ON EINSTEIN'S ALLEGED PROOF OF INVARIANCE OF SPHERICAL LIGHT WAVES IN SPECIAL RELATIVITY

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The subject of this brief note is Einstein's alleged proof of the Lorentz invariance of spherical light waves in his special relativity theory. Our purpose is to describe a positive gap in above mentioned proof, and to further demonstrate the invalidity of its conclusion. Recall that the purpose of Einstein's alleged proof is to reconcile the fundamental assumptions of special relativity, namely

(A1) that the laws of physics are the same in all nonaccelerated reference frames, i.e. if K' is a coordinate system moving uniformly (and devoid of rotation) with respect to a coordinate system K, then natural phenomena run their course with respect to K' according to exactly the same laws as with respect to K.

(A2) that light in vacuum propogates along straight lines with constant velocity $c \approx 300,000$ metres per second.

Assumption (A2) was termed "the Law of Propogation of Light", and being a *law* was therefore subject to assumption (A1).

In Einstein's own words [Ein19, Ch.7, 11]:

> "*There is hardly a simpler law in physics than that according to which light is propogated in empty space. Every child at school knows, or believes he knows, that this propogation takes place in straight lines with velocity* $c = 300000m/sec$*.... "Who would imagine that this simple law has plunged the conscientiously thoughtful physicist into the greatest intellectual difficulties?"*

Notice the occurence of "law" in the above quote. Einstein assumes that (A2) *is* a *law*, namely the law of propogation of light. To be simultaneously consistent with (A1) then requires (A2) to be satisfied in every nonaccelerated reference frame. However the conjunction of (A1) and (A2), abbreviated (A12), is contradictory according to classical mechanics and Fizeau's law of addition of velocities, to say the least. However Einstein claims – and this popularly lauded as among his greatest intellectual achievements – that this is only an *apparent* incompatibility, and which

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is resolved by postulating that Lorentz transformations relate the measurements of space- and time-coordinates in the reference frames K, K'.

But we submit that Einstein's argument is incomplete, and his conclusion incorrect. Einstein's error in the first steps of his theory has been elaborated by other authors, notably [Bry], [Cro19]. The present note arises from the author's own study of the controversy, and his attempt to clearly identify the incompatibility in plain mathematical terms.

The positive gap has a twofold source. Firstly from Einstein's confusing quadratic expressions like

$$\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2 \tag{1}$$

with "the equation of a sphere". Nay, (1) is a three-dimensional cone in the four variables ξ, η, ζ, τ. Yes, it contains numerous "spherical" two-dimensional subsets. For instance, the standard geometric sphere S centred at the origin satisfies the further condition that

$$\partial\tau/\partial\xi = 0,\ \partial\tau/\partial\eta = 0, \partial\tau/\partial\zeta = 0,$$

where $\partial\tau/\partial\xi$ denotes the directional derivative constrained to S. In short, a geometric sphere requires that $\xi^2+\eta^2+\zeta^2$ be *constant*. However the Lorentz $SO(3,1)$-invariance of the quadratic form $h = x^2+y^2+z^2-c^2t^2$ in no way implies the Lorentz invariance of the forms $h_1 = x^2+y^2+z^2$ and $h_2 = c^2t^2$. Consequently we find Einstein's attempt to resolve the incompatibility of (A12) invalidated by basic computations.

To repeat, Minkowski's form $h = x^2+y^2+z^2-c^2t^2$ is invariant with respect to Lorentz transformations, i.e. the Lie group $G = SO(h) = SO(3,1)$ where invariance means $\xi^2+\eta^2+\zeta^2-c^2\tau^2$ is numerically equal to $x^2+y^2+z^2-c^2t^2$ for every Lorentz transform $(\xi,\eta,\zeta,\tau) = (x,y,z,t).\phi$. As we argue below, Minkowski's form is essentially the *only* Lorentz invariant quadratic form on $\mathbb{R}^4$.

Now Einstein's argument concerns the geometry of a wave front generated by a light pulse, i.e. the action of the Lorentz group on subsets of the null cone $N = \{h=0\}$. Obviously N is G-invariant and satisfies $x^2+y^2+z^2 = c^2t^2$. Let $\mathscr{C}$ be the vector space of (possibly degenerate) quadratic forms q on $\mathbb{R}^4$, and let $h \in \mathscr{C}$ be Minkowski's form.

Theorem 1. *If $q \in \mathscr{C}$ is a quadratic form on $\mathbb{R}^4$ such that the restriction $q|_N$ is G-invariant, then q is proportional to h. That is, the Minkowski form is the unique Lorentz invariant quadratic form (modulo scalars) on $\mathbb{R}^4$.*

Proof. The G-action on N is transitive on nonzero vectors. Therefore if $q|_N$ is G-invariant, then $q|_N$ is constant. By continuity it follows that $q|_N$ is identically zero since $q|_N = q(0) = 0$. So the zero locus of q contains the zero locus of h, i.e.

$N \subset \{q=0\}$. Finally a theorem of J. H. Elton implies q, h are indeed proportional, c.f. [Elt10]. The details are as follows: let $q \otimes_{\otimes} \mathbb{R}\mathbb{C}$ and $h \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the real quadratic forms q, h. Thus $q \otimes_{\mathbb{R}} \mathbb{C} : \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is a complex-valued quadratic form. Elton's argument depends on establishing the following inclusion

$$\{h \otimes \mathbb{C} = 0\} \subset \{q \otimes \mathbb{C} = 0\}. \tag{2}$$

According to the tensor construction we have $h \otimes \mathbb{C}(x+iy) = h(x) - h(y) + 2ih(x,y)$. Therefore $h \otimes \mathbb{C}(x+iy) = 0$ if and only if $h(x) = h(y)$ and $h(x,y) = 0$, where $h(\cdot,\cdot)$ is the bilinear form canonically defined by h. Once the inclusion (2) is established, Hilbert's Nullstellensatz [Eis13] implies $q \otimes \mathbb{C} = \lambda h \otimes \mathbb{C}$ for some $\lambda \in \mathbb{C}$. But obviously $\lambda \in \mathbb{R}$ and the theorem follows. $\square$

Now we return to the subject at hand, namely Einstein's alleged proof that (A12) are compatible with respect to Lorentz symmetry. The substance of Einstein's second error is this:

> **the Lorentz invariance of h, N does not imply the Lorentz invariance of the forms $h_1 = x^2 + y^2 + z^2$ and $h_2 = c^2 t^2$.**

The two-dimensional case in xt-variables is illustrative, where $h = x^2 - c^2t^2$ is invariant with respect to the group $G = SO(1,1)$ generated by $a_\theta := \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$. We see $SO(1,1)$ is isomorphic to the split multiplicative torus $\mathbb{R}^\times_{>0}$ using the logarithm. The unit sphere includes two vectors $\langle 1,1\rangle$, $\langle -1,1\rangle$, and which are mapped by $a_\theta \in SO(1,1) \simeq \mathbb{R}^\times_{>0}$ to

$$\langle \xi, \tau \rangle = \langle \cosh\theta + \sinh\theta, \cosh\theta + \sinh\theta \rangle, \ \langle -\cosh\theta + \sinh\theta, \cosh\theta - \sinh\theta \rangle.$$

But evidently $\xi^2 \neq x^2 = 1$ and $\tau^2 \neq t^2 = 1$ when $\theta \neq 0$. That is, the image of the unit sphere does not correspond to a sphere in $\xi\tau$ coordinates. In other words the equation $x^2 = c^2t^2$ is not the equation of sphere when x, t are variable, and likewise $\xi^2 = c^2\tau^2$ is not the equation of sphere when ξ, τ are both variable and nonconstant.

The author submits that the above argument positively demonstrates a gap in Einstein's purported proof. The reader may consult Einstein's own words, e.g. [Ein19][Ch.11, pp.39] and verify that Einstein does not treat the general case but restricts himself to a velocity parallel to x-axis.

The above gap in Einstein's argument has been investigated by several critical authors, notably [Bry], [Cro19]. Our goal in this article has been to describe this gap in simple mathematical terms. It appears likely to this author that item (A2) is incorrect for another reason, namely that *light* is not something that *travels through space*. A critical examination of this assumption can be found in R.Sansbury's book [San12]. Moreover the incompatibility of (A2) with the hypothesis of "wave–particle" duality is analyzed by A.K.T. Assis,[Ass99, §7.2.4, pp.133].

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