \geq 0$ and all $\alpha \in C$.

Proof: [Thurston, Minimal Stretch Maps preprint]

2 Well-Rounded Retract of Teichmueller Space

Here is our proposal for constructing well-rounded retracts of $Teich(S)$.

Definition: For a given metric g, let $C = C(g)$ be the set of geodesic nonseparating π_1-essential curves on (S, g).

Definition: The C-systole of (S, g) consists of those curves in C which minimize g-lengths

Definition: We say a collection of curves C' fills the surface S if the complement $S - C'$ is a disjoint union of topological disks.

Notation: For metric g, let $S_0(g)$ be the minimal geodesic subsurface constructed in Lemma 1.

Remark: Evidently $|\chi(S_0(g))| \leq |\chi(S)|$ with equality if and only if the C-systoles of g are filling. Otherwise if C' does *not* fill, then by Lemma 1 there exists a unique minimal geodesic subsurface $(S_0, \partial S_0)$ such that C' is contained in the interior of S_0 and disjoint from ∂S_0. This is the basic construction defining the retract.

Definition: Let W be the subvariety of $Teich$ consisting of hyperbolic metrics whose C-systoles fill the surface.

Definition: For every index $0 < j \leq |\chi(S)|$, let W_j be the subvariety of $Teich$ consisting of hyperbolic metrics whose C-systoles generate a minimal subsurface S_0 with $|\chi(S_0)| \geq j$.

Evidently $W_1 = Teich$ by construction and $W_{|\chi(S)|} = W$ as defined above.

Theorem: *The Teichmueller space $Teich$ continuously and equivariantly retracts onto W. Moreover W is a minimal dimension spine of $Teich$.*

Proof: The retract $Teich \to W$ is defined as a composition of retracts

$$W_1 \to W_2 \to \cdots \to W_{|\chi(S)|}.$$

The general retract $W_j \to W_{j+1}$ is defined as follows:

Let (S, g) be a hyperbolic surface in W_j with $|\chi(S_0(g))| = j < |\chi(S)|$. Let $\{g_t\}$ be the unique one-parameter deformation of hyperbolic metrics constructed in Belt Tightening Lemma which *contracts* the geodesic boundary $\partial S_0(g)$ and simultaneously expands the lengths of C' in $S_0(g)$.

Claim: There exists a minimal stopping time $\tau = \tau(g)$ which depends continuously on g such that $g_\tau \in W_{j+1}$. Equivalently τ is the unique minimal time such that a new *independant* C-systole appears and which *strictly increases the genus* of the supporting minimal subsurface $S_0(g_\tau)$). Analytically τ is defined as the least time t such that

$$|\chi(S_0(g_t))| > |\chi(S_0(g_0))|.$$

Claim: The one-parameter deformation defines a continuously well-posed global retraction $g \mapsto g_\tau$ from W_j to W_{j+1}.

Claim: The subvariety W_{j+1} is a codimension one subvariety of W_j. Therefore $W_{|\chi|}$ is a codimension $2g - 1$ subvariety of $Teich$. This is the minimal possible dimension according to Bieri-Eckmann homological duality. QED.

Remark: Geometric minimality requires a further homological duality argument a la [Souto-Pettet]. [Insert details]