COUNTEREXAMPLES TO GUTH'S SPONGE PROBLEM

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Abstract.

1.

Let D^n be the n-dimensional unit radius disk in euclidean $\mathbb{R}^n$. Suppose U, V are open subsets of $\mathbb{R}^n$.

Definition 1. A continuously differentiable map $f : U \to V$ is called an expanding embedding (EE-map) if f is injective continuously differentiable map and $||D_x f(v)|| \geq ||v||$ for every $x \in U$ and tangent vector $v \in T_x U$.

Here $|| \cdot ||$ refers to Euclidean length. The second local condition holds iff the symmetric matrix ${}^t Df.Df$ has all eigenvalues ≥ 1. Equivalently f is EE-map iff f increases the induced path length between all points and curves in U.

Larry Guth's *Sponge Problem* asks:

> *Does there exist a constant $\epsilon = \epsilon_n$ such that every open subset $U \subset \mathbb{R}^n$ satisfying $vol(U) < \epsilon_n$ admits an EE-map $f : U \hookrightarrow D^n$?*

This article considers the negation of Guth's question:

> *Does there exist open subsets U of $\mathbb{R}^n$ with arbitrarily small volume which do not admit EE-maps $f : U \to D$ into the round unit disk ?*

The question of which open subsets U of $\mathbb{R}^n$ admit EE-maps into the unit disk depends on the *linear diameter* of the open subset $U \hookrightarrow \mathbb{R}^n$.

Definition 2 (Linear Diameter)**.** The linear diameter of an open subset $U \hookrightarrow \mathbb{R}^n$ is defined as

$$diam(U) := \max_{x,y \in U} ||x - y||.$$

In this article we always use $diam(U)$ to designate the *linear diameter* of the open subset U. We make some trivial observations: we obviously need the ambient Euclidean structure to define $||x - y||$ for $x, y \in U$. The linear diameter of $f(U)$ is the diameter of the smallest disk which contains $f(U)$, and it's almost tautological that an open subset can be rigidly mapped into the unit disk if and only if it's linear

Date: September 1, 2022.

diameter is ≤ 1. If U is an open subset of $\mathbb{R}^n$ with $diam(U) > 1$, then there does *not* exist an isometry (rigid motion) of U into the unit disk nor into any open subset V with smaller diameter $diam(V) \leq 1$.

In general, the linear diameter of an open subset U can *decrease* in the image $f(U)$ of an EE-map and its possible that $diam(U) > diam(f(U))$. Of course the induced path length diameter of $f(U)$ is greater than the path length diameter of U, but this is no obstruction to being EE-mapped into D. Indeed the unit disk can contain arbitrarily long curves, but these curves need rotate around the disk and cannot all share a common direction. Rotations and non parallel directions cause the linear diameter to diminish in the image.

2.

We propose to study open subsets U via hard disk packings P of U. For a general random disk packing P of U, there are "rattlers" and disks are likely going to have degrees of freedom inside U, i.e. directions where the hard disks can be displaced. It's useful to use the terminology of force loads $\mathbf{b}$ on the packing P. Formally a force load is a vector-valued map $\mathbf{b} : P \to \mathbb{R}^n$. The force loads are useful to define various types of rigidity. Under various assumptions of rigidity on the neighboring particles or the boundary ∂U, we obtain different types of *jamming*. The following three jamming definitions are taken from [Don+04, §2.5].

Definition 3. A packing $P = \{P_i\}$ of an open subset U is *locally jammed* if each disk P_i can support any load $\mathbf{b}$ with it's neighbors being held fixed.

The packing is *collectively jammed* if it can support without rearrangement any load $\mathbf{b}$ of the disks with the condition that the boundary ∂U is pointwise fixed.

The packing is *strictly jammed* if it can support without rearrangement any load $\mathbf{b}$ with compressive global boundary component (so-called "positive macroscopic pressure") on ∂U.

To quote from [Ibid]:

> "A collectively jammed packing can support any load without rearrangements of the particles as long as the boundary is held fixed externally. Strictly jammed packings on the other hand can support any load with a compressive global (boundary) component (i.e., positive macroscopic pressure). Note, however, that a packing may be able to support all compressive global loads even though it is not strictly jammed, as it may be unstable due to the existence of collective unjamming mechanisms."

We say the packing P is rigid relative to U if P is strictly jammed as per the above definition. The contrapositive is that the packing has nontrivial rearrangement in response to the load only if the force load is expansive with respect to ∂U. This means the domain needs its volume expanded in order for the rigid disks to admit nontrivial displacement.

So if we are given a random packing P what category of jamming, if any, does it satisfy? The definition (3) means we consider random loads $\mathbf{b} = (\mathbf{b_i})_\mathbf{i} \in (\mathbb{R}^\mathbf{n})^\mathbf{P}$ which is a distribution of forces $\mathbf{b_i}$ over the various disks P_i in the packing. Given a random load $\mathbf{b}$ the question is: how does the packing P react to the load? Can P support the load in its given state, or does the load cause the packing P to *rearrange* itself into a new packing Q? Since we assume the disks in the packing are hard and rigid, the load causes a stress and the packing reacts and displaces to a new stress minimizing position if possible.

Lemma 1. *Let P be a disk packing of an open subset U of $\mathbb{R}^n$. If P is rigid relative to U, then $diam(f(U)) \geq diam(U)$ for all EE-maps $f : U \to \mathbb{R}^n$.*

Proof. (Proof needed. It's possible that we need further hypothesis on U, i.e. that U is contained $P \subset U \subset B$ where B is the minimal round disk containing P.) ☐

[To Be Continued ...]

References

[Don+04] A. Donev et al. "A linear programming algorithm to test for jamming in hard-sphere packings". In: *Journal of Computational Physics* 197.1 (2004), pp. 139–166.

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