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Abstract.

Contents

1. Introduction

We seek elementary and intelligible mathematical theories. Our emphasis is always on concrete structures, on positive explicit constructions, as opposed to the abstract, implicit, presumptive methods of many "modern" mathematicians. Thus our research is much informed by classical physics and mechanics, and we especially

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Date: May 7, 2021.

are interested in foundational issues underlying Einstein's Special and General Relativity. Applications of Optimal Transport and Algebraic Topology is a persistent theme in our investigations.

2. Special Relativity

2.1. **Lorentz Non Invariance of Spherical Light Waves.** Our research [ref] has identified a critical error in the foundations of Einstein's special relativity. The error essentially consists in this: that "radius" is not Lorentz invariant variable, e.g. the class of radial solutions of the homogeneous wave equation is not invariant under Lorentz transformations. This is related to a gap in Einstein's attempt to reconcile the principle of relativity (applied to non accelerated inertial frames K, K') with the law of propapagation of light. In simplest terms, the hypothesis that spherical light waves remain spherical in every inertial frame K, is untenable, and not a Lorentz invariant property. The error, present in almost every treatment of Einstein's special theory, is easily overlooked when one mistakes the expression

$$\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2$$

with the equation of a sphere. In itself, such an expression, which is indeed Lorentz invariant, represents only a null cone. To specify a sphere requires a second equation, for example,

$$d(\xi^2 + \eta^2 + \zeta^2) = 0.$$

But then we observe that the expression $\xi^2 + \eta^2 + \zeta^2$ and its differential, is definitely *not* Lorentz invariant. Again, this is critical error at the basis of SR.

2.2. **R. Sansbury's experiment: Does Light Travel Through Space?**

3. General Relativity

3.1. **Einstein's Topological PCA versus Gromov's Probabilistic PCA.** For Einstein, the only objective reality is the topological coincidence of points. This is the motivation for his seeking tensor equations, wherein the zeros $a_{ij} = 0$ are invariant with respect to change of variable, as the basis for a general theory of gravitation. For example, the left hand side of Einstein's field equations

$$R_{ij} - Rg_{ij} = \kappa T_{ij}$$

is evidently a tensor quantity, since the Ricci tensor $Ric = (R_{ij})$ is indeed tensorial, as is its trace R (scalar curvature). In our research [ref] we investigate a probabilistic formulation of Einstein's PCA, which is much closer to Gromov's category of finite probability spaces. In this work, we replace "tensor equations" – which in our mind are much too idealized objects over the real numbers $\mathbb{R}$ – with stochastic "frequency equations".

Idea: the basic tensor expression

$$d\xi = \frac{\partial \xi}{\partial x_1} dx_1 + \cdots + \frac{\partial \xi}{\partial x_n} dx_n$$

has a probabilistic interpretation when we replace $d\xi$ with the time-rate $d\xi/dt$. [Compare Gromov's paper on Mendel, where tensors and distributions are identified].

3.2. **"Is Energy Well-Defined in GR?".** The title of this subsection is taken from an unpublished essay of Erik Curiel. The right hand side of Einstein's field equations κT_{ij} describes the contribution of matter (or "mass-energy") to the system, including the electromagnetic energies. See models for T_{ij} representing a perfect fluid, or electric field, etc.. However Einstein's equivalence principle views gravitional potential energy as coincident (in an accelerated frame) with the motion or inertial mass of an object. [INCOMPLETE]

- What are the physical units in Einstein's field equations?
- Does general covariance permit the use of physical units?
- In stable low energy conditions, the map

$$\text{matter} \to \text{mass}$$

is constant, and the conservation of matter (which can neither be created nor destroyed) manifests as the conservation of mass. In this stable setting, one is justified to interpret the stress-energy tensor T_{ij} as satisfying a continuity equation. In this case, we then find T_{ij} is divergence free.

3.3. **GR and Gravitational Waves.**

3.4. **GR and Thermodynamics.** The role of physical coordinates in GR is very controversial. Since the work of Hilbert [ref], mathematicians have developed the hazardous habit of discarding apparent singularities as "merely artifacts of the given coordinate system". However thermodynamics contains strict rules for evaluating and comparing physical units. E.g., it relatives temperature to average kinetic energy. And indeed, the theory distinguishes kinetic energy from potential energy.

- is general covariance incompatible with the existence of objective physical units (e.g. temperature and thermodynamics)?

3.5. **GR and Black Holes.**

4. Dold-Thom, Sweepouts, and Optimal Transportation

In the article [ref] we develop further applications of OT to algebraic topology, and especially the problem of constructing sweepouts and homology cycles via solutions to OT programs. Our starting point is the greatest theorem you've probably never heard of, namely Dold-Thom's theorem. If X is topological space, and G is finitely-generated abelian group, then the Dold-Thom group is the kernel $AG_0(X;G) := \ker(\epsilon)$ of the augmentation map

$$\epsilon : G(X) \to G,$$

defined by $\epsilon(\sum' g_x.x) = \sum' g_x$. Therefore $AG_0(X;G)$ consists of all finitely-supported G-valued distributions on X with zero net sum (i.e. mean zero). Here $\emptyset$ represents the constant 0-valued distribution on X. The vacuum state $\emptyset$ serves as canonical basepoint on $AG_0(X)$.

Theorem (Dold-Thom)**.** *The singular reduced homology functor $X \mapsto \tilde{H}_*(X;G)$ is naturally equivalent to the functor of $\emptyset$-pointed homotopy groups $X \mapsto \pi_*(AG_0(X;G), \emptyset)$.*

Corollary. *If Y is a Moore space, e.g. $Y = \mathbb{S}^q$ is a q-sphere, then $AG_0(Y;G)$ is a model of an Eilenberg-Maclane classifying space $K(G,q)$.*

According to the obstruction methods of Steenrod, Eilenberg, Maclane, it follows that there is a natural equivalence between free homotopy classes $[X, AG_0(Y;G)]$ and singular cohomology groups $H^q(X;G)$. This leads us to the following topological problem:

Problem 1. *Construct and classify homotopically nontrivial continuous maps*

$$f : X \to AG_0(Y)$$

for given topological space X and q-sphere $Y = \mathbb{S}^q$.

For example, construct and classify continuous maps $f : \Sigma_g^2 \to AG_0(\mathbb{S}^1)$, where Σ_g^2 is a connected Riemann surface.

Remark. If f is a continuous map solving 1, then regular fibres $f^{-1}(pt)$, where pt represents a distribution on Y, are cycles in X. These cycles are nontrivial whenever f is homotopically nontrivial, [ref].

Evidently the group $AG_0(Y)$ can be viewed as a discretized version of the subspace H of $L^2(Y)$ consisting of all measurable functions f satisfying $\int_Y 1.f = 0$. Thus $H = 1^\perp$ is the orthogonal complement to the subspace of constant functions. Likewise we are replacing the set of Borel-Radon measures $\mathscr{M}(X \times Y)_{\geq 0}$ with the set of signed Borel-Radon measures having zero total measure $\int 1.d\mu(x) = 0$.

From the viewpoint of OT, it is necessary to first construct interesting costs c between the a given source space X and the space of distributions on spheres.

Problem 2. *(Discrete Version) Construct and classify geometric costs c associated with correlating probability measure σ on X and probability measures τ on $AG_0(\mathbb{S}^q)$;*

(Continuous Version) Construct and classify the natural costs c associated with correlating probability measures σ on X and probability measures τ on the orthogonal subspace H?

To be more specific about the meaning of "natural" costs: we say a cost is natural if it represents a reasonable interaction energy between a unit source mass at x and a unit target mass at y. We allow the possibility – as experience shows us – of the interaction energy $c(x, y)$ depending on the relative positions of x, y relative to the target measure τ. In this case the interaction energy has a Machian type interpretation. The costs should also satisfy some basic regularity assumptions. See [ref].

4.1. **Sweepouts and OT.** Fix a source space (X, σ). Concretely we assume X is Riemannian, with metric g, and source measure $\sigma = vol_X$ equal to canonical volume measure (modulo scalars). Let Y be a k-dimensional target Riemannian space, with target measure $\tau << vol_Y$. We assume $\int_X \sigma \geq \int_Y \tau$, and say the source is abundant with respect to the target. To begin the study of optimal transports from source to target, we require a choice of cost function $c : X \times Y \to \mathbb{R}$. For our geometric applications, it is convenient to make the following assumptions on c:
(A0)
(A1)
(A2)
(A3)
In our thesis [ref] we compared and contrasted the properties of attractive costs, e.g. $c = d^2/2$ when $Y \subset X$, with a repulsive costs. The recent article [McCann–Kim] has studied interaction potentials which have the form. Another interesting alternative is via Weber's potentials V, as discussed by AKTA (c.f. W.E.Weber's completed works).

Practically, the author finds the best results are obtained when the target Y is given as a subset of the source, or say by some canonical embedding $Y \hookrightarrow X$. Interesting applications arise when $Y = \partial X[t]$, where $X[t] \subset X$ is a "rational excision" of X.

Attractive costs measure the interaction energies of oppositely charged particles on the source and target, e.g. when source has all negative charges, and target has exclusively positive charges.

The repulsive costs represent the interaction energies of negative source charges to negative target charges. Given a cost c satisfying the assumptions (A), we obtain a contravariant functor $Z : 2^Y \to 2^X$ defined by $Z(Y_I) := \cap_{y \in Y_I} \partial^c \psi(y)$, where $\psi = \psi^{cc}$ is the Kantorovich potential maximizing the dual program. Our hypotheses on c imply the cells $Z(y)$ are $(n-k)$-dimensional cycles in X for almost every point $y \in Y$.. In other words, the OT program defined by the data (σ, τ, c) generates – via Kantorovich duality – a *nontrivial contravariant functor* $Z : 2^Y \to 2^X$. And as we vary y over the target (Y, τ), restricting the functor Z to the singletons $Y \hookrightarrow 2^Y$, we obtain Y-parameter measurable family of closed subsets $Z(y)$ on X. Our proposal is $y \mapsto Z(y)$ is an interesting "measure-sweepout". Our idea is that these maps are actually topological sweepouts continuous in the necessary topologies. Explicitly, this requires proving: if y_0, y_1 are sufficiently close in Y, then the cycles $Z(y_0)$ and $Z(y_1)$ bound a Lipschitz chain of small area, i.e. there exists a chain C such that $\partial C = Z(y_1) - Z(y_0)$ and C has arbitrarily small area.

Problem 3. *Given a source space (X, σ), study costs c and target spaces (Y, τ) for which the Y-parameter family of subsets $Z(y)$ defines topological sweepouts of the source. More concretely, under the assumptions (A), prove the Y-parameter family of cycles Z is continuous in Almgren's flat topology on chains.*

Thus we propose using the regularity theory of OT to generate continuous topological objects from the measure theoretic objects arising from Monge-Kantorovich duality.

4.2. **Application to Guth's Width Inequalities.** The previous section introduced the possibility of constructing topological sweepouts via solutions of OT programs. Now we study the possibility of representing minimal sweepouts by such solutions. For applications, we were motivated by Guth-Gromov's width inequality [ref]:

> *If M^n is a closed Riemannian manifold, then there exists a universal constant $C(n)$ depending only on the dimension n such that $width_k(X, g)^{1/k} \leq C(n) vol(X, g)^{1/n}$.*

We recall that the k-width is defined by a min-max problem, namely

$$width_k(X, g) := \min_{\{z_t\}} \max_t vol_k(z_t),$$

where the minimum ranges over all k-parameter sweepouts z of X. Estimates on $width_k$ imply every k-sweepout contains at least one cycle of large volume. Now our task is to interpret $width_k$ in terms of optimal transportation. Let σ, τ, c be as above, with $Z(y) := \partial^c \psi(y)$. The assumptions (A) imply the existence of a measurable map $T : X \to Y$ defined σ-a.e. satisfying $T\#\sigma = \tau$, and such that

$$g(y) = \int_{T^{-1}(y)} \frac{1}{||DT||} f(x) d\mathscr{H}^{n-k}(x)$$

for τ-a.e. $y \in Y$. Evidently

$$vol_{n-k}[T^{-1}(y)] = vol_{n-k}[\partial^c \psi(y)] = \int_{T^{-1}(y)} 1.f(x) d\mathscr{H}^{n-k}(x).$$

Now trivially, if the derivative DT had constant magnitude along the fibres $T^{-1}(y)$, then we could immediately compare the density $g(y) = d\tau/d\mathscr{H}^k|_y$ with the $(n-k)$-volume of the fibre $T^{-1}(y)$. However the width of the sweepout only depends on the fibre of *maximal* $(n-k)$ volume. [INCOMPLETE].

5. Souls and Spines

Many years ago a professor (J.S) told me about spines of $PGL(\mathbb{Z}^n)$, and encouraged me to try and construct spines for $Sp(\mathbb{Z}^{2g}, \omega)$ and various other discrete groups, e.g. arithmetic groups, and mapping class groups of Riemann surfaces. A method for constructing spines, and producing a wide variety of candidate spines, was the subject of our PhD thesis [ref]. Our idea was to exhibit the spines in the singularities (locus of discontinuity) of an OT program, and this reduction to singularity was the main topological tool developed via the regularity theory of OT. Beyond the construction of small dimensional classifying spaces, which was our original motivation, the construction of spines, or souls, as the singularity of an optimal transport map appears to have wider applications. We discuss some extensions of our ideas below.

5.1. **Medial Axis Transform.** Blum's medial axis transform $M(A)$ of an open subset $A \subset \mathbb{R}^n$...

5.2. **Souls in Singular Alexandrov Spaces.** Let (X, d) be an open complete finite-dimensional Alexandrov space with nonnegative sectional curvature $\kappa \geq 0$. We do not assume X is everywhere regular, i.e. that X is a manifold, and allow the possibility that X is singular. The problem of constructing souls S of X is the problem of finding a compact totally geodesic subspace $S \hookrightarrow X$ such that X strongly deformation retracts onto S. When X is smooth manifold, then G. Perelman [ref] proved that X is homeomorphic to a disk bundle over S. The construction of S, in the case of smooth manifolds, is due to Cheeger-Gromoll [ref] and Sharafutdinov [ref].

6. Mapping Class Groups

6.1. **Closing the Steinberg Symbol.**

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