A SHORT PROOF THAT MEDIAL AXIS TRANSFORM IS HOMOTOPY-ISOMORPHISM

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The goal of this article is to present a simple proof of a well-known folk theorem concerning Blum's Medial Axis Transform [Blu67]. Let A be a bounded open subset of $\mathbb{R}^N$. The medial axis $M(A)$ defined by Blum consists of all $x \in A$ for which $dist(x, \partial A)$ is attained by at least two distinct points,

$$M(A) := \{x \in A \mid \#argmin_{y \in \partial A}\{d(x, y)\} \geq 2\}. \tag{1}$$

Blum conjectured that the inclusion $M(A) \hookrightarrow A$ is a homotopy-isomorphism. This implies $M(A)$ contains all the topology of A, and is connected whenever A is. A formal proof of the conjecture for bounded open subsets of $\mathbb{R}^N$ is established in [Lie04]. The present article describes a short proof of the above homotopy-isomorphism.

A "ball" in this article designates some Euclidean open ball contained in $\mathbb{R}^N$. The open ball centred at x with radius $r > 0$ is denoted $B_r(x)$. The directed geodesic segment between a pair of points x, y is denoted $[x, y]$. The Riemannian exponential function is denoted exp_x for every basepoint x.

For $x \in A$, let $r(x)$ be the maximal radius of those balls $B_r(x)$ which are contained in A, i.e.

$$r(x) := \sup\{r \mid B_r(x) \subset A\}.$$

Balls centred at x can also be contained by balls $B_R(m)$ centred at $m \in A$ and with $B_R(m) \subset A$.

The maximal ball $B_{r(x)}(x)$ with centre at x and contained in A is unique. There are many more balls contained in A and containing $B_{r(x)}(x)$ as a subset; the next Lemma shows there exists a unique maximal such ball for every $x \in A$.

Lemma 1. *Let A be open bounded subset of $\mathbb{R}^N$. For every $x \in A$ there exists a unique maximal ball in A which both contains $B_{r(x)}(x)$ and is contained in A. I.e., there exists unique $m = m(x) \in A$ and radius $R > 0$ such that*

$$B_{r(x)}(x) \subset B_R(m) \subset A.$$

Definition 2 (Max-Centre Map)**.** Let A be open bounded subset of $\mathbb{R}^N$. For every $x \in A$, let $m(x) \in A$ be the centre of the unique maximal ball containing $B_{r(x)}(x)$ and contained in A.

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The max-centre $x \mapsto m(x)$ defines a self map $m : A \to A$. The self-map $m : A \to A$ is not one-to-one, and the inverse m^{-1} is not everywhere uniquely defined. These properties are very convenient.

Lemma 3. *Let A be a bounded open subset of $\mathbb{R}^N$. The max-centre map $m : A \to A$ is continuously differentiable.*

Definition 4. Let $V = V(A) \subset A$ be the locus of non-differentiability of the inverse m^{-1} of the max-centre map $m : A \to A$.

The Inverse Function Theorem [Spi71, pp.35] says the locus of non-differentiability $V(A)$ coincides with zero locus $Jac(m) = det(Dm) = 0$ of the Jacobian of m, and is therefore a closed subset of A by 3.

The max-centre construction has direct interpretation in terms of shape operator, principal curvatures, and Tube Formula as explained by Gromov [Gro91, pp.16-17], [MS09, Appendix, pp.118]. If $\mathbf{n}_0, \mathbf{n}_1$ are inward pointing unit normals at $y_0, y_1 \in \partial A$ (where inward is relative to A), then inward equidistant deformations

$$exp_{y_0}(\epsilon \mathbf{n}_0) \text{ and } exp_{y_1}(\epsilon \mathbf{n}_1) \tag{2}$$

are disjoint for sufficiently small $\epsilon > 0$. If A is locally concave at y_0, y_1, then inward deformations will diverge and be disjoint for every $\epsilon > 0$. However convexity of A (positive curvature) will refocus the normals $\mathbf{n}_0, \mathbf{n}_1$, and inward deformations will converge, and the exponentials in (2) will coincide. But now it is convenient to recall an important fact from Riemannian geometry, that "focalization is impossible before the cut locus", [Vil09, pp.180-1]. No focalization before the cut locus implies a useful No-Crossing result:

Lemma 5 (No-Crossing)**.** *In the above notation, if y_0, y_1 are distinct boundary points, then the geodesic segments $[y_0, m(y_0)]$ and $[y_1, m(y_1)]$ are disjoint except when $m(y_0) = m(y_1)$, in which case*

$$[y_0, m(y_0)] \cap [y_1, m(y_1)] = \{m(y_0)\}.$$

No Crossing says the union

$$A = \cup_{y \in \partial A} [y, m(y)]$$

is a type of "needle decomposition" of A. No Crossing also implies the existence and continuity of a map $h : A \times [0, 1] \to A$ defined by

$$h(x, s) = [x, m(x)]_{s.dist(x,m(x))}. \tag{3}$$

Here $z = [x, y]_s$ denotes means the unique point z on the geodesic segment joining x, y and with $dist(x, z) = s.dist(x, y)$.

Proposition 6. *The map $h : A \times [0,1] \to A$ defined by equation (3) is a continuous strong deformation retract of A onto $V(A)$ (Def.4). That is h satisfies:*
(i) $h(x,0) = x$ for all $x \in A$.
(ii) $h(x,1) \in V(A)$ for all $x \in A$.
(iii) $h(x,s) = x$ for all $x \in V(A)$, $0 \leq s \leq 1$.

Corollary 7. *The inclusion $V(A) \hookrightarrow A$ is a homotopy-isomorphism.*

Thus we find the *closed* subset $V(A)$ contains all the topology of A. Strictly speaking $V(A)$ contains Blum's $M(A)$. Oftentimes $M(A)$ is not a closed subset of A, and in fact we find $V(A)$ is the topological closure $V(A) = \overline{M(A)}$.

The question of applying the above arguments to complete Riemannian manifolds which admit unique geodesics between the necessary points, is the subject of ongoing study.

References

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