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Abstract.

Contents

1. Introduction 1
2. Sweepouts and Optimal Transportation 1
2.1. Dold-Thom Theorem 2
2.2. Regularity Assumptions on Cost 3
2.3. Sweepouts and Optimal Transport 4
2.4. What are the consequences of positive solutions to Problem 3? 4
2.5. Guth's Width Inequalities 5
References 5

1. Introduction

We seek elementary and intelligible mathematical theories. Our emphasis is always on concrete structures, on positive explicit constructions, as opposed to the abstract, implicit, presumptive methods of many "modern" mathematicians. Thus our research is much informed by classical physics and mechanics, and we especially are interested in foundational issues underlying Einstein's Special and General Relativity. Applications of Optimal Transport and Algebraic Topology is a persistent theme in our investigations.

2. Sweepouts and Optimal Transportation

Henceforth we assume X is a complete finite dimensional Riemannian manifold, possibly noncompact, with Riemannian metric g, and source measure σ on X proportional to the volume measure vol_X. For applications we suppose X is a fixed

Date: May 9, 2021.

source space. Let Y be a k-dimensional target Riemannian space, with target measure $\tau << vol_Y$. We assume $\int_X \sigma \geq \int_Y \tau$, and say the source is abundant with respect to the target. We allow the target Y to be variable, as we discuss below.

2.1. **Dold-Thom Theorem.** In the article [ref] we develop further applications of OT to algebraic topology, and especially the problem of constructing sweepouts and homology cycles via solutions to OT programs. Our starting point is a version of Dold-Thom's theorem.

Definition 1. *If X is topological space, and G is finitely-generated abelian group, then the Dold-Thom group is the kernel $AG_0(X;G) := \ker(\epsilon)$ of the augmentation map $\epsilon : G(X) \to G$, defined by $\epsilon(\sum' g_x.x) = \sum' g_x$.*

Therefore $AG_0(X;G)$ consists of all finitely-supported G-valued distributions on X with zero net sum (zero net charge). Here $\emptyset$ represents the constant 0-valued distribution on X. The vacuum state $\emptyset$ serves as canonical basepoint on $AG_0(X)$.

Theorem (Dold-Thom)**.** *The singular reduced homology functor $X \mapsto \tilde{H}_*(X;G)$ is naturally equivalent to the functor of $\emptyset$-pointed homotopy groups $X \mapsto \pi_*(AG_0(X;G), \emptyset)$.*

Corollary. *If Y is a Moore space, e.g. $Y = \mathbb{S}^q$ is a q-sphere, then $AG_0(Y;G)$ is a model of an Eilenberg-Maclane classifying space $K(G,q)$.*

According to the obstruction methods of Steenrod, Eilenberg, Maclane, it follows that there is a natural equivalence between free homotopy classes $[X, AG_0(Y;G)]$ and singular cohomology groups $H^q(X;G)$. Thus Dold-Thom allows the construction of cohomology cycles via the construction of topologically nontrivial maps $X \to AG_0(Y;G)$ whenever $H^q(X;G) \neq 0$. This leads us to a general topological problem:

Problem 1. *Construct and classify homotopically nontrivial continuous maps*

$$f : X \to AG_0(Y)$$

for given topological space X and q-sphere $Y = \mathbb{S}^q$ whenever $H^q(X) \neq 0$.

For example, construct and classify continuous maps $f : \Sigma_g^2 \to AG_0(\mathbb{S}^1)$, where Σ_g^2 is a connected Riemann surface.

Maps f solving Problem 1 generate homological cycles on X. The argument is described in R. Kirby's book [Kir06]. If f is a continuous map solving 1, then regular fibres $f^{-1}(pt)$, where pt represents a distribution on Y, are cycles in X. These cycles are nontrivial whenever f is homotopically nontrivial, and even Poincaré dual to the cocycle generated by f (assuming X is compact oriented, and Y is a Moore space).

Trivial observation: the topological abelian group $AG_0(Y)$ is a discretization of the subspace H of $L^1(Y)$ consisting of measurable functions f satisfying $\int_Y 1.f = 0$. Thus $H = 1^\perp$ is the orthogonal complement to the subspace of constant functions on Y. Every mapping $f : X \to L^1(Y)$ can be uniquely decomposed into $f = f_0 + f_1$, where f_0 is a constant-valued function on Y, and $f_1 \in H$.

For applications of optimal transport, it is necessary to first construct interesting costs c between a given source space X and the space of distributions on spheres.

Problem 2. *Construct and classify natural geometric costs c associated with correlating probability measure σ on X and probability measures τ on $AG_0(\mathbb{S}^q)$.*

What is a natural geometric cost? We say a cost is natural if it represents a reasonable interaction energy between a unit source mass at x and a unit target mass at y. We allow the possibility – as experience shows us – of the interaction energy $c(x, y)$ depending on the relative positions of x, y relative to the target measure τ. In this case the interaction energy has a Machian type interpretation. The costs should also satisfy some basic regularity assumptions. See [ref]. To fix ideas, it is already interesting problem to construct interesting costs between a surface Σ_g^2 and, say, $AG_0(S^1)$.

2.2. **Regularity Assumptions on Cost.** To begin the study of optimal transport from source (X, σ) to targets (Y, τ), we require a choice of cost function $c : X \times Y \to \mathbb{R}$ which satisfies some geometric assumptions. In this article it is sufficient to assume:

- **(A0)** The cost c is twice-continuously differentiable jointly in the source and target variables (x, y), nonnegative, and proper. Thus $c(x, y) \geq 0$ and all proper sublevels are compact.
- **(A1)** For every $y \in Y$, we assume $x \mapsto c(x, y)$ is nonconstant on every open subset of X.
- **(A2)** The cost satisfies (Twist) condition with respect to the source variable throughout $dom(c)$: for every $x' \in X$ the rule $y \mapsto \nabla_x c(x', y)$ defines an injective mapping $dom(c_{x'}) \to T_{x'}X$.

The (Twist) condition (A2) is properly understood through Kantorovich duality, as we will discuss below. It essentially guarantees the uniqueness a.e. of c-optimal transports. It moreover gives important equation describing the fibres of the optimal transport mapping.

À priori, it is difficult to construct costs c between spaces X, Y which occupy different spaces, and which have no spatial relationship, i.e. have no measure of either "near" and "far", or "hot" and "cold". In case X, Y occupy different universes, then they might be said to be infinitely far apart and the only canonical cost appears to be a constant zero (or infinite) cost. Practically speaking, the author finds the best results are obtained when the target Y is given as a subset of the source, or say by some canonical embedding $Y \hookrightarrow X$. Most interesting applications arise when $Y = \partial X[t]$, where $X[t] \subset X$ is a "rational excision" of X (see our thesis [ref:chapter] for illustrations).

Abstractly, the author is not convinced one can invent an interesting geometric cost. Rather we turn to physical models for inspiration. In our view, cost always represents a cost of energy, and not necessarily of dollars, or kilometers. In our thesis [ref] we compared the properties of *attractive* costs, e.g. the quadratic geodesic cost $c(x,y) = d(x,y)^2/2$ when $Y \subset X$, with the class of so-called *repulsive costs*. Heuristically, the attractive costs represents interaction energies between oppositely charged positive source and and negatively charged target configurations. The repulsive cost represents interaction energies between, say, positive source and positive target configurations. We recall that "opposite charges attract" and "like charges repel", hence the terminology.

2.3. **Sweepouts and Optimal Transport.** Now suppose we have a source σ, target τ, and cost c satisfying the assumptions (A012). The OT program defined by the data (σ, τ, c) generates (via Kantorovich duality) a *nontrivial contravariant functor* $Z : 2^Y \to 2^X$ defined by $Z(Y_I) := \cap_{y \in Y_I} \partial^c \psi(y)$, where $\psi = \psi^{cc}$ is the Kantorovich potential maximizing the dual program. If the c-optimal transport is regular and continuously single-valued, then the functor Z reduces to a map $Y \to 2^X$ defined by $Z(y) = \partial^c \psi(y)$. As we vary y over the target (Y, τ) we obtain Y-parameter family of closed subsets $Z(y)$ on X. Our hypotheses on c imply the cells $Z(y)$ are $(n-k)$-dimensional Lipschitz subvarieties in X for pointwise every $y \in Y$, and even smooth $(n-k)$-submanifolds for a.e. every $y \in Y$. [ref]

We propose that the parameterization $y \mapsto Z(y)$ defines even nontrivial topological sweepouts, and continuous in the necessary Almgren topologies. Explicitly this requires proving: if y_0, y_1 are sufficiently close in Y, then the cycles $Z(y_0)$ and $Z(y_1)$ bound a Lipschitz $(n-k+1)$-chain of small area, i.e. there exists a chain C such that $\partial C = Z(y_1) - Z(y_0)$ and C has arbitrarily small area. One also needs proves that the cycles $Z(y)$, $y \in Y$, assemble to the fundamental class $[X]$ of the source space, although we think this follows from the hypotheses on σ, τ.

Problem 3. *Given a source space (X, σ), construct and study costs c and target spaces (Y, τ) for which the Y-parameter family of subsets $y \mapsto Z(y)$ defines topological sweepouts of the source X. More concretely, under the assumptions (A012), prove the Y-parameter family of cycles Z is continuous in Almgren's flat chain topology.*

2.4. **What are the consequences of positive solutions to Problem 3?** Sucessfully identifying the combinations of costs c and targets (Y, τ) for which $Z(y)$ defines a sweepout of X means identifying a robust method for positively constructing homologically interesting sweepouts and families of cycles. As we describe in the next section, this also leads to a path for applications of optimal transport to systolic geometry, i.e. for constructing optimizers of min-max programs.

2.5. **Guth's Width Inequalities.** The previous section introduced the possibility of constructing topological sweepouts via solutions of OT programs. Now we study the possibility of representing minimal sweepouts by such solutions. For applications, we were motivated by Guth-Gromov's width inequality [Gut09]:

> *If $X = (X, g)$ is a closed n-dimensional Riemannian manifold, then there exists a universal constant $C(n)$ depending only on the dimension such that $width_k(X, g)^{1/k} \leq C(n) vol(X, g)^{1/n}$.*

Recall that the k-width of (X, g) is defined by a min-max problem, namely

$$width_k(X, g) := \min_{\{z_t\}} \max_t vol_k(z_t),$$

where the minimum ranges over all k-parameter sweepouts z of X, and the volume is with respect to the metric g. Estimates on $width_k$ imply every k-sweepout of X contains at least one cycle of large volume.

Our goal is to interpret $width_k$ in terms of optimal transportation. Let σ, τ, c be as above, with $Z(y) := \partial^c \psi(y)$. The assumptions (A012) imply the existence of a measurable map $T : X \to Y$ defined σ-a.e. satisfying $T\#\sigma = \tau$, and such that

$$g(y) = \int_{T^{-1}(y)} \frac{1}{|JT|} f(x) d\mathscr{H}^{n-k}(x)$$

for τ-a.e. $y \in Y$. Here $|JT|$ denotes Jacobian of T defined by $|JT| = \sqrt{DT.^t DT}$. To compare the volume of the fibres, we write

$$vol_{n-k}[T^{-1}(y)] = vol_{n-k}[\partial^c \psi(y)] = \int_{T^{-1}(y)} 1.f(x) d\mathscr{H}^{n-k}(x).$$

If the Jacobian JT has constant magnitude along the fibre $T^{-1}(y)$, then we can immediately compare the density $g(y) = \frac{d\tau}{d\mathscr{H}^k}(y)$ with the $(n-k)$-volume of the fibre $T^{-1}(y)$. But in general, we cannot expect $|JT|$ to be constant along fibres.

Furthermore the width of the sweepout depends on the fibre of *maximal* volume.

Problem 4. *Under the above hypotheses, determine method such that the volume of the fibres $T^{-1}(y) = \partial^c \psi(y)$ can be estimated and bounded by $g(y)$. That is, approximate the $(n-k)$-width of the sweepout $Z(y)$ in terms of the optimal transport data.*

References

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