ON EINSTEIN'S ALLEGED PROOF OF INVARIANCE OF SPHERICAL LIGHT WAVES IN SPECIAL RELATIVITY

J. H. MARTEL

1. Einstein's Attempted Resolution of an "Apparent Incompatibility"

The subject of this brief note is Einstein's alleged proof of the Lorentz invariance of spherical light waves in his special relativity theory. Our purpose is to describe a positive gap in above mentioned proof, and to further demonstrate the invalidity of its conclusion. We recall that the purpose of Einstein's alleged proof is to reconcile the fundamental assumptions of special relativity, namely

(A1) that the laws of physics are the same in all nonaccelerated reference frames, i.e. if K' is a coordinate system moving uniformly (and devoid of rotation) with respect to a coordinate system K, then natural phenomena run their course with respect to K' according to exactly the same laws as with respect to K.

(A2) that light in vacuum propagates along straight lines with constant velocity $c \approx 300,000$ kilometres per second.

In Einstein's own words [Ein19, Ch.7, 11]:

> "*There is hardly a simpler law in physics than that according to which light is propagated in empty space. Every child at school knows, or believes he knows, that this propagation takes place in straight lines with velocity* $c = 300,000km/sec$*.... Who would imagine that this simple law has plunged the conscientiously thoughtful physicist into the greatest intellectual difficulties?*"

Notice the mention of "law" in the above quote. Einstein presents (A2) as the *law* of propagation of light. To be simultaneously consistent with (A1) then requires (A2) to be satisfied in every nonaccelerated reference frame. However the conjunction of (A1) and (A2), abbreviated (A12), appears contradictory according to classical mechanics and Fizeau's law of addition of velocities, to say the least. However Einstein claims – and this popularly lauded as among his greatest intellectual achievements – that this is only an *apparent* incompatibility, and which is resolved by postulating that Lorentz transformations relate the space- and time-measurements in the

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reference frames K, K'. But we submit that Einstein's argument is incomplete and his conclusion incorrect. Einstein's error in the first steps of his theory has been elaborated by other authors, notably [Bry], [Cro19]. The present note arises from the author's own study of the controversy, and his attempt to identify the incompatibility in plain mathematical terms. In section [ref] we respond to some possible objections, and submit that these two errors invalidate Einstein's argument.

2. Lorentz Invariant Tensors: Existence and Nonexistence

The positive gap has a twofold source. Firstly from Einstein's confusing quadratic expressions like

$$\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2 \tag{1}$$

with "the equation of a sphere". But no, for (1) is a three-dimensional cone in the four variables ξ, η, ζ, τ. The cone does contain numerous "spherical" two-dimensional subsets. For instance, the standard geometric sphere S centred at the origin simultaneously satisfies (1) and the further condition

$$\frac{1}{2} d(\xi^2 + \eta^2 + \zeta^2) = \xi d\xi + \eta d\eta + \zeta d\zeta = 0.$$

In short, the standard geometric sphere requires that *two*(!) quadratic forms $\xi^2 + \eta^2 + \zeta^2$ and $c^2\tau^2$ be *simultaneously constant*. This leads us to Einstein's second error, namely that the Lorentz $SO(3,1)$ invariance of the quadratic form $h = x^2 + y^2 + z^2 - c^2t^2$ in no way implies the Lorentz invariance of the forms $h_1 = x^2 + y^2 + z^2$ and $h_2 = c^2t^2$.

To elaborate, Minkowski's form $h = x^2 + y^2 + z^2 - c^2t^2$ is invariant with respect to Lorentz transformations, i.e. the Lie group $G = SO(h) = SO(3,1)$. Invariance says $\xi^2+\eta^2+\zeta^2-c^2\tau^2$ is numerically equal to $x^2+y^2+z^2-c^2t^2$ for every Lorentz transform $(\xi, \eta, \zeta, \tau) = \phi.(x, y, z, t)$. But as we argue below, Minkowski's form is essentially the *only* Lorentz invariant quadratic form on $\mathbb{R}^4$. Now Einstein's argument concerns the geometry of a wave front generated by a light pulse, and the action of the Lorentz group on subsets of the null cone $N = \{h = 0\}$. Obviously N is G-invariant and defined by the equation $x^2 + y^2 + z^2 = c^2t^2$. Let $\mathscr{C}$ be the vector space of (possibly degenerate) quadratic forms q on $\mathbb{R}^4$, and let $h \in \mathscr{C}$ be Minkowski's form.

Theorem. *If $q \in \mathscr{C}$ is a quadratic form on $\mathbb{R}^4$ such that the restriction $q|_N$ is G-invariant, then q is proportional to h.*

Proof. The G-action on N is transitive on nonzero vectors. Therefore if $q|_N$ is G-invariant, then $q|_N$ is constant. By continuity it follows that $q|_N$ is identically zero since $q|_N = q(0) = 0$. So the zero locus of q contains the zero locus of h, i.e.

$$N \subset \{q = 0\}. \tag{2}$$

Now we use a theorem of J. H. Elton to conclude q, h are proportional, c.f. [Elt10]. We outline his argument. Let $q \otimes_{\mathbb{R}} \mathbb{C}$ and $h \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the real quadratic forms q, h. Thus $q \otimes_{\mathbb{R}} \mathbb{C} : \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is a complex-valued quadratic form. Elton's proof establishes the following inclusion

$$\{h \otimes \mathbb{C} = 0\} \subset \{q \otimes \mathbb{C} = 0\}. \tag{3}$$

According to the tensor construction we have $h \otimes \mathbb{C}(x+iy) = h(x) - h(y) + 2ih(x,y)$. Therefore $h \otimes \mathbb{C}(x+iy) = 0$ if and only if $h(x) = h(y)$ and $h(x,y) = 0$, where $h(\cdot,\cdot)$ is the bilinear form canonically defined by h. Elton's proof reduces to establishing the implication: if (2) is satisfied, then $h(x) = h(y)$ and $h(x,y) = 0$ implies $q(x) = q(y)$ and $q(x,y) = 0$ for all $x, y \in \mathbb{R}^4$. That q vanishes on the null cone N implies q is indefinite if it is not identically zero.

Once the inclusion (3) is established, Hilbert's Nullstellensatz [Eis13] implies $q \otimes \mathbb{C} = \lambda \cdot h \otimes \mathbb{C}$ for some $\lambda \in \mathbb{C}$. But then obviously $\lambda \in \mathbb{R}$ and the theorem follows. $\square$

That is, the Minkowski form is the unique Lorentz invariant quadratic form (modulo scalars) on $\mathbb{R}^4$ which vanishes on the null cone. Another argument is possible [Arm+18] which establishes nonexistence of any other Lorentz invariant $(0,2)$-tensors (the proof is long algebraic computation). Naturally the question arises whether there exist any other invariant (p,q)-tensors which arise independantly of the Minkowski form h.

Now we return to the subject at hand, namely Einstein's alleged proof that (A12) are compatible with respect to Lorentz transforms. For illustration, consider the two-dimensional case in the variables x, t. Here we find $h = x^2 - c^2t^2$ is invariant with respect to the group $G = SO(1,1)$ generated by $a_\theta := \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$. We see $SO(1,1)$ is isomorphic to the split multiplicative torus $\mathbb{R}^\times_{>0}$ using the logarithm. The unit sphere includes two vectors $\langle 1,1\rangle$, $\langle -1,1\rangle$, and which are mapped by $a_\theta \in SO(1,1) \simeq \mathbb{R}^\times_{>0}$ to

$$\langle \xi, \tau\rangle = \langle \cosh\theta + \sinh\theta, \cosh\theta + \sinh\theta\rangle,\ \langle -\cosh\theta + \sinh\theta, \cosh\theta - \sinh\theta\rangle.$$

But evidently $\xi^2 \neq x^2 = 1$ and $\tau^2 \neq t^2 = 1$ when $\theta \neq 0$. Thus the quadratic forms $h_1 = x^2$ and $h_2 = t^2$ are not invariant. Likewise we find the image of the unit sphere $x^2 = 1$ does not correspond to a sphere in $\xi\tau$ coordinates.

Thus we find a positive gap in Einstein's argument, c.f. [Bry], [Cro19].

3. Objections and Response

To conclude we briefly recall some popular objections to the above line of reasoning.

First one might object that spheres in the frame K are well known to correspond to ellipsoids in K'. But the critics neglect that the Lorentz contraction is assumed to be a property of *material* objects, where material spheres in K are seen as material ellipsoids in K'. But light spheres are not material, and not subject to Lorentz contraction.

Secondly critics might object that (A12) only requires the consistent *measurement* of c in arbitrary reference frames K, K'. But we remind critics that space- and time-measurements are always dependant on material objects, and often non local, i.e. the source and receiver might be separated by large distance. The impossibiliy of synchronizing non local clocks leads to the apparent impossibility of measuring the "one-way" velocity of light. It strikes the author that the incompatibility of (A12) is not merely "apparent" but actually *essential*, and even natural evidence that (A2) is not a proper natural law. That all measurements of "c" only succeed in measuring the "two-way" velocity of light, where the source and target coincide is discussed in [Zha97], [Pér11]. See [Ver] for entertaining introduction.

It appears likely to this author that (A2) is incorrect for another reason, namely that *light* is not something that *travels through space*. A critical examination of this assumption can be found in R. Sansbury's book [San12]. Moreover the incompatibility of (A2) with the hypothesis of "wave–particle" duality is analyzed by A.K.T. Assis [Ass99, §7.2.4, pp.133]. This is subject of future investigations.

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