ON EINSTEIN'S ALLEGED PROOF OF INVARIANCE OF SPHERICAL LIGHT WAVES IN SPECIAL RELATIVITY

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The subject of this brief note is Einstein's alleged proof of the Lorentz invariance of spherical light waves in his special relativity theory. Our purpose is to describe a positive gap in above mentioned proof, and to further demonstrate the invalidity of its conclusion. We recall that the purpose of Einstein's alleged proof is to reconcile the fundamental assumptions of special relativity, namely

(A1) that the laws of physics are the same in all nonaccelerated reference frames, i.e. if K' is a coordinate system moving uniformly (and devoid of rotation) with respect to a coordinate system K, then natural phenomena run their course with respect to K' according to exactly the same laws as with respect to K.

(A2) that light in vacuum propagates along straight lines with constant velocity $c \approx 300,000$ kilometres per second.

In Einstein's own words [Ein19, Ch.7, 11]:

> "*There is hardly a simpler law in physics than that according to which light is propagated in empty space. Every child at school knows, or believes he knows, that this propagation takes place in straight lines with velocity* $c = 300,000km/sec$*.... Who would imagine that this simple law has plunged the conscientiously thoughtful physicist into the greatest intellectual difficulties?*"

Notice the occurence of "law" in the above quote. Einstein presents (A2) as a *law*, namely the law of propagation of light. To be simultaneously consistent with (A1) then requires (A2) to be satisfied in every nonaccelerated reference frame. However the conjunction of (A1) and (A2), abbreviated (A12), appears contradictory according to classical mechanics and Fizeau's law of addition of velocities, to say the least. However Einstein claims – and this popularly lauded as among his greatest intellectual achievements – that this is only an *apparent* incompatibility, and which is resolved by postulating that Lorentz transformations relate the space- and time-measurements in the reference frames K, K'. But we submit that Einstein's argument is incomplete and his conclusion incorrect. Einstein's error in the first steps of his

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theory has been elaborated by other authors, notably [Bry], [Cro19]. The present note arises from the author's own study of the controversy, and his attempt to identify the incompatibility in plain mathematical terms.

The positive gap has a twofold source. Firstly from Einstein's confusing quadratic expressions like

$$\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2 \tag{1}$$

with "the equation of a sphere". Nay, but (1) is a three-dimensional cone in the four variables ξ, η, ζ, τ. Yes, it contains numerous "spherical" two-dimensional subsets. For instance, the standard geometric sphere S centred at the origin simultaneously satisfies (1) and the further condition

$$\frac{1}{2}d(\xi^2 + \eta^2 + \zeta^2) = \xi d\xi + \eta d\eta + \zeta d\zeta = 0 \quad \text{on } S.$$

In short, the standard geometric sphere requires that $\xi^2 + \eta^2 + \zeta^2$, and consequently $c^2\tau^2$, be *constant*. This leads us to Einstein's second error, namely that the Lorentz $SO(3,1)$ invariance of the quadratic form $h = x^2 + y^2 + z^2 - c^2t^2$ in no way implies the Lorentz invariance of the forms $h_1 = x^2 + y^2 + z^2$ and $h_2 = c^2t^2$. These two errors invalidate Einstein's argument.

To elaborate, Minkowski's form $h = x^2 + y^2 + z^2 - c^2t^2$ is invariant with respect to Lorentz transformations, i.e. the Lie group $G = SO(h) = SO(3,1)$. Invariance says $\xi^2+\eta^2+\zeta^2-c^2\tau^2$ is numerically equal to $x^2+y^2+z^2-c^2t^2$ for every Lorentz transform $(\xi, \eta, \zeta, \tau) = \phi.(x, y, z, t)$. But as we argue below, Minkowski's form is essentially the *only* Lorentz invariant quadratic form on $\mathbb{R}^4$. Now Einstein's argument concerns the geometry of a wave front generated by a light pulse, and the action of the Lorentz group on subsets of the null cone $N = \{h = 0\}$. Obviously N is G-invariant and defined by the equation $x^2 + y^2 + z^2 = c^2t^2$. Let $\mathscr{C}$ be the vector space of (possibly degenerate) quadratic forms q on $\mathbb{R}^4$, and let $h \in \mathscr{C}$ be Minkowski's form.

Theorem. *If $q \in \mathscr{C}$ is a quadratic form on $\mathbb{R}^4$ such that the restriction $q|_N$ is G-invariant, then q is proportional to h.*

Proof. The G-action on N is transitive on nonzero vectors. Therefore if $q|_N$ is G-invariant, then $q|_N$ is constant. By continuity it follows that $q|_N$ is identically zero since $q|_N = q(0) = 0$. So the zero locus of q contains the zero locus of h, i.e.

$$N \subset \{q = 0\}. \tag{2}$$

Now we use a theorem of J. H. Elton to conclude q, h are proportional, c.f. [Elt10]. We outline his argument. Let $q \otimes_{\mathbb{R}} \mathbb{C}$ and $h \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the real quadratic forms q, h. Thus $q \otimes_{\mathbb{R}} \mathbb{C} : \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is a complex-valued quadratic form. Elton's argument depends on establishing the following inclusion

$$\{h \otimes \mathbb{C} = 0\} \subset \{q \otimes \mathbb{C} = 0\}. \tag{3}$$

According to the tensor construction we have $h\otimes\mathbb{C}(x+iy)=h(x)-h(y)+2ih(x,y)$. Therefore $h\otimes\mathbb{C}(x+iy)=0$ if and only if $h(x)=h(y)$ and $h(x,y)=0$, where $h(\cdot,\cdot)$ is the bilinear form canonically defined by h. Elton's proof reduces to establishing the implication: if (2) is satisfied, then $h(x)=h(y)$ and $h(x,y)=0$ implies $q(x)=q(y)$ and $q(x,y)=0$ for all $x,y\in\mathbb{R}^4$. That q vanishes on the null cone N implies q is indefinite if it is not identically zero.

Once the inclusion (3) is established, Hilbert's Nullstellensatz [Eis13] implies $q\otimes\mathbb{C}=\lambda\cdot h\otimes\mathbb{C}$ for some $\lambda\in\mathbb{C}$. But then obviously $\lambda\in\mathbb{R}$ and the theorem follows. ☐

That is, the Minkowski form is the unique Lorentz invariant quadratic form (modulo scalars) on $\mathbb{R}^4$ which vanishes on the null cone. If the reader examines the irreducible linear representations of G, then it's possible to find invariant tensors of arbitrarily large degree. However the above theorem establishes the nonexistence of any other invariant quadratic tensors.

Now we return to the subject at hand, namely Einstein's alleged proof that (A12) are compatible with respect to Lorentz transforms. For illustration, consider the two-dimensional case in the variables x,t. Here we find $h=x^2-c^2t^2$ is invariant with respect to the group $G=SO(1,1)$ generated by $a_\theta:=\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}$, where $\theta\in\mathbb{R}$. We see $SO(1,1)$ is isomorphic to the split multiplicative torus $\mathbb{R}^\times_{>0}$ using the logarithm. The unit sphere includes two vectors $\langle 1,1\rangle$, $\langle -1,1\rangle$, and which are mapped by $a_\theta\in SO(1,1)\simeq\mathbb{R}^\times_{>0}$ to

$$\langle\xi,\tau\rangle=\langle\cosh\theta+\sinh\theta,\cosh\theta+\sinh\theta\rangle,\ \langle-\cosh\theta+\sinh\theta,\cosh\theta-\sinh\theta\rangle.$$

But evidently $\xi^2\neq x^2=1$ and $\tau^2\neq t^2=1$ when $\theta\neq 0$. Thus the quadratic forms $h_1=x^2$ and $h_2=t^2$ are not invariant. Likewise we find the image of the unit sphere $x^2=1$ does not correspond to a sphere in $\xi\tau$ coordinates.

Thus we find a positive gap in Einstein's argument, c.f. [Bry], [Cro19]. It appears likely to this author that (A2) is incorrect for another reason, namely that *light* is not something that *travels through space*. A critical examination of this assumption can be found in R. Sansbury's book [San12]. Moreover the incompatibility of (A2) with the hypothesis of "wave–particle" duality is analyzed by A.K.T. Assis [Ass99, §7.2.4, pp.133]. This is subject of future investigations.

To conclude we briefly recall the popular objections to the above line of reasoning. Firstly one might object that spheres in the frame K are well known to correspond to ellipsoids in K'. But the critics neglect that the Lorentz contraction is assumed to be a property of *material* objects, and indeed material spheres in K become material ellipsoids in K'. However light spheres are not material and not subject to Lorentz contraction. Secondly critics might object that (A12) refers to the consistent *measurement* of c in frames K,K'. But velocity is ex definitio the ratio of *distance* over *time*. It's possible (in logical sense) that the combination of length contraction and time dilation might yield the same measurement for all observers K, K'. However the material properties of both rulers and clocks make these measurements untenable. For a history of this controversy, the reader might investigate the problem of the so-called "one-way" measurement of the speed of light. See [Ver] for an amusing presentation.

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