Applications of Optimal Transport to Algebraic Topology

How to build Spines from Singularity...

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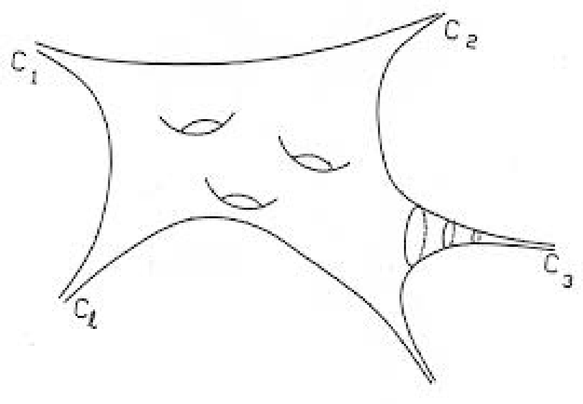
Thesis available: https://github.com/jhmartel

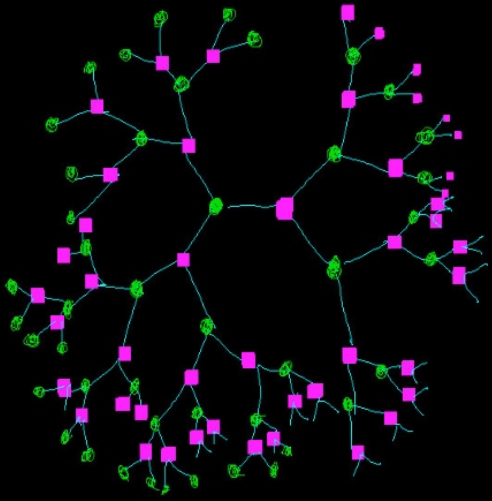
Our favourite problem of algebraic topology:

IF (X,d,vol_X) is a finite-dimensional, complete,
finite-volume NPC space,
THEN:
construct a *minimal* subvariety Z of X,
s.t. the inclusion Z $\rightarrow$ X is a homotopy-isomorphism,
and strong-deformation retract.



DEF: maximal-codimension retracts *Z* are called *Spines* / *Souls* *of (X,d,vol_X).*
($\kappa \leq 0$) / *($\kappa \geq 0$)*

"subvariety" means satisfying a finite number of locally-lipschitz equations ("local DC")
[definition from Poincare's ``Analysis Situs"]



X EΓ-model

=Universal Cover of space M

$\pi_1(M) = \Gamma$

← ← ***Goal: construct Spines from EΓ models** (κ ≤ 0)*

i.e. construct minimal Universal Covering Spaces for π1=Γ

X / Γ quotient with $\pi_1 = \Gamma$

Poincare ~1895 constructs Algebraic-Topology:

Our thesis introduces and develops a general method
for explicitly constructing **SMALL**-dimensional $E\Gamma$ classifying spaces.

Hypothesis: Γ is infinite, discrete, Bieri-Eckmann duality group

with finite cohomological dimension $cd(\Gamma)=\nu < +\infty$ and dualizing module **D**.

Our method presumes a user has explicit geometric $E\Gamma$ model X:

i.e. (X,d) is finite-dimensional Cartan-Hadamard space (NPC contractible)

where group action $\mathbf{X} \times \Gamma \to \mathbf{X}$ is isometric, proper discontinuous, free,

and quotient X/Γ finite volume w.r.t. volume measure vol_X.

Our arguments are general, with applications to many popular "geometric" groups:

Ex: Γ = **Z**, **Z^2**, **Z^3**, …. *torsion-free abelian groups*

= GL(**Z**^2), GL(**Z**^3), Sp(**Z**^4), Sp(**Z**^6), … *arithmetic subgroups* ***G(Z)*** *of* ***Q****-reductive groups,*

= MCG(Σ_g) … *mapping class groups of closed orientable surfaces,*

= *knot groups*,

= *braid groups,* etc.

……. all models not created equal…..

Ex: - MCG(Σ_g) acts isometrically on Teich(Σ_g) with *Weil-Petersson* geometry.

- **G(Z)** acts isometrically on spaces of quadratic (or hermitian) forms.

- GL(**Z**^2) acts isometrically on Voronoi's cone, projectivizes to H^2 *(Poincare disk)*

Bieri-Eckmann homological duality:

If X is geometric $E\Gamma$ model, then typically $\dim(X) \gg \ldots. \gg cd(\Gamma)$.

Bieri-Eckmann formula: $\boxed{cd(\Gamma) = \dim(X) - (q + 1),}$

q=spherical-dimension(**D**).

Eilenberg-Ganea: *$cd(\Gamma)$ == minimal dimension of $E\Gamma$-models ** if $cd(\Gamma) > 2$*

Ex: *Teichmueller space T_g*

$\dim(T_g) = 6g-6$, $\quad\quad$ $\mathrm{vcd}(MCG(\Sigma_g)) = 4g-5$,

(Harer/Ivanov)

– *Open Problem: where is explicit (4g-5)-dimensional model of $EMCG(\Sigma_g)$??*

(g=2) Seek 3-dimensional retract of 6-dimensional $X=T_2$

– Our “final solutions” are obstructed by a problem we call “Closing the Steinberg symbol”

New General Method for Large-Codimension Retracts:

Let Γ be Bieri-Eckmann duality group, and (X,d,vol_X) a geometric EΓ model.

Our thesis constructs **Γ-invariant closed subsets Z of X, with dim(Z) ≈ cd(Γ)**,

for which the **inclusion Z → X is homotopy-isomorphism**,

and explicit **Γ-equivariant continuous deformation retracts X onto Z;**

and moreover we describe a technique

(based on specific solutions of a problem called **“Closing the Steinberg symbol”**)

for **achieving Z with MAX codimension dim(Z) == cd(Γ)**.

Spines and Souls : Tradition in Geometric-Homology.:

Klein, Minkowski, Poincare, Steenrod, Thom, Lefschetz, Wall, Eilenberg, Ganea, Borel, Serre, Thurston, Gromov, Neeman, Mumford, Gromoll-Cheeger-Perelman, Soule, Ash, McConnell, ….

...how to construct NEW models of "old" spaces, and as explicit as possible?

"Textbook" constructions of $E\Gamma$ are abstract/external/dislocated

- Requires perfect knowledge of Γ, i.e. generators and relations.
- Milnor: $E\Gamma$=joins($\Gamma,\Gamma,\Gamma,\ldots$).
- Wall: inductive wedges of spherical-complexes and attaching maps.
- Postnikov towers, Cayley graphs, Rips complex, … .

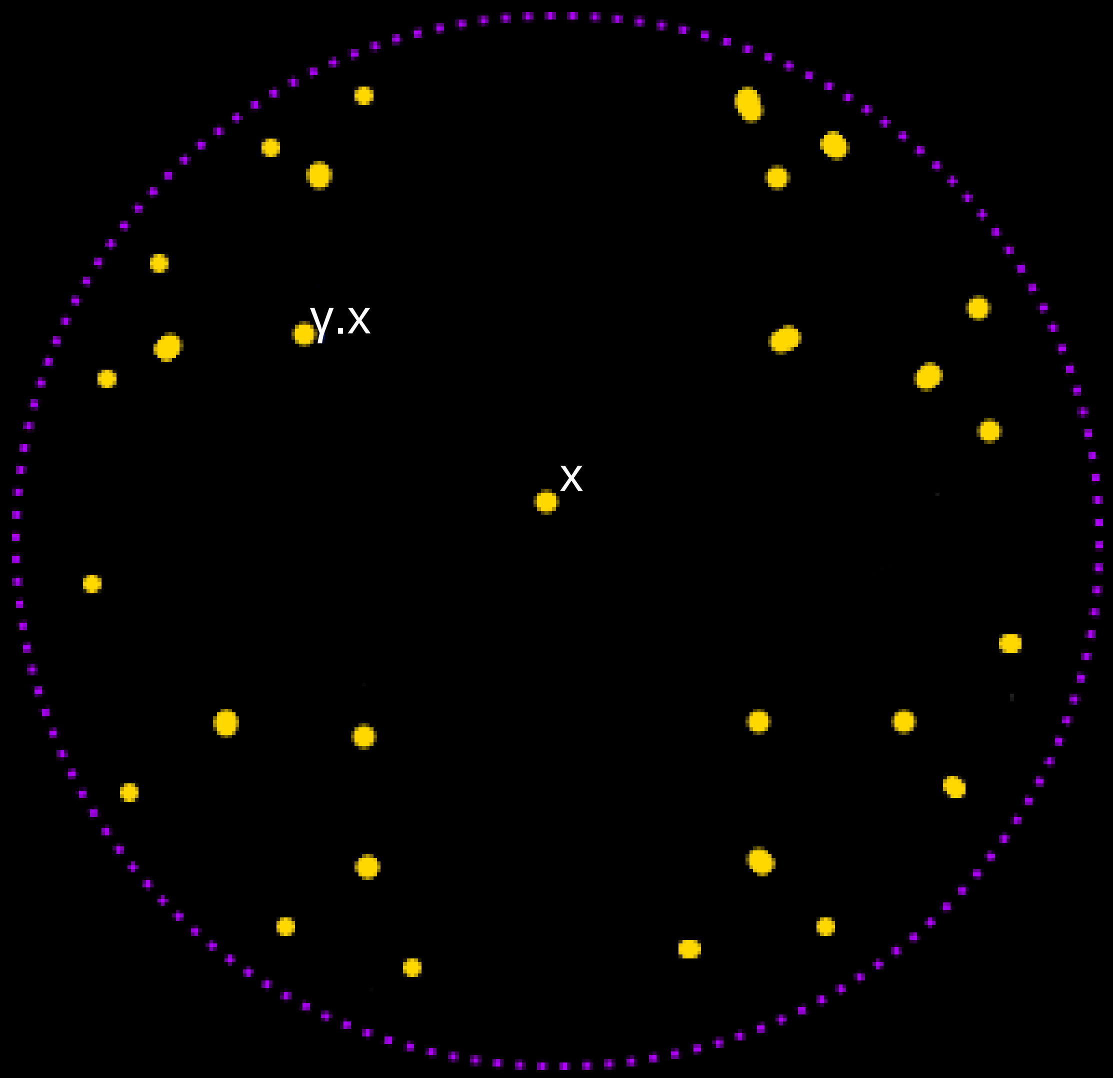
We presume limited knowledge of Γ, but *require explicit geometric* $E\Gamma$-model (X,d,vol_X).

Our thesis develops a new program for In Situ Reduction-to-Spine,
- exhibits Spines as explicit subsets of initial model X
- nonlinear extension of Soule-Ash's Well Rounded Retract
[1980s, explicit Spines for EGL(Z^N), N≥1]

...illustrating our approach:

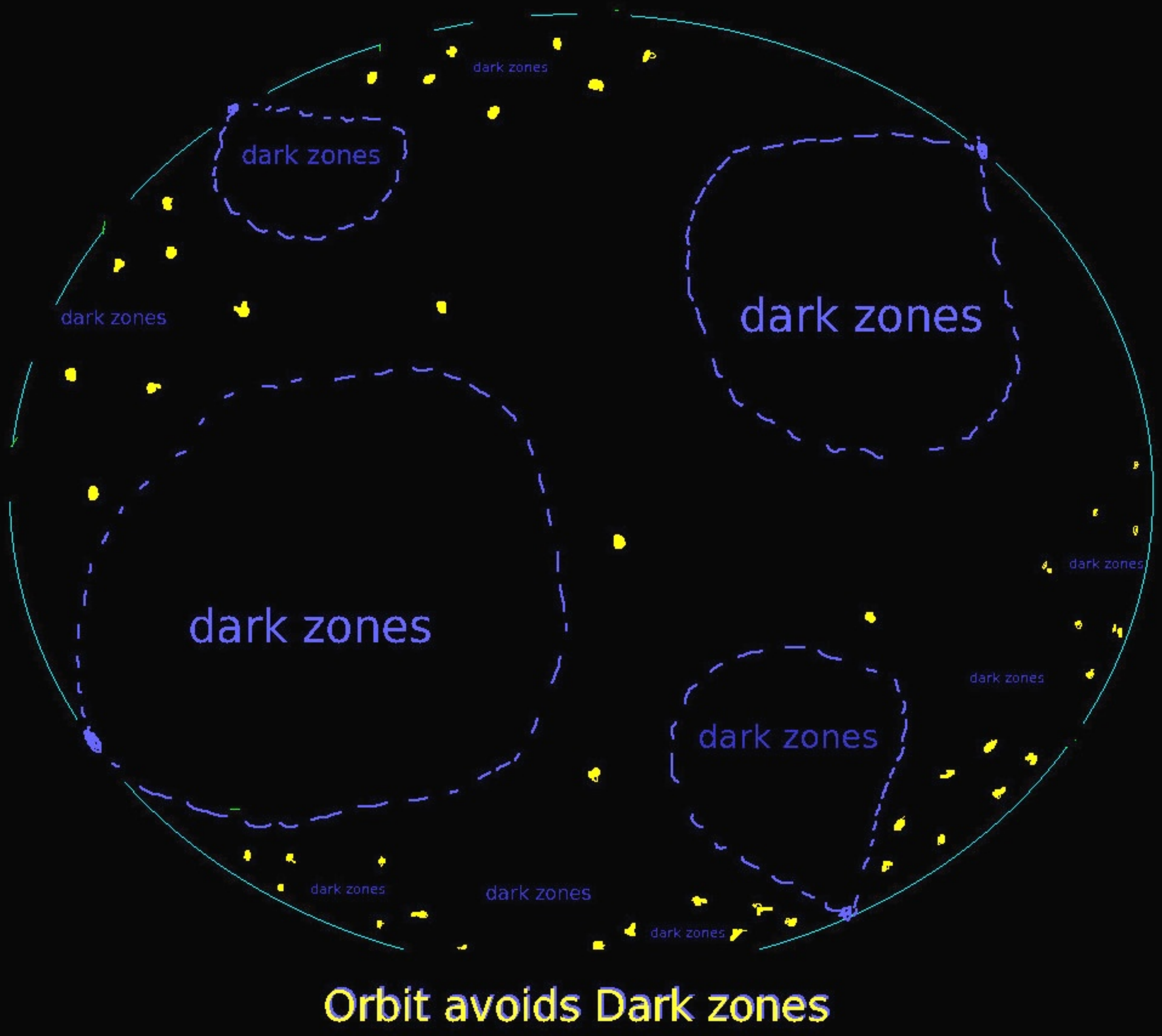
Begin with initial geometric
EΓ model (X,d,volX).

Yellow dots = Γ orbit of pt. x

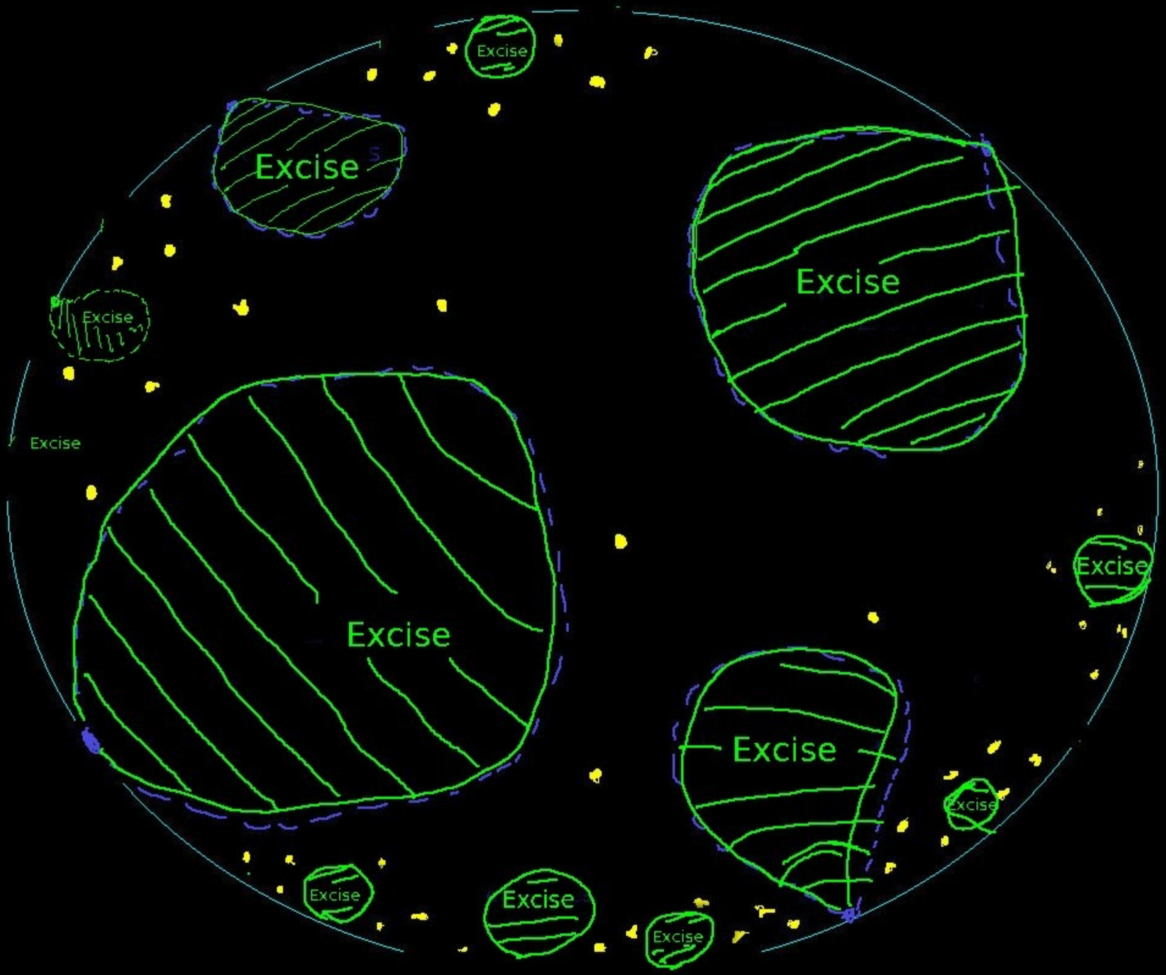


Observe:
Γ-orbit avoid dark zones.

*Dark zones =
countable family of
Γ-rational horoballs V[t]*



Excision X[t] obtained by scooping out / excising the dark zones from X.



Orbit avoids Dark zones

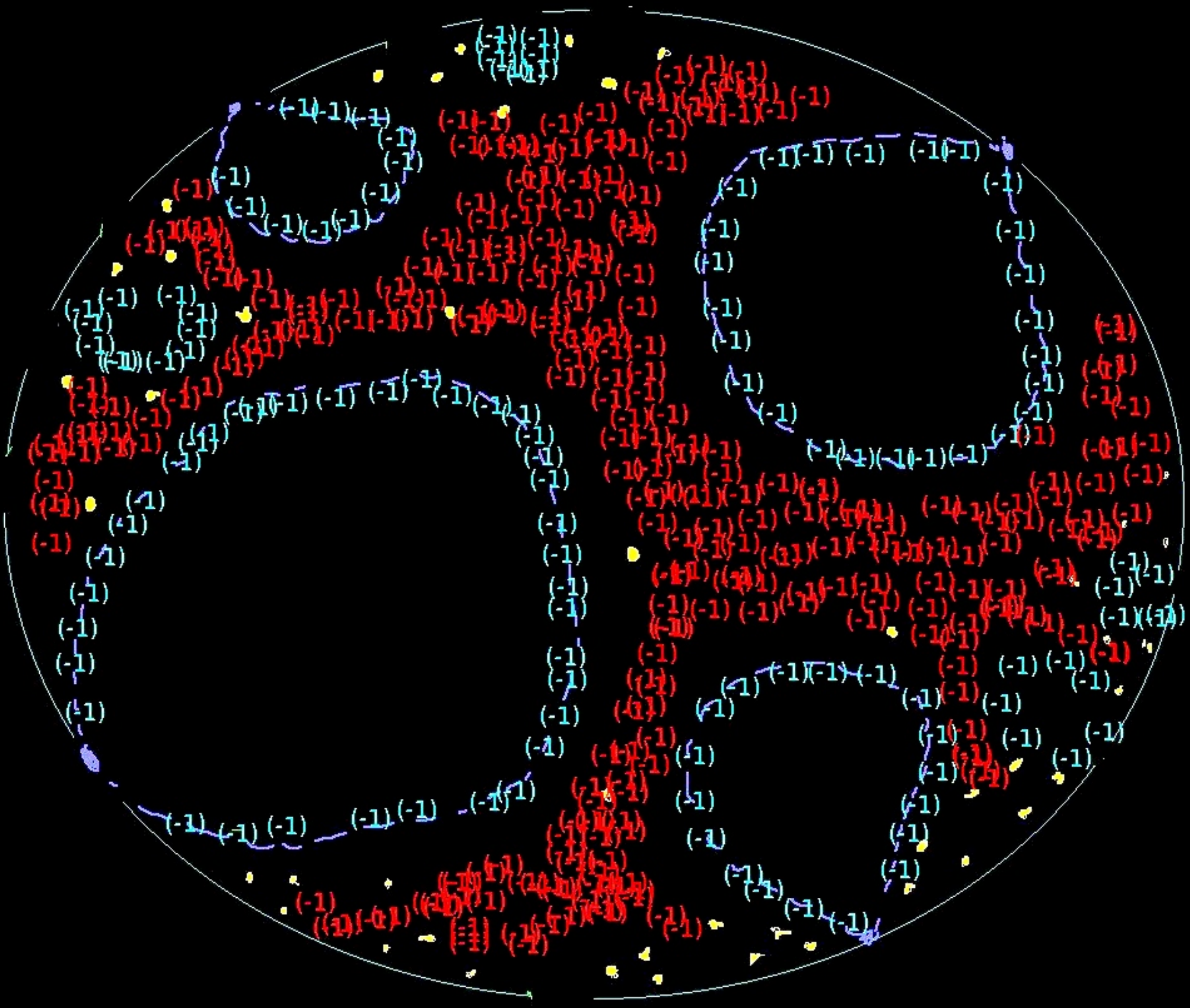
Scoop/Excise convex deep dark zones at-infinity.

Excision ==>
manifold-with-corners
$X[t] \times \delta X[t]$.

Define:

(-1) source measure σ on X[t]

(-1) target measure τ on δX[t]

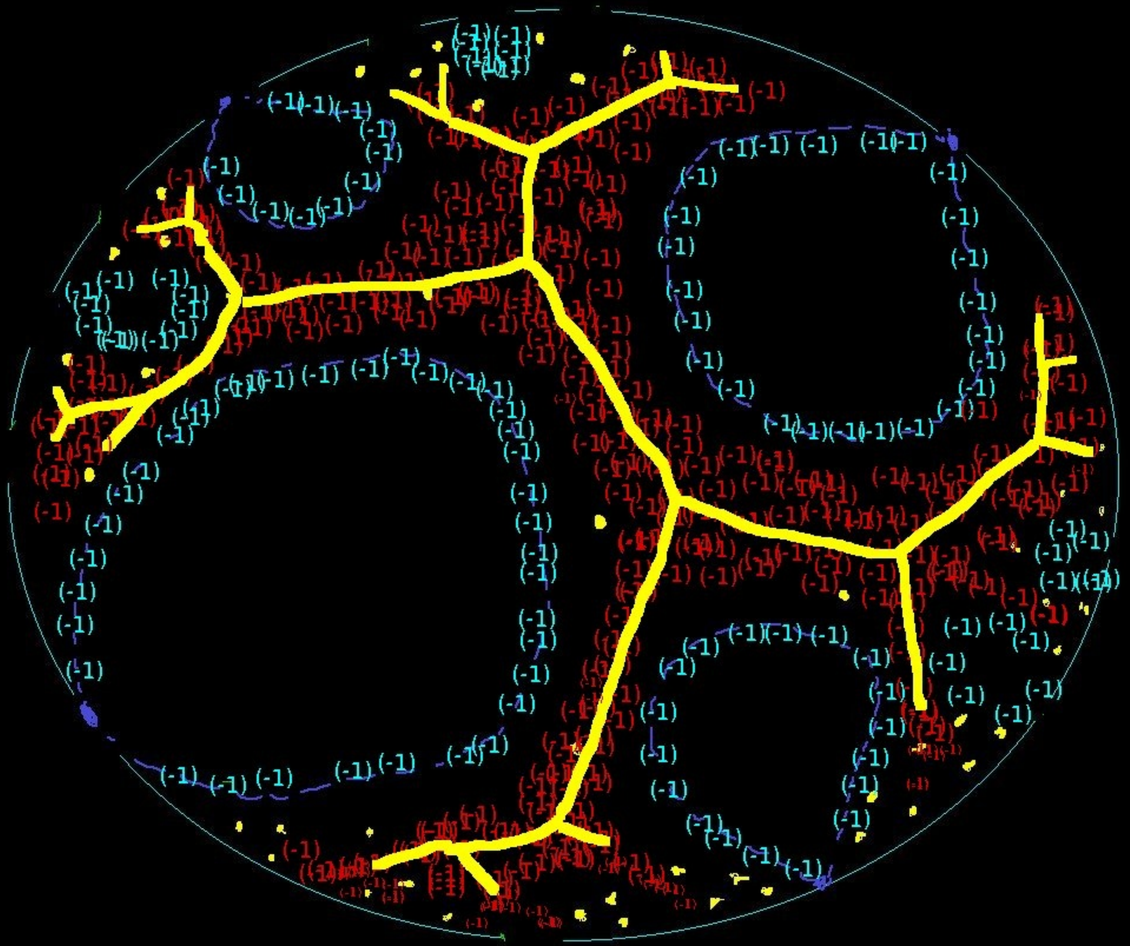


With measures σ, τ, we next study the Singularity structure of "Energy-minimizing" semicouplings from source to target.

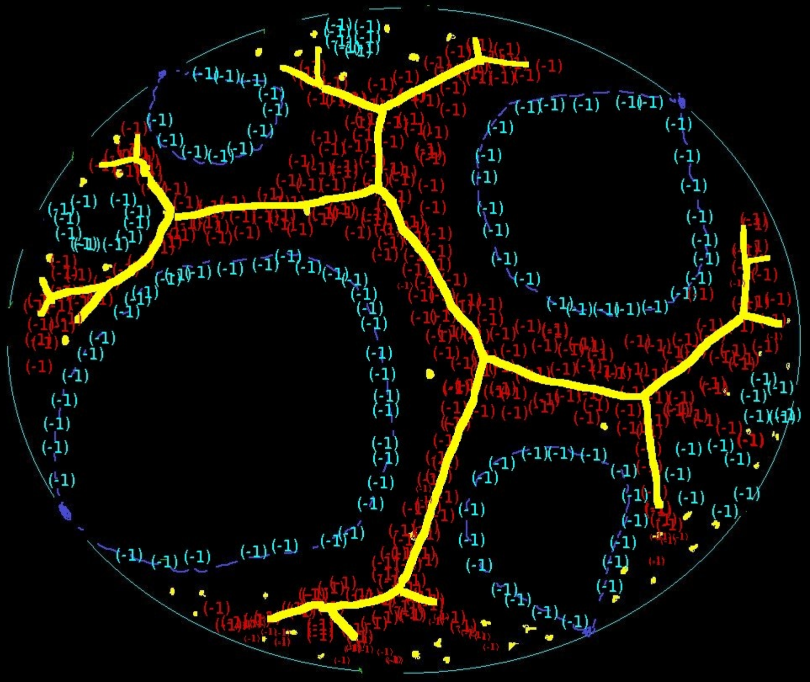
$$Z: 2^{\delta X[t]} \rightarrow 2^{X[t]}$$

We propose:

Spines are readily displayed in the locus-of-discontinuity (Singularities) of optimal semicouplings from X[t] to δX[t]



Singularity structure on the
activated energy-minimizing configuration



Remarks: this one-dimensional tree T is long-known, and many retractions of X onto T exist.

Except our construction (via Singularity structure) applies <u>verbatim</u> to higher dimensions...

...Spines are not hidden…

...Spines are not hidden…

“the idea”

Spines are readily displayed
in *locus-of-discontinuity Z*
of “deformation retracts $r : X[t] \to \delta X[t]$ ”

...Spines are not hidden…

“the formalization”

Spines readily displayed in
Kantorovich’s Contravariant Singularity Functor $Z(\sigma,\tau, v)$
of v-optimal semicouplings π from $(X[t], \sigma)$ to $(\delta X[t], \tau)$

source target

… *where* $v: X[t] \times \delta X[t]$ *is the ``visible repulsion” cost on a* Γ*-rational excision* $X[t]$

Terms to define:

Singularity functor $Z(\sigma,\tau, \nu)$, $Z : 2^{\delta X[t]} \to 2^{X[t]}$

of ν-optimal semicouplings π

from source $(X[t], \sigma)$ to target $(\delta X[t], \tau)$

of Γ-rational excision $X[t]$

where $\nu: X[t] \times \delta X[t] \to R$ is “visible repulsion cost”.

Terms to define:

Topology:

Source excision models $(X[t], \sigma)$

Target $(\partial X[t], \tau)$.

Steinberg modules $D := \tilde{H}_q(\partial X[t]; \mathbb{Z})$.

Steinberg symbols $B \in H_q(\partial X[t]; \mathbb{Z})$

and $FILL[B] = H_{q+1}(X[t], \partial X[t]; \mathbb{Z})$.

Chain sums $\underline{F} = \sum_{\gamma \in \Gamma} F.\gamma$

with well-separated gates $\{G\} = \{$FILL[B].γ $\mid \gamma \in \Gamma\}$.

Terms to define:

Mass transport:

Costs $c : X[t] \times \partial X[t] \to \mathbb{R}$.

Two-pointed repulsion and visibility costs c* , v

c-optimal semicouplings π.

c-concave potentials $\psi^{cc} = \psi$.

c-subdifferentials $\partial^c \psi(y) \subset X[t]$ for $y \in \partial X[t]$.

Monge-Kantorovich duality: c-optimal semicouplings π supported on graph of $\partial^c \psi$.

Kantorovich Singularity functor $Z : 2^{\partial X[t]} \to 2^{X[t]}$.

Filtrations $Z_0 \hookleftarrow Z_1 \hookleftarrow Z_2 \hookleftarrow \cdots$.

Kantorovich's Contravariant Singularity Functor

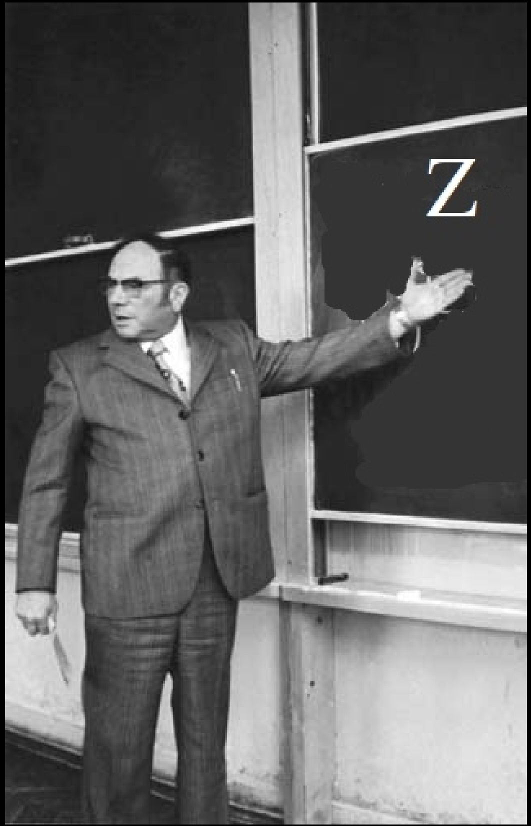
$$Z : 2^{\partial X} \to 2^X, \quad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y) = \text{“} \cap_{y \in Y_I} r^{-1}(y) \text{”}.$$

… but everything summarized in: *Kantorovich's Contravariant Singularity Functor*

$$Z : 2^{\partial X} \to 2^{X},$$

$$Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

$$= \text{“} \cap_{y \in Y_I} r^{-1}(y) \text{”}.$$

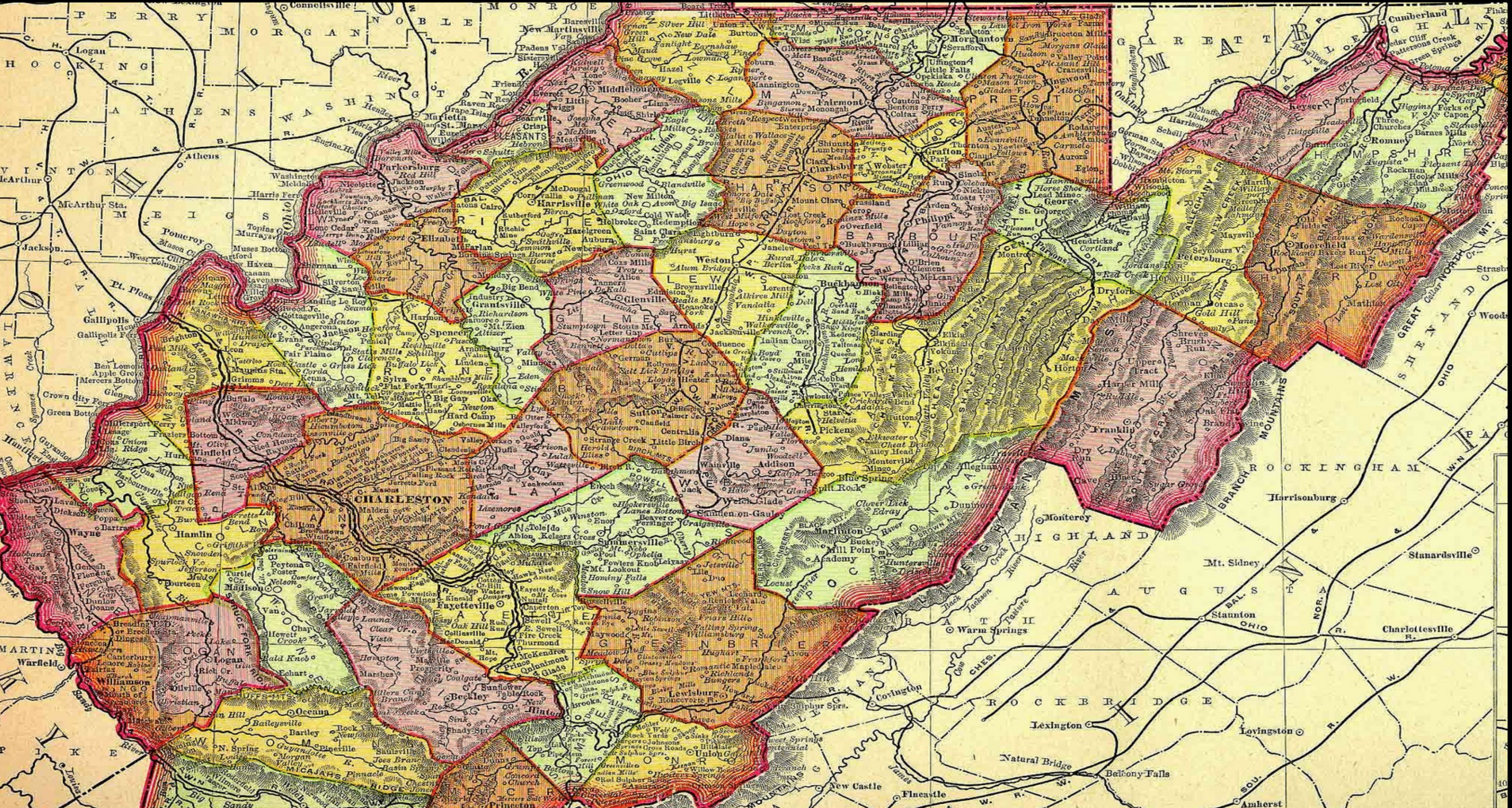


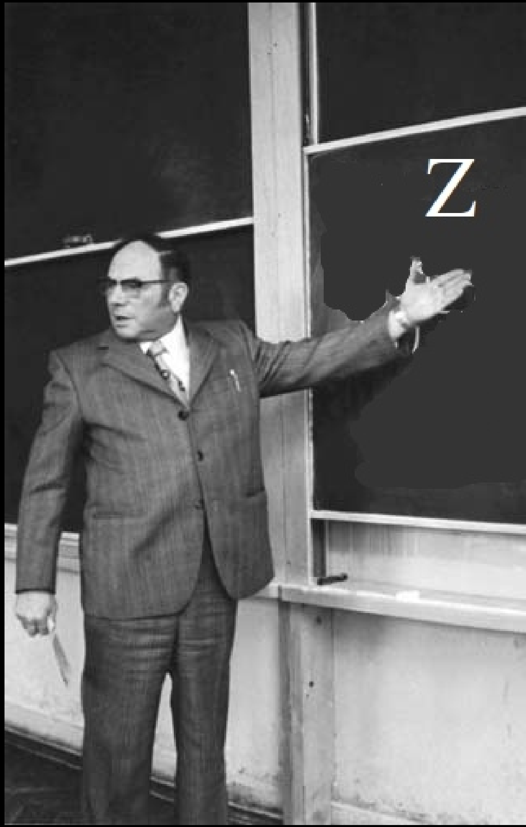
$\psi^{cc} = \psi$ is c-concave potential, $\psi: \delta X[t] \to R$, *and* Y_I < δX[t] *closed subset.*

$$Z : 2^{\partial X} \to 2^X, \qquad Z(Y_I) = \cap_{y \in Y_I} \partial^c \psi(y)$$

***Singularity** is overburdened term.*

<u>Economic Definition:</u>
Singularity arises wherever there is competition for limited common resources.
<u>Singularity is Why countries exist with borders.</u>



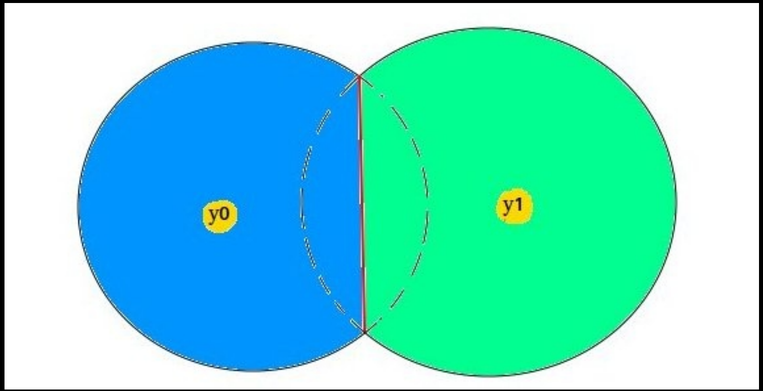


Singularity: exists *wherever competition for limited common resources.*

c=d**2/2 quadratic cost
(+)→ ← (–) attraction

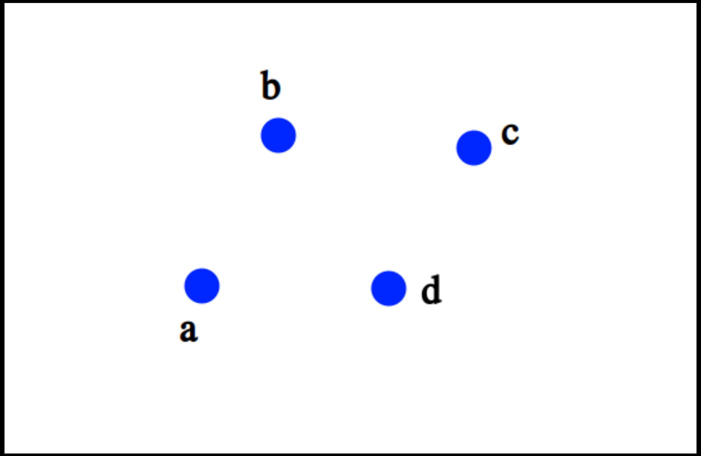
- y0,y1 no competition
  (no interact)

- *Singularity* = Empty



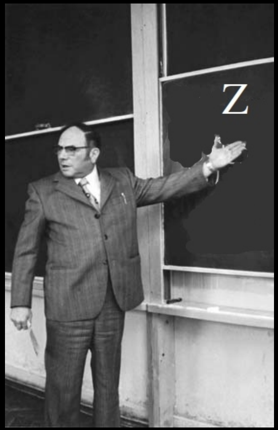
- y0,y1 compete/interact

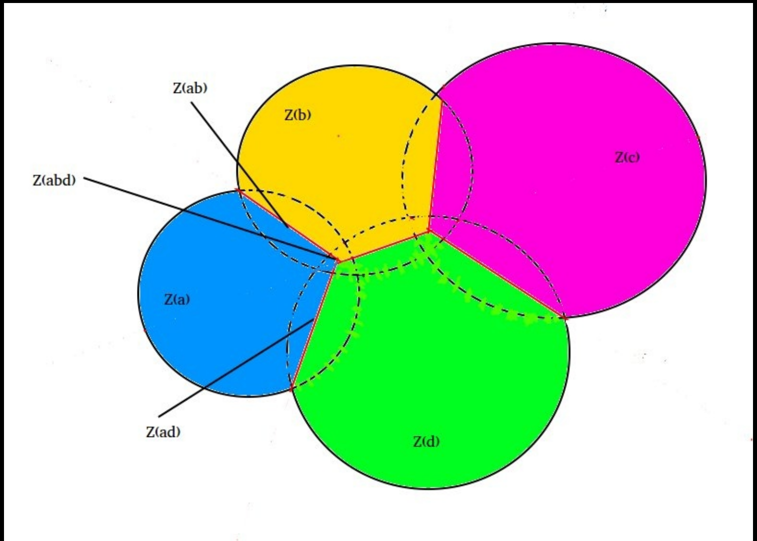
- *Singularity* nonempty
  & *<u>stable/persistent</u>*
  w.r.t. target measure τ

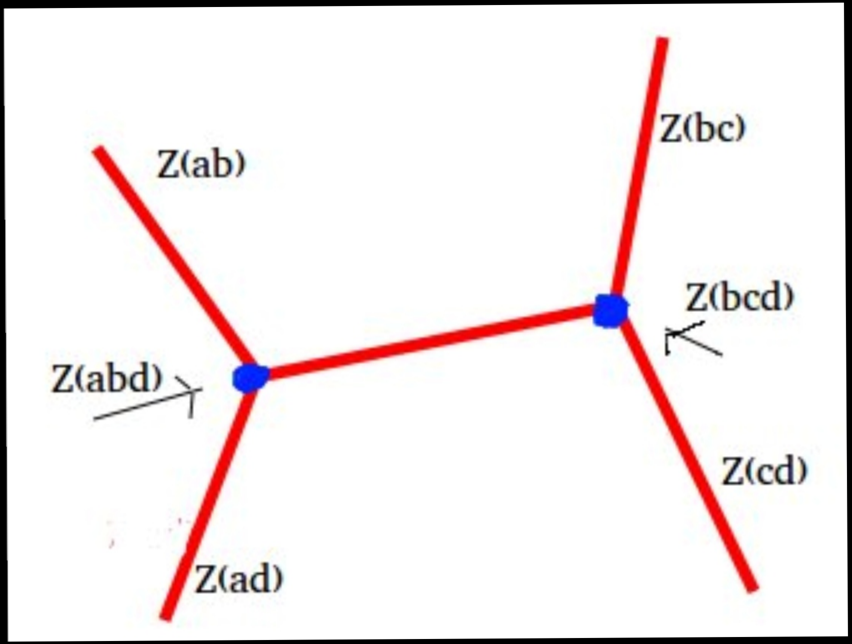


Source = 2-dim background field

Target = +a+b+c+d (4 point masses)



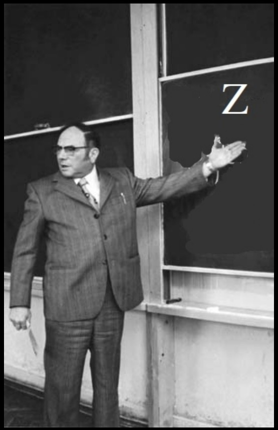




c=d**2/2
quadratic
cost

(+)→ ← (–)
attraction

...for every cost c: X x δX → ***R,*** *there is c-Legendre-Fenchel transform*



Key definitions: - c-concave potentials $\psi^{cc} = \psi$.

- c-subdifferentials $\partial^c\psi(y)$ are subset of X for every yε δX

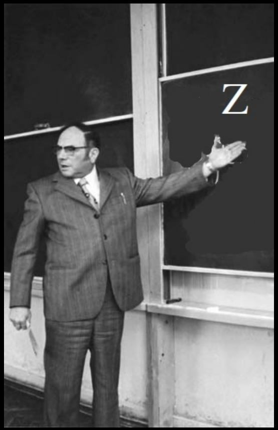
Kantorovich says *``c-optimal semicouplings π are supported on the graphs of c-subdifferentials of c-concave potentials* $\psi^{cc} = \psi$ *"*

Monge-Kantorovich Duality:

$$\max_{\psi\ c\text{-concave}}\left[\int_X -\psi^c(x)d\sigma(x)+\int_{\partial X}\psi(y)d\tau(y)\right] = \inf_{\pi\in SC(\sigma,\tau)}\int_{X\times\partial X} c(x,y)d\pi(x,y)$$

$-\psi^c(x) + \psi(y) =< c(x,y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.



- c-concavity $\psi^{cc} = \psi$ of a potential $\psi : \partial X[t] \to \mathbb{R}$ represents a pointwise inequality

$$-\psi^c(x) + \psi(y) \le c(x, y)$$

for all $(x, y) \in X[t] \times \partial X[t]$,

with equality $\psi(y) - \psi^c(x) = c(x, y)$

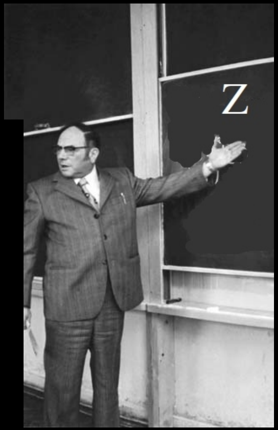
if and only if $y \in \partial^c \psi^c(x)$ iff $x \in \partial^c \psi(y)$

iff $y \in argmax[\{\psi(y_*) - c(x, y_*) + c(x, y) | y_* \in Y\}]$.

Variational defn. of c-subdifferential

Define Z({y}) := $\partial^c \psi(y)$

Kantorovich's Contravariant Singularity Functor IS EXPLICIT.



$Z(Y_I)$ consists of $x \in X$ for which

Variational defn. of c-subdifferential

$argmax[\{\psi(y_*) - c(x,y_*) + c(x,y) | y_* \in Y\}]$ contains Y_I ,

where $y_0 \in Y_I$ is reference point.

Abbreviate $c_\Delta(x; y, y') := c(x,y) - c(x,y')$ ....two-pointed cross difference.

Implies equations

$$Z(Y_I) = \{0 = \psi(y_0) - \psi(y) - c_\Delta(x; y_0, y) \mid y, y_0 \in Y_I, y \neq y_0\}$$

Reduces to $\#(Y_I) - 1$ equations. Symmetry y_0, y_1.

Applications to Algebraic Topology:

Contravariance says Z(Y_I)'s are local cells in X[t], parameterized contravariantly by subsets Y_I of Y=δX[t]

if $Y_I \hookrightarrow Y_J$. then $Z(Y_I) \hookleftarrow Z(Y_J)$

Theorem (Local Reduction)

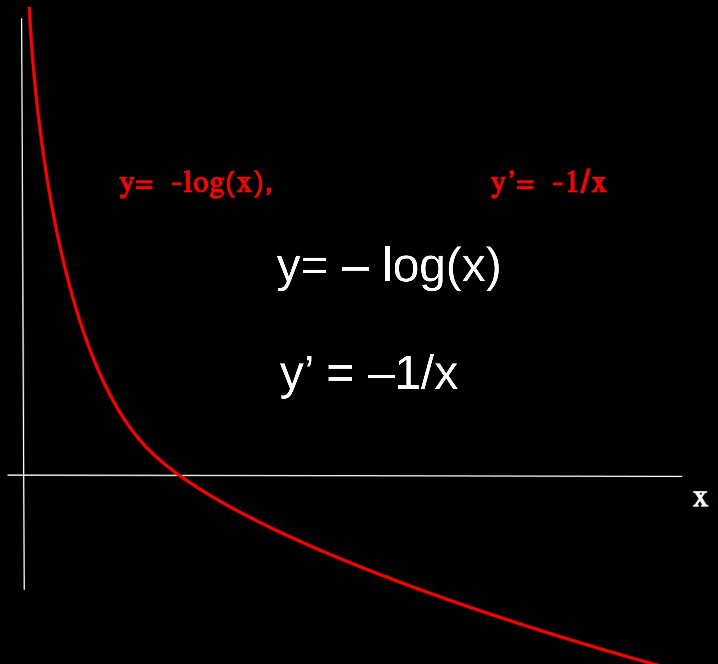
Let Z=Z(σ,τ,c) be Kantorovich functor for cost c, source σ, target τ, with σ[X] > τ[δX[t]].

Then we find local criterion (UHS conditions) which ensures Z(Y_I) ← Z(Y_J) is homotopy-isomorphism, and construct explicit continuous deformation retracts wherever (UHS) satisfied.

Proof: - Variational definition of c-subdifferentials, and gradient flow toward positive poles (not zeros!).

- Model: gradient flow to x=0 of f(x) = – log(x), x>0 .

- flow accelerates into the cusp.



Applications to Algebraic Topology:

If we "skewer the cube diagonally" and filtrate according to dimension, we obtain descending chain of closed subsets

$$X[t] \leftarrow Z\{1\} \leftarrow Z\{2\} \leftarrow Z\{3\} \leftarrow \ldots, \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{where codim } Z\{k\}=k-1$$

-Contravariance implies Z{1}, Z{2}, Z{3},… are homology-cycles in X, $\delta\, Z\{k\}=0$ (consequence of adjunction formula)

Theorem (Global Reduction):

Let $Z=Z(\sigma,\tau,c)$ be Kantorovich functor for cost c, source σ, target τ, with $\sigma[X] > \tau[\delta X[t]]$.

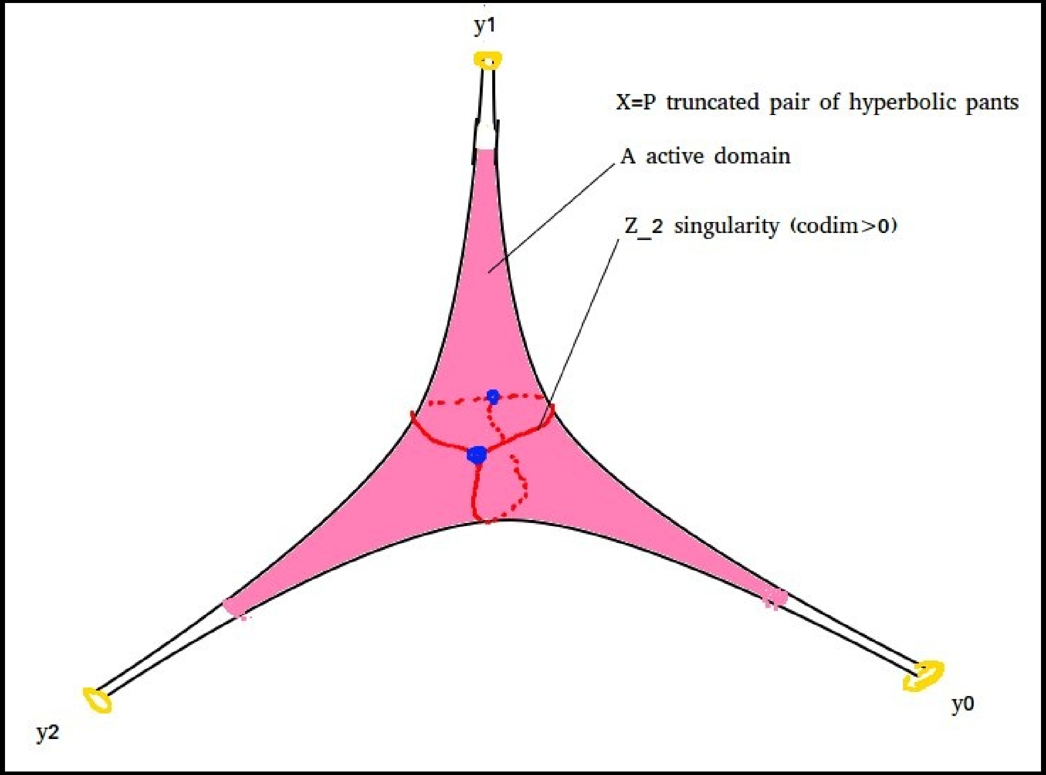
We identify index $J >= 0$ such that local cells $\{Z(Y\_I) \mid Y\_I \rightarrow Y\}$ and their local homotopy reductions assemble into global continuous reductions

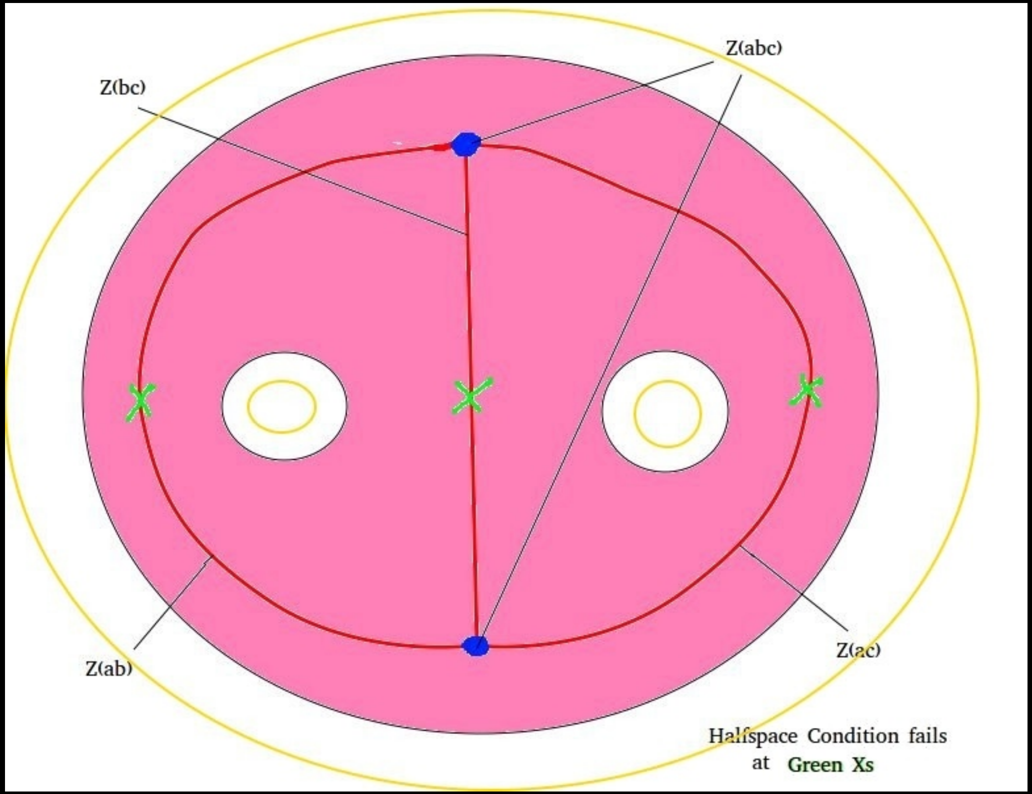
$$X[t] \rightarrow Z\{1\} \rightarrow Z\{2\} \rightarrow \ldots \rightarrow Z\{J+1\}$$

and such that Z{J+1} is a codimension-J closed subvariety of X, with inclusion $Z\{J+1\} \rightarrow X$ a continuous homotopy-isomorphism.

- *index J defined by max. codimension of cells Z(Y_I) where (UHS) satisfied.*

Global Reduction Theorem ==>
Singularity of Repulsion cost between excised source pant P[t] and target boundary δP[t] produces the familiar Θ-graph

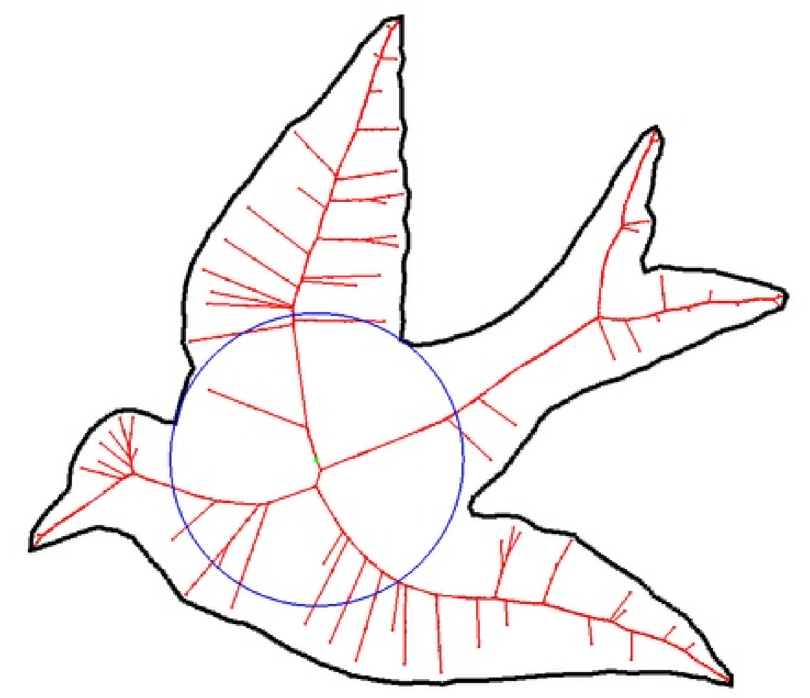




Local & Global Reduction Theorems are "generalizations" of following "folk-theorem":

Thm [Leutier, 2004]
If A is open subset of R^N,
then the "medial axis" M(A) includes into A as homotopy-isomorphism
(and is actually strong def. retract).

Recall M(A):={ x ε A | dist(x, δA) attained
by at least two points y1,y2 ε δA }



- M(A) unstable w.r.t. perturbations of A

- M(A) generally codimension 1 hypersurface

Our Local and Global Reduction theorems are general.

- Valid for every choice of cost c: $X[t] \times \delta X[t] \to R$

Applications require index J be large as possible.

- Many obstructions exist, i.e. local (UHS) conditions.

*Ex. Quadratic cost c=d**2/2*

- *inclusion $Z\{1\} \to (X=Z\{0\})$ is generally* ***NOT*** *homotopy-isomorphism*

Best results obtained with anti-quadratic ``repulsion" costs

We illustrate in next few slides….

Example: [Attractive (–1)+ (+1) Quadratic Cost]

Consider closed unit interval X = [0,1] with boundary δX={0, 1}.

σ is uniform distribution of (-1) charges. mass(σ) =15(-)

τ is uniform distribution of (+1) charges. Mass(τ) = 6(+)

mass(σ) > mass(τ)

Ground state *Ground state*

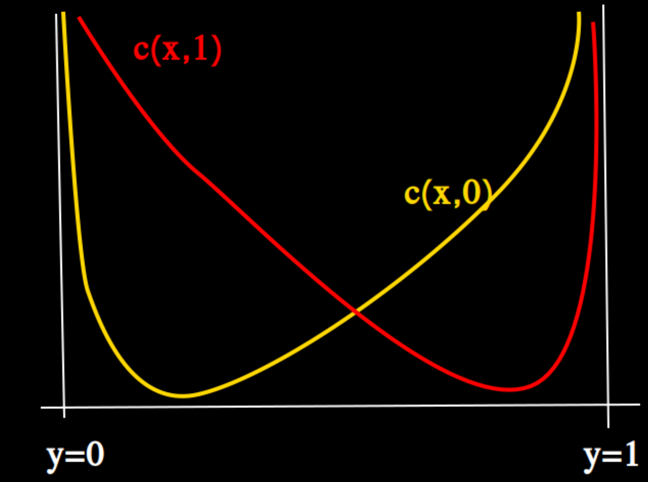
One-dimensional Repulsion cost c:*

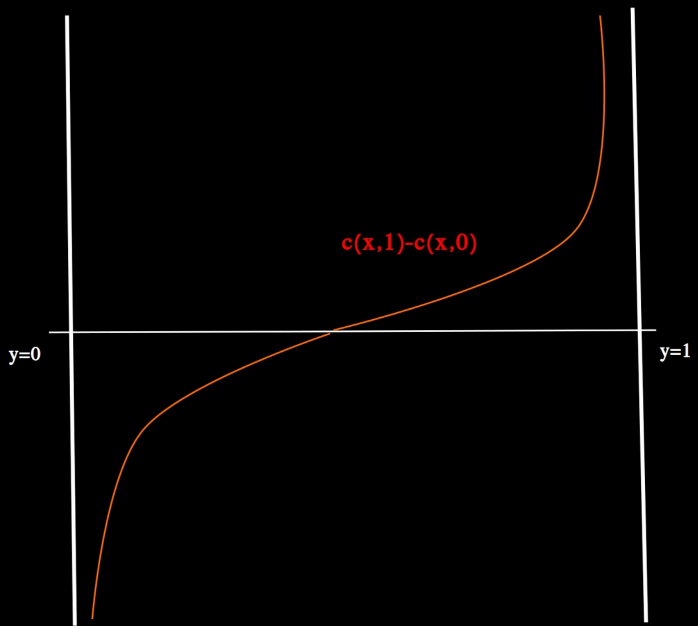
Unit interval $X = [0,1]$, boundary $\delta X = \{0,1\}$

$$c^*(x,0) = |x|^{-2} + 2|1-x|^{-2}$$

$$c^*(x,1) = 2|x|^{-2} + |1-x|^{-2}$$

$$c^*(x,1) - c^*(x,0) = |1-x|^{-2} - |x|^{-2}$$





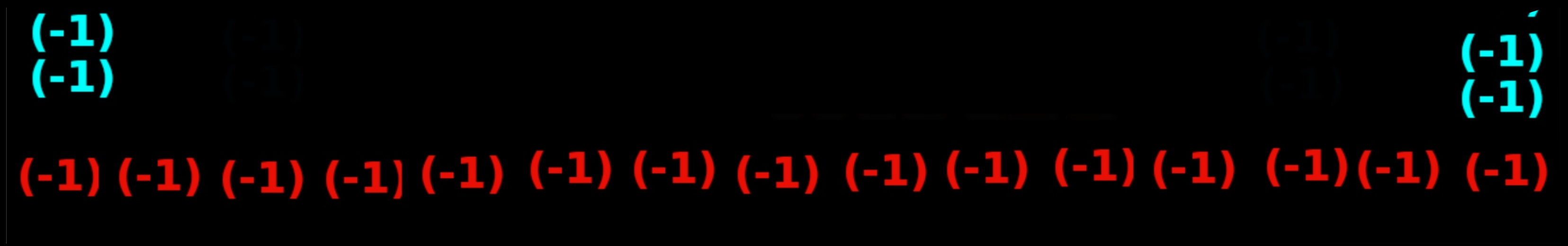
- cross-difference is critical-point free!
(critical points at poles x=0,1)

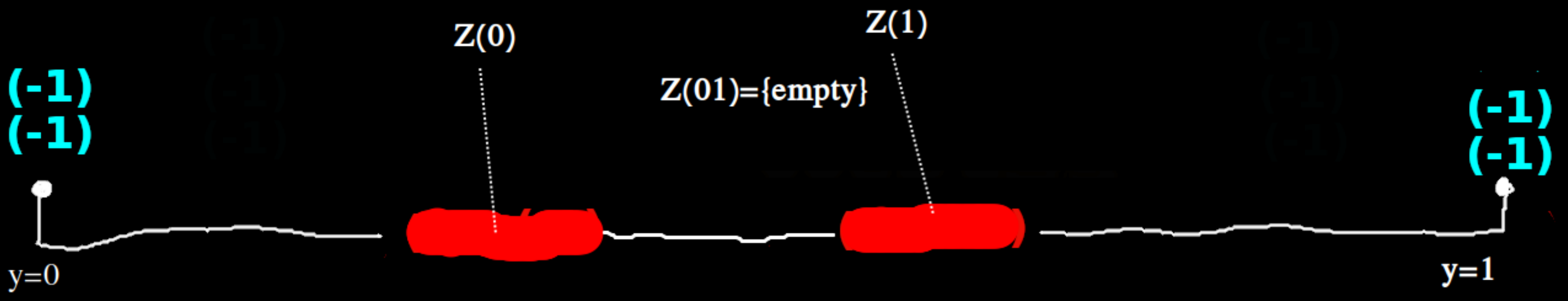
- every fibre is connected.

Consider closed unit interval $X = [0,1]$ with boundary $\delta X=\{0, 1\}$.

σ is uniform distribution of (-1) charges. mass(σ) =15(-)
τ is uniform distribution of (-1) charges. Mass(τ) = 4(-)

mass(σ) >>...>> mass(τ)



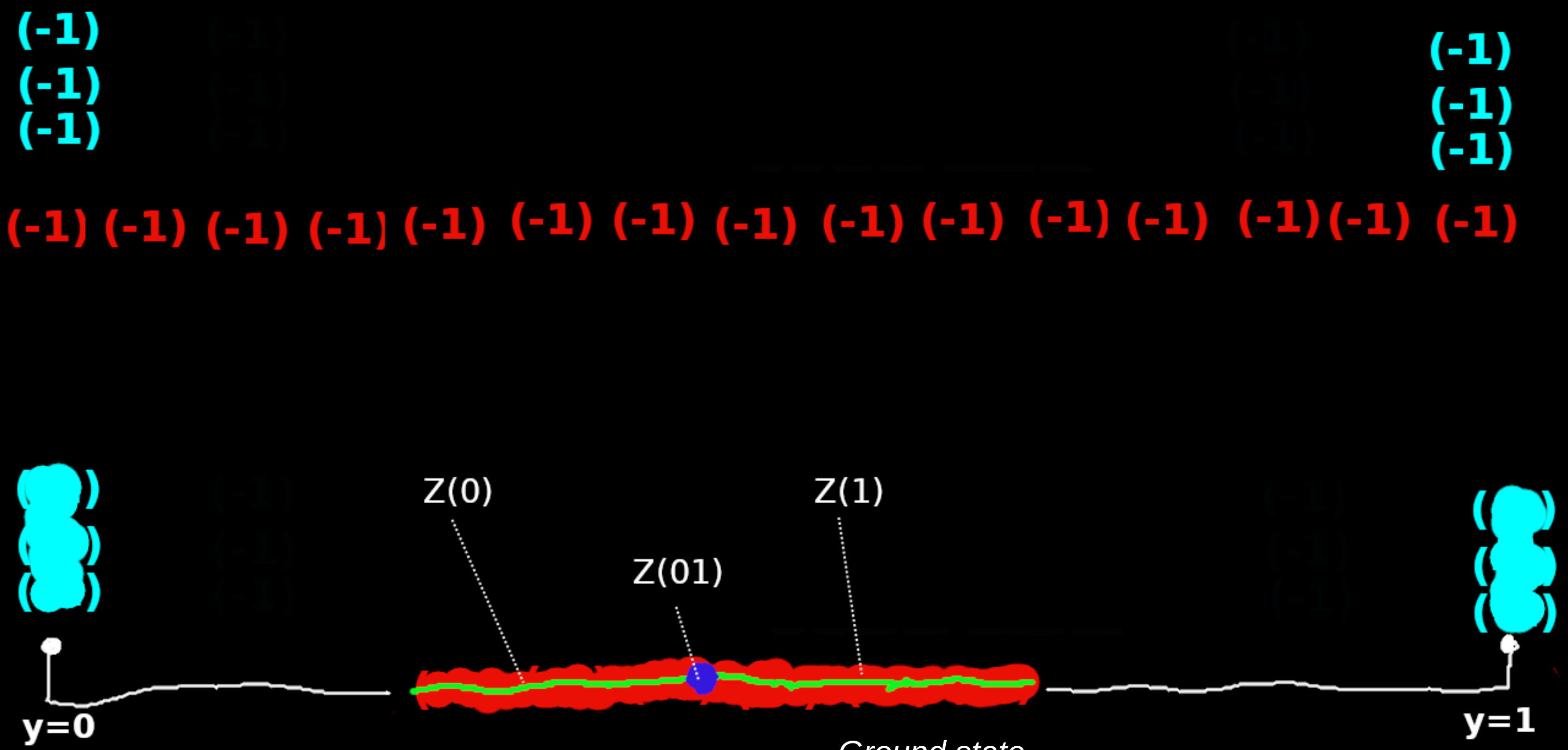


Ground state

Consider closed unit interval X = [0,1] with boundary δX={0, 1}.

σ is uniform distribution of (-1) charges. mass(σ) =15(-)
τ is uniform distribution of (-1) charges. Mass(τ) = 6(-)

mass(σ) >>> mass(τ)



Ground state

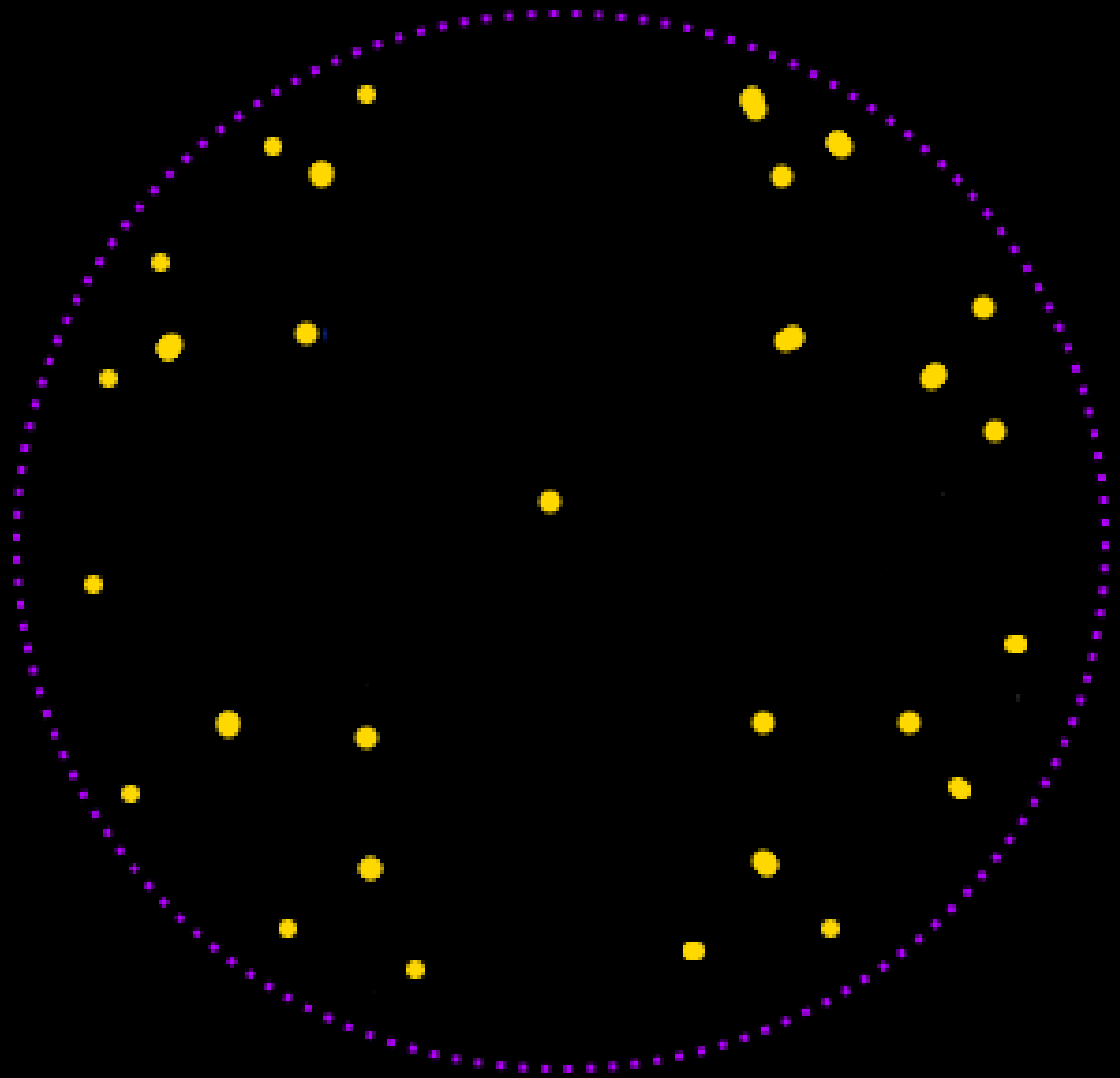
– Kantorovich functor $Z=Z(\sigma,\tau,c)$ defined for general costs c,
– we have defined the one-dimensional repulsion cost $c=c^*$

Now we describe an applications to a topological Extension Problem.

– requires <u>“Closing the Steinberg symbol”</u> and <u>visibility cost</u> v^*

Recall our excision $X[t]$, $\delta X[t]$ and the hypothesis that Γ satisfies Bieri-Eckmann homological duality:

Initial geometric EΓ-model X



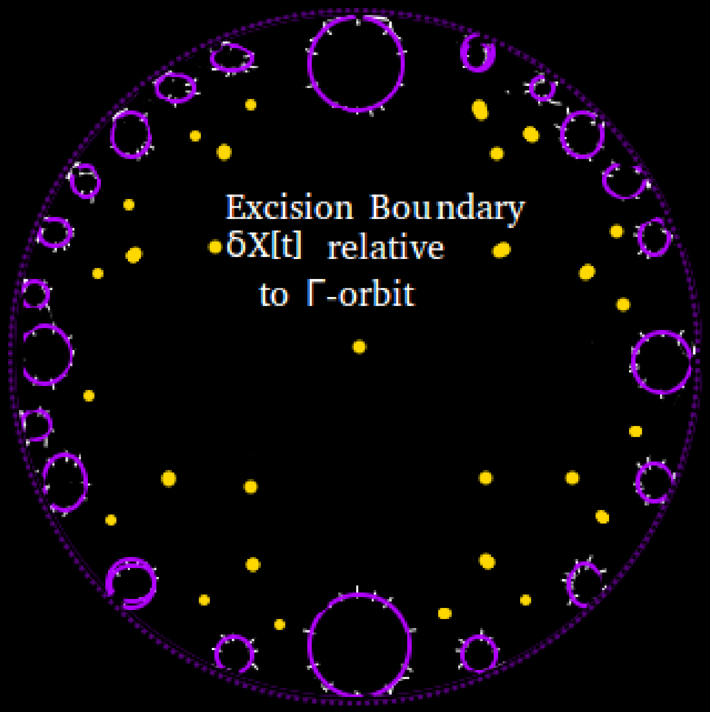
Γ-rational Excision X[t] x δX[t]

Excision X[t] := X – U V[t]

-obtained by scooping-out/excising from X a countable family of Γ-rational horoballs V[t] which are "nearly" at-infinity.

Obtain: Source (X[t] , σ) / Target (δX[t], τ)

– *Γ-rationality implies X[t] and boundary δX[t] are Γ-invariant subsets, and inherit proper-discontinuous Γ-actions.*



Theorem [Curtis-Solomon-Tits]:

The excision boundary $\delta X[t]$ of maximal Γ-rational excision has the homotopy-type of a countable wedge of q-spheres.

Theorem [Bieri-Eckmann]:

The reduced singular homology group, with natural ZΓ-module structure

$\boxed{\mathbf{D} = \tilde{H}_*(\delta X[t])}$ is homological-dualizing module (a.k.a. “Steinberg module”).

*Steinberg module **D** is infinite cyclic **Z**Γ-module*
– generated by boundary spheres B ”at-infinity”

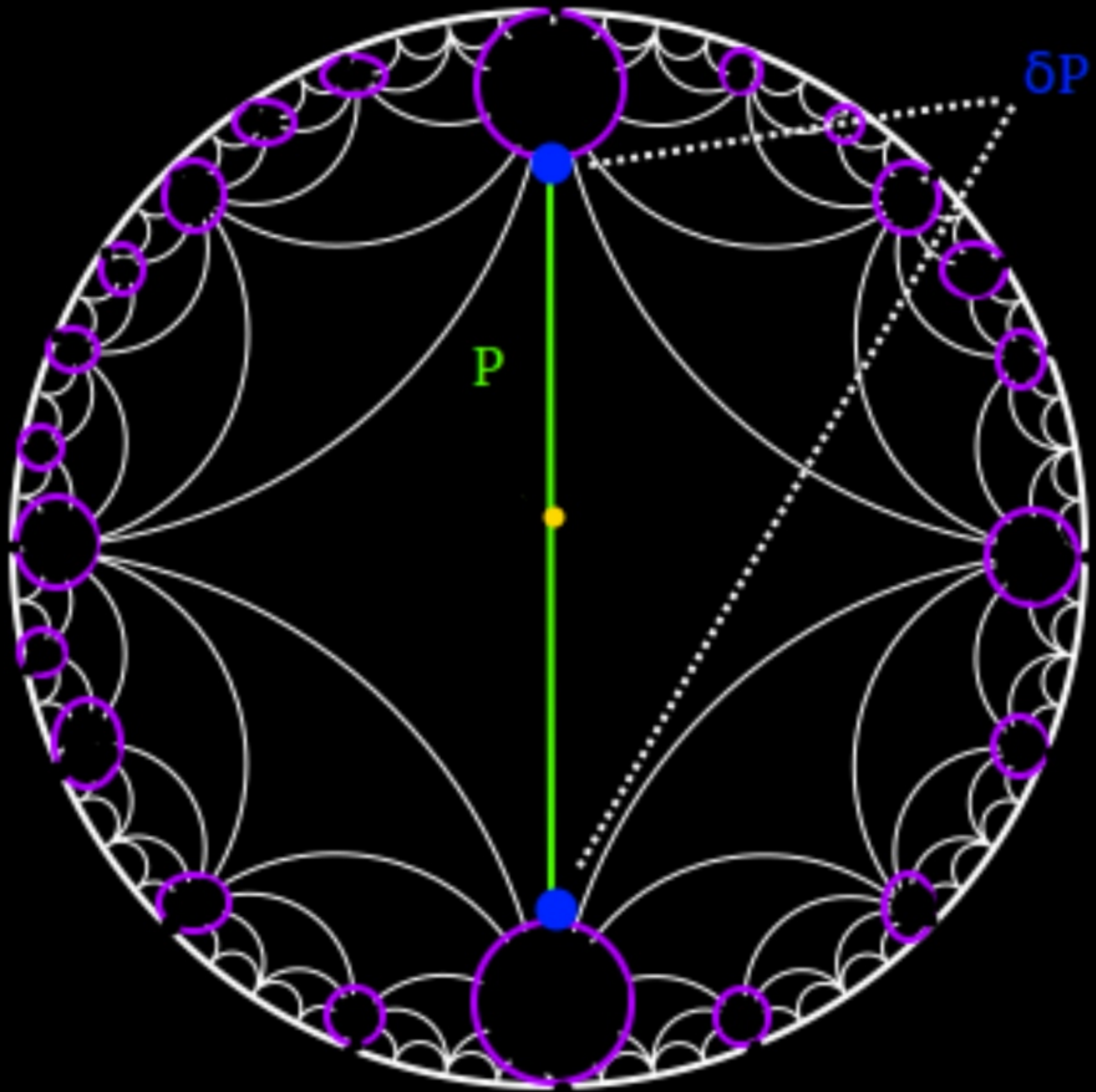
{X[t] Cartan-Hadamard} + {LES relative homology}   ==>

The boundary map ("connecting homomorphism") is isomorphism,
with canonical inverse δ^(-1) defined by "Flat Filling " on singular chains

$$\partial : H_{q+1}(X[t], \partial X[t]) \to \tilde{H}_q(\partial X[t])$$

P=FILL[B]   *B*

Steinberg symbol P is
relative 1-cycle; with boundary
δP a boundary 0-sphere.

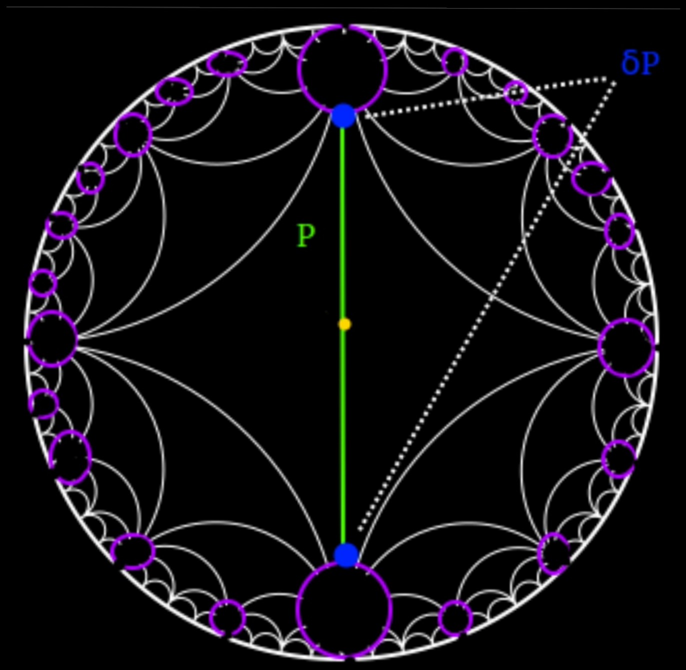


Homological duality:

- Bieri-Eckmann duality implies FILL[B] is dual cycle to minimal spines.

- dim(FILL[B]) is max codimension of minimal spines (homological duality formula)

Duality ==> $cd(\Gamma) + \dim(\text{FILL}[B]) = \dim(X)$



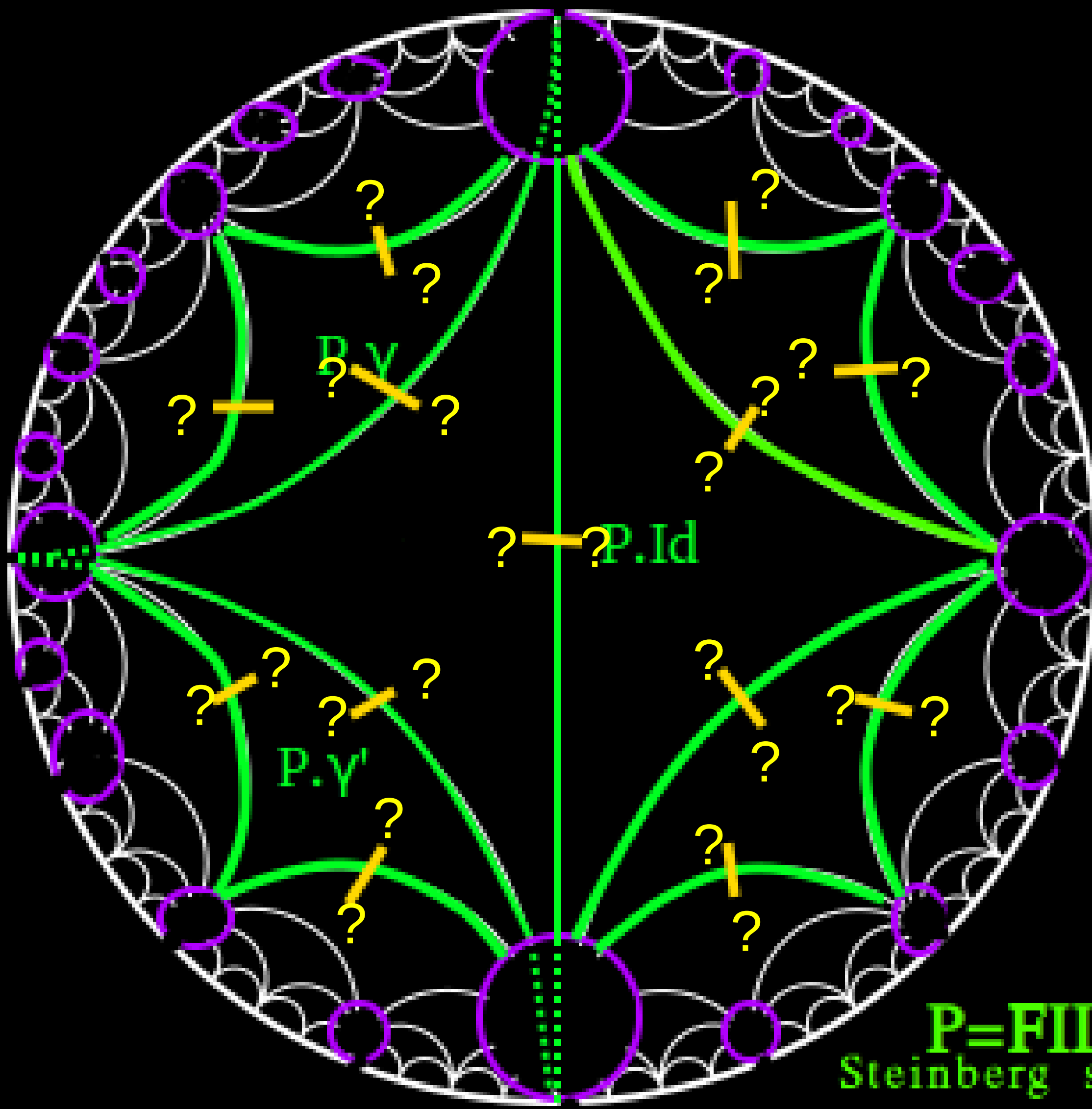
- Observe FILL[B] deformation retracts to {pt}, ………...FILL[B] → {pt}...

<u>*Extension Problem:*</u>

Find a continuous extension of the local reductions

{ P.γ → {pt}, γ ε Γ }

and obtain a global continuous reduction of X[t] onto Spine Z



Today: we cannot announce a general solution to this Extension Problem.

- But numerical evidence confirms a Conjecture contingent on two hypotheses:

  - *that we have a finite subset I of Γ which successfully "Closes Steinberg", and*

  - *that the definition of visible-repulsion cost v satisfies a (Twist) condition familiar from optimal transport theory*

<u>*Conjecture / (Work-In-Progress)*</u>:

Let (X,d,vol_X) be a geometric EΓ model, with Γ a Bieri-Eckmann duality group.

IF:

- **I** is finite subset of Γ which successfully “Closes Steinberg symbol”,

- and if v is the visible repulsion cost defined on the corresponding chain sum <u>F</u> replacing the Γ-rational excision X[t],

---

THEN:

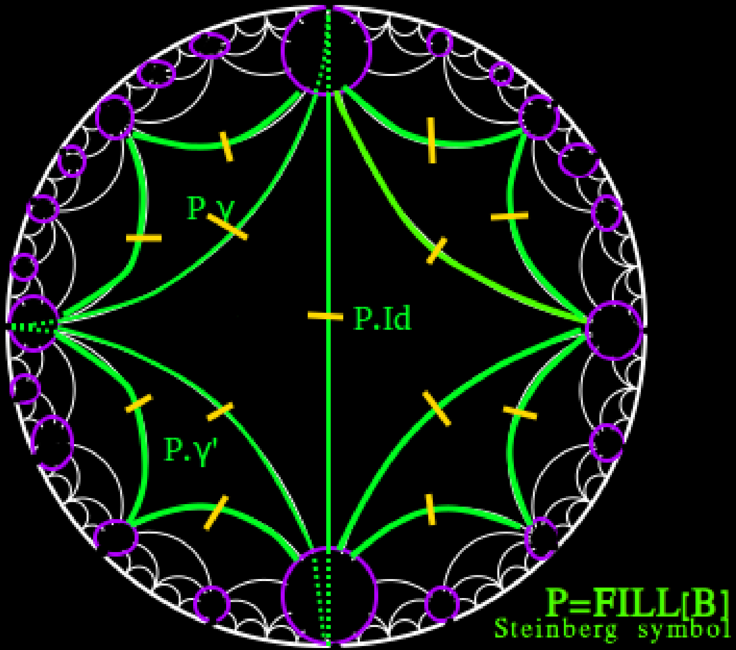
- *the Global Reduction Theorem applied to functor Z(σ,τ,v) gives positive solution to Extension Problem.*

- *(UHS) conditions are satisfied up to index J:=q+1=dim(**FILL[B]**).*

- *the closed subvariety Z{J+1} is a maximal codimension retract of X.*

<u>*We propose:*</u>

{Closing Steinberg} + {Visible Repulsion Cost v} + {Global Reduction Thm.} ==> {Spine}



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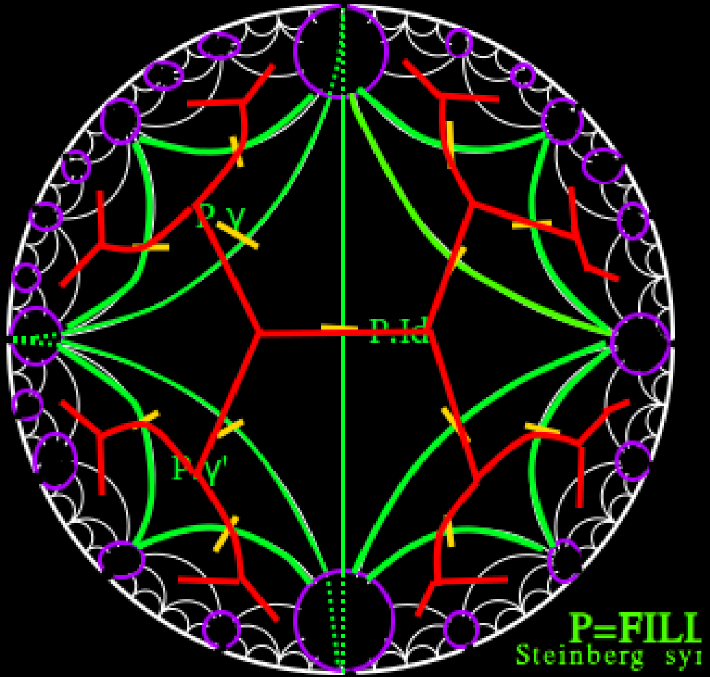


Closing Steinberg (CS):

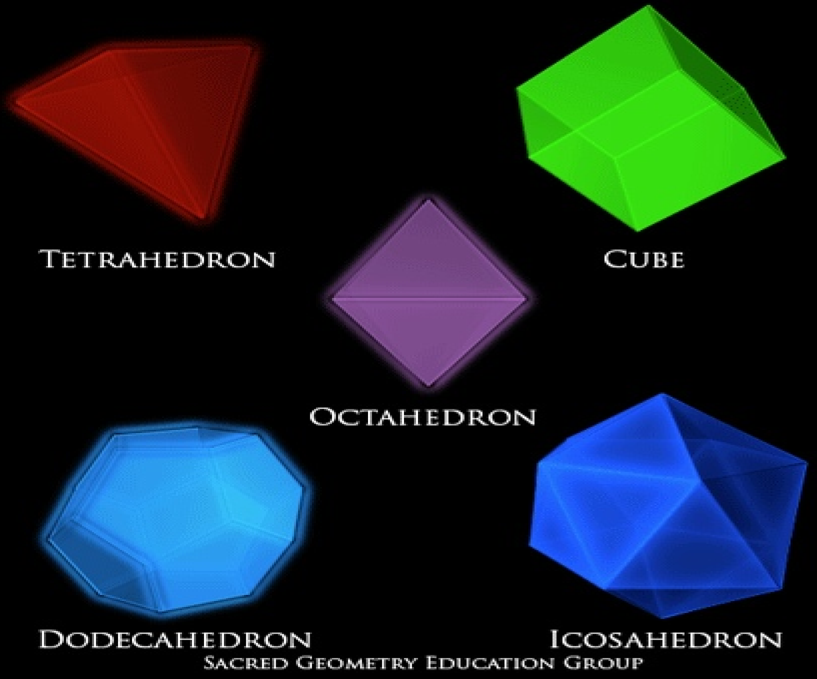
In low dimensions (CS) is the problem of stitching a football from collection of panels {P}.

In applications: the panels $\{P\} = \{ \mathrm{FILL}[B].\gamma \mid \gamma \in \Gamma \}$ are flat-filled Steinberg symbols.

Ex: Regular platonic solids solve (CS) with panels P = triangle, square, pentagon.

P=Triangle

P=Square



P=Pentagon

Polyhedra
“At-infinity” !

Panels = P = FILL[B]

Flat-filled Steinberg symbols

Closing Steinberg (Definition):

- *A finite subset **I** of Γ successfully "Closes Steinberg" if :*

(CS1) the chain sum $\sum_{\gamma \in I} P.\gamma \neq 0 \bmod 2$ (nontrivial over $\mathbb{Z}/2$ coefficients).

(CS2) the chain sum $\sum_{\gamma \in I} \partial P.\gamma = 0 \bmod 2$ (vanishing boundary over $\mathbb{Z}/2$ coefficients).

(CS3) there exists $x \in X[t]$ which is simultaneously visible from $P.\gamma$, $\gamma \in I$, in $X[t]$ (well-defined closed convex hull).

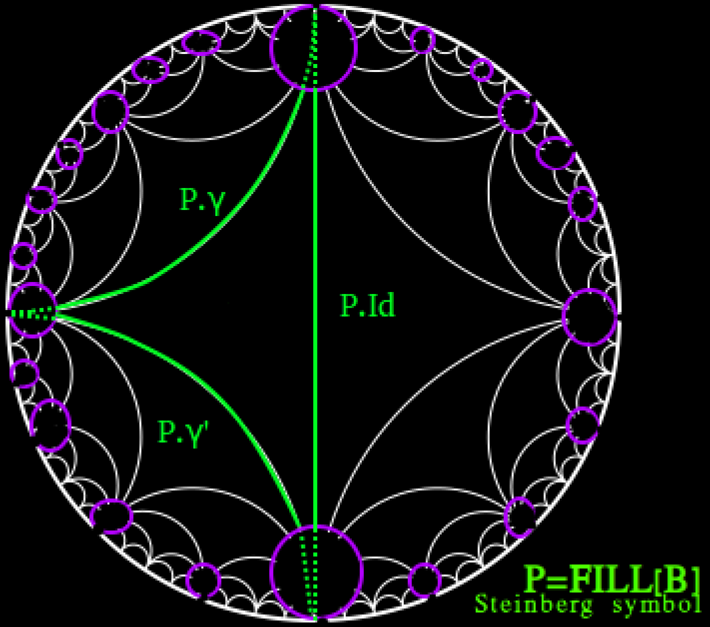
(CS4) if we define $F := \overline{conv}\{P.\gamma | \ \gamma \in I\}$, then the convex chain sum $\underline{F} = \sum_{\gamma \in \Gamma} F.\gamma$ has well-seperated gates structure with gates $\{G\} = \{P.\gamma | \ \gamma \in \Gamma\}$. (well-separated gates)

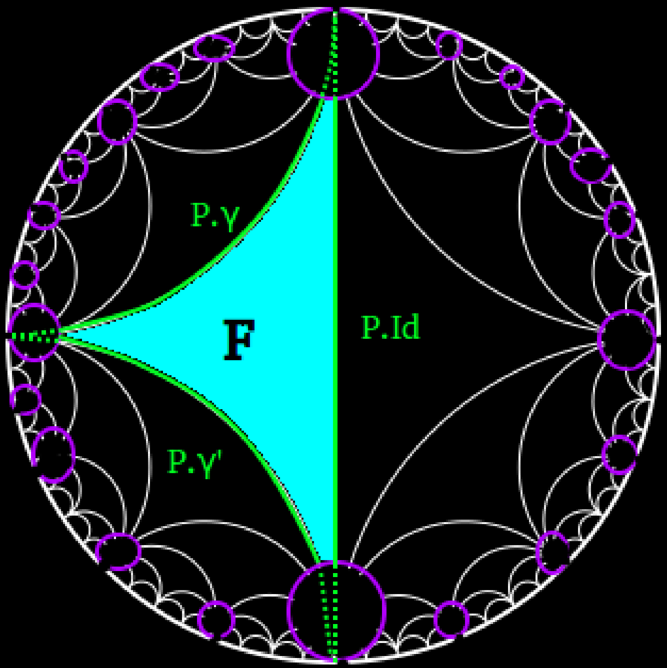
Closing Steinberg:

CS1, CS2 ==> constructing nonzero $\xi \ \varepsilon\ H_0 (\Gamma, \mathbf{Z}_2 \Gamma \times \mathbf{D}) \neq 0$

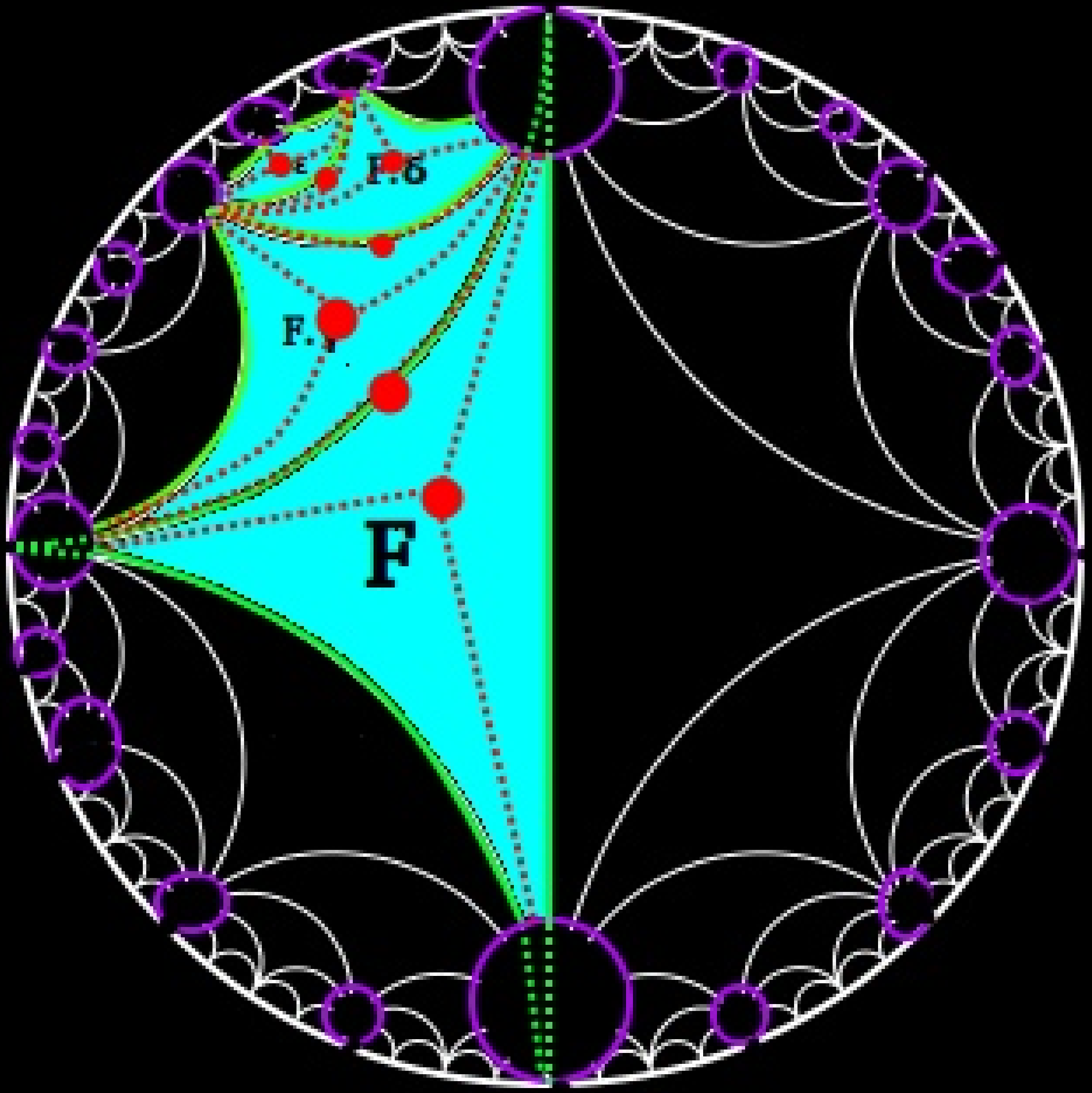
- equivalent to a syzygy in projective homological resolution of **D**.

- implies formal solutions ξ satisfying CS1, CS2 exist.





I={Id, γ, γ'}



Chain sum F with well-separated gates:

Well-separated gates ==>>
summands F, F' have
either trivial intersection
or $F \cap F' = P$

- the translates F. Γ define chain sum
  $\underline{F} = \Sigma\, F.\gamma$

- Γ acts on chain summands of $\underline{F}$ like
  "shift operator" equivalent to
  translation action $\Gamma \times \Gamma \to \Gamma$

- the translates F.γ, γ ε Γ,
  do not necessarily fill X[t] !

Ex: Γ=PGL(Z^3)

The following symbol ξ' is a solution to (CS):

$$\xi' := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

I := {3x3 minors of ξ' } is the finite subset PGL(Z^3) which solves (CS).

Solutions to (CS) replace X, or X[t], with chain sums:

Proposition 37. *The convex hull of the rank-one states spanned by the columns of ξ', i.e.*

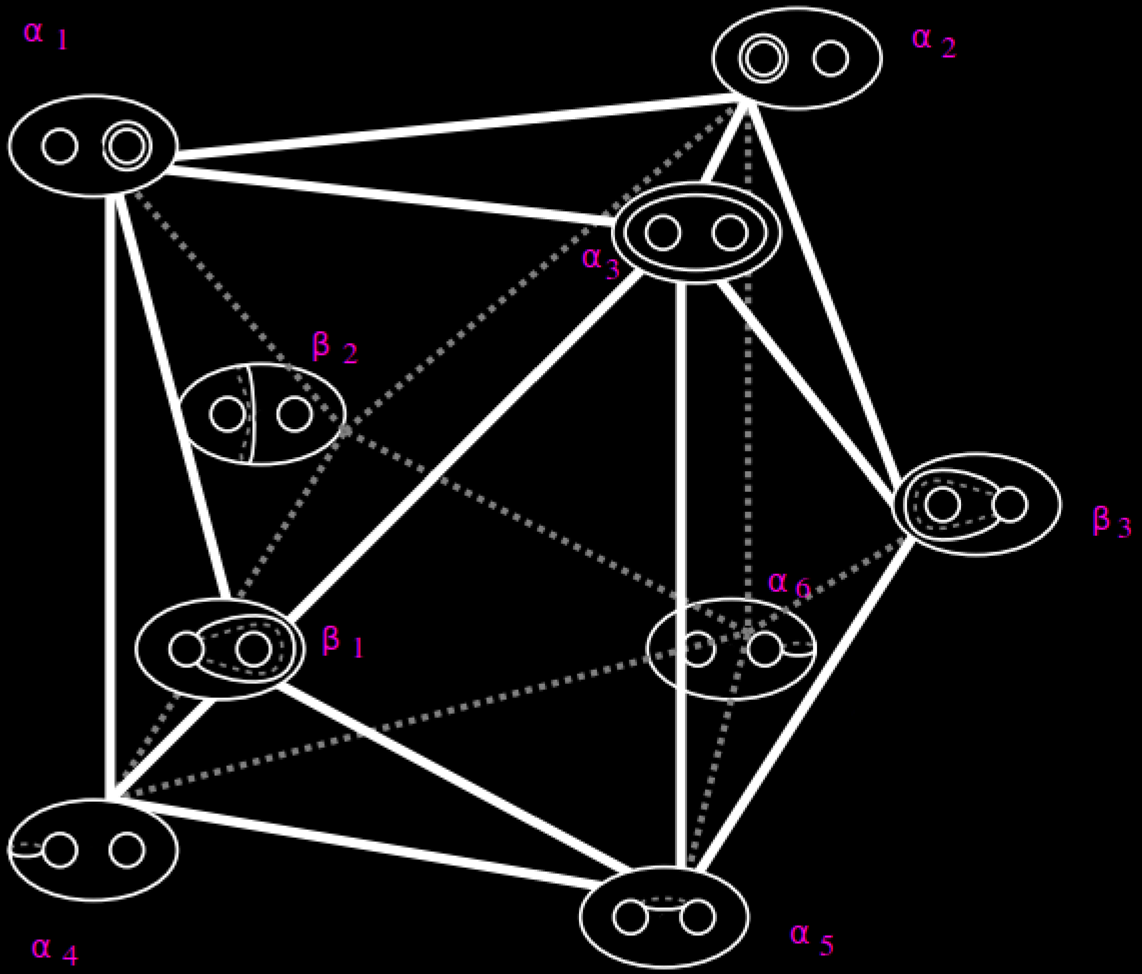
$$F := conv[\{x^2, y^2, z^2, (x+z)^2, (y+z)^2, (x+y+z)^2\}] \subset Q$$

forms a convex set F, whose translates $F.GL(\mathbb{Z}^3)$ tessalate Voronoi's cone Q of three-dimensional positive-semidefinite real states.

(CS) for Γ=MCG(Σ_2): Open Problem

N. Broaddus [2012]
identifies the panel P=3-ball
in genus 2 Teichmueller space.

(CS) is primary obstruction
to building explicit spine of Teich_2

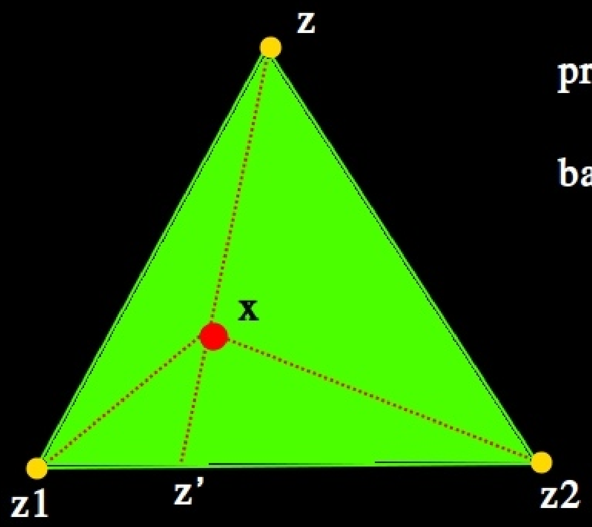


Next: the visible repulsion costs v, v^*...

For brevity we describe the ‘’visible repulsion’’ cost v in the simplest case of a finite compact 2-simplex ▲.

Source = F

Target = Extreme points E[F]
= {z,z1,z2}



proj(z,x)=:z'

bar(λ.z1 + (1-λ).z2)=z'

$0 \le \lambda \le 1$

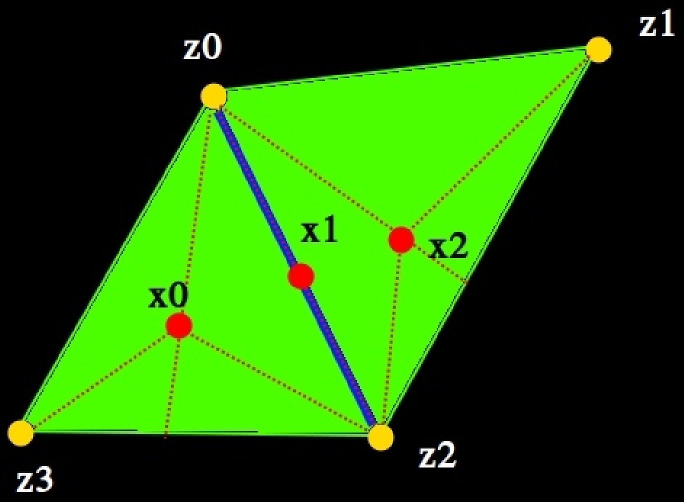
$$v(x,z)=1/2.\|x-z\|^{(-2)} + \lambda.\|x-z1\|^{(-2)} + (1-\lambda).\|x-z2\|^{(-2)}$$

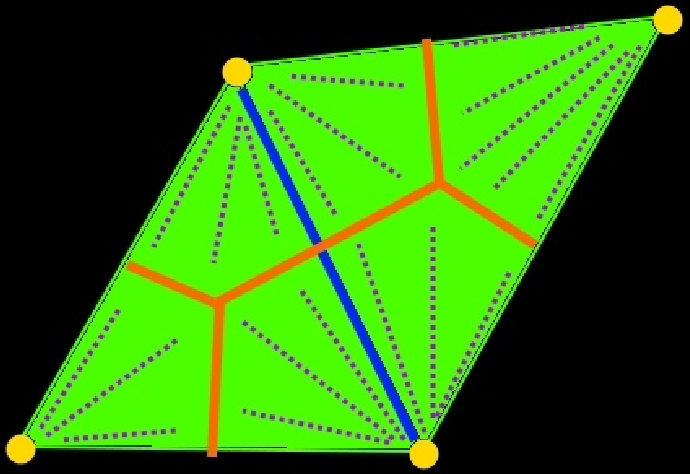
v: F x E[F] -> R visibility cost

- the definition of v extends to repulsion cost v^* on chain sum $\underline{F}$ with well-separated gates $\{G\}$

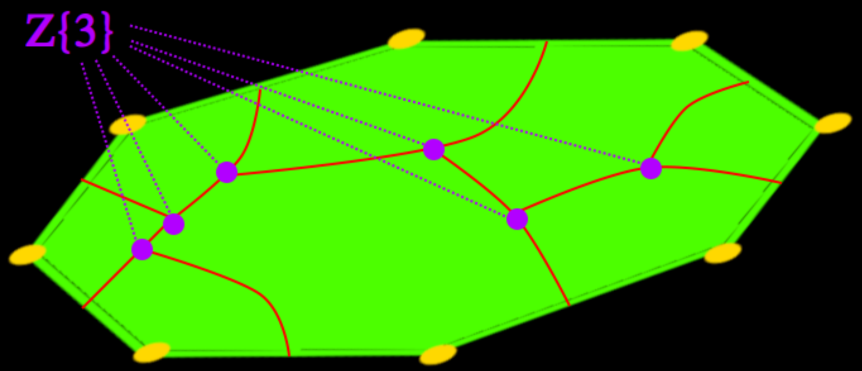
- gates G geodesically convex imply v^* is continuous extension of the restricted repulsion costs $v^*|G=v$

- Singularity structures $Z(\sigma,\tau, v^*)$ are continuous interpolations of restricted singularity structures $Z(\sigma, \tau, v^*|G)$ over gates $\{G\}$ of $\underline{F}$

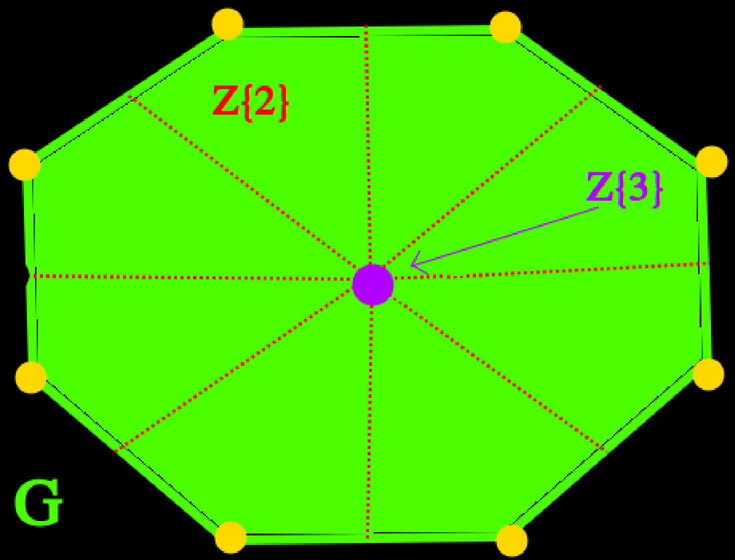




Symmetry of Gates ==> (UHS) Conditions satisfied:

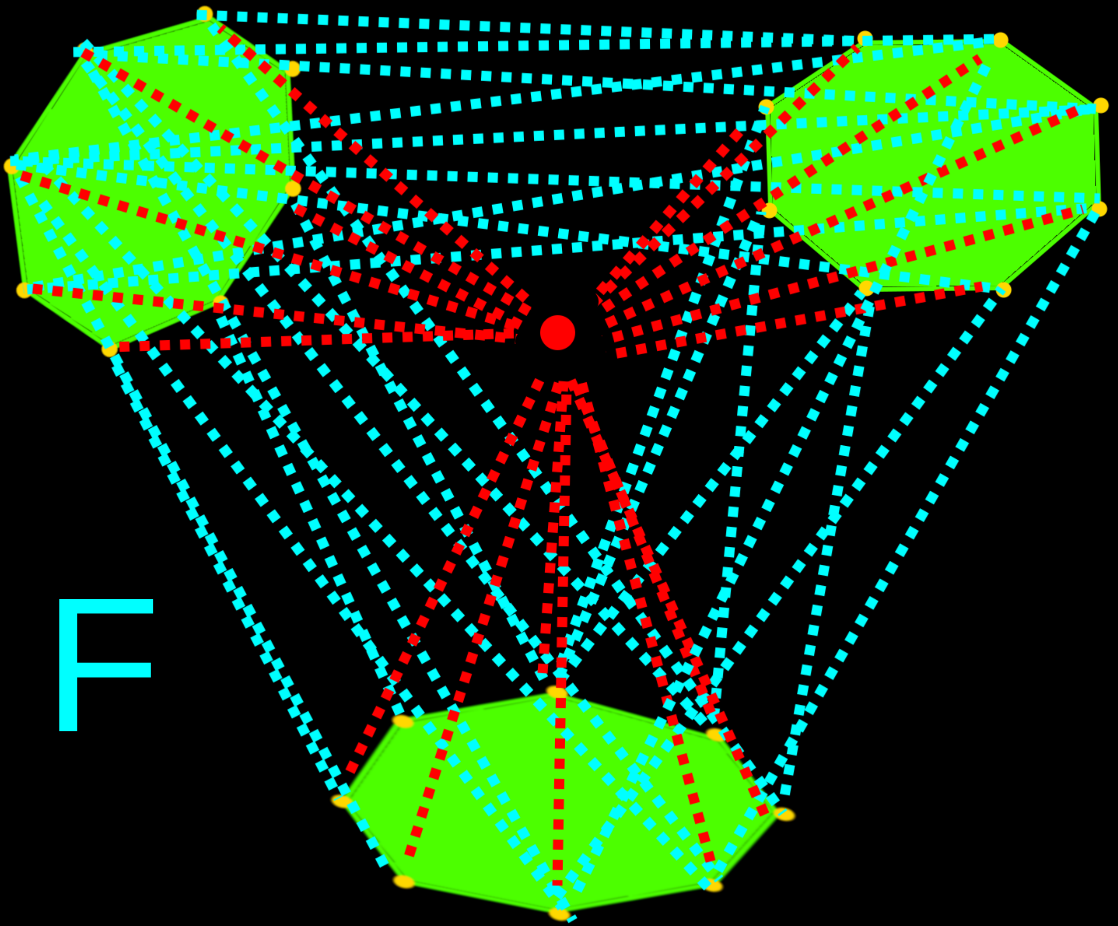


G deformation retracts to Z{2}



- Z{3} is disconnected for *generic* convex panels G

- Symmetry implies G homotopy-reduces to Z{3}={pt}



Successfully Closing Steinberg symbol replaces X with a chain sum

$$\underline{F} = \Sigma\, F.\gamma$$

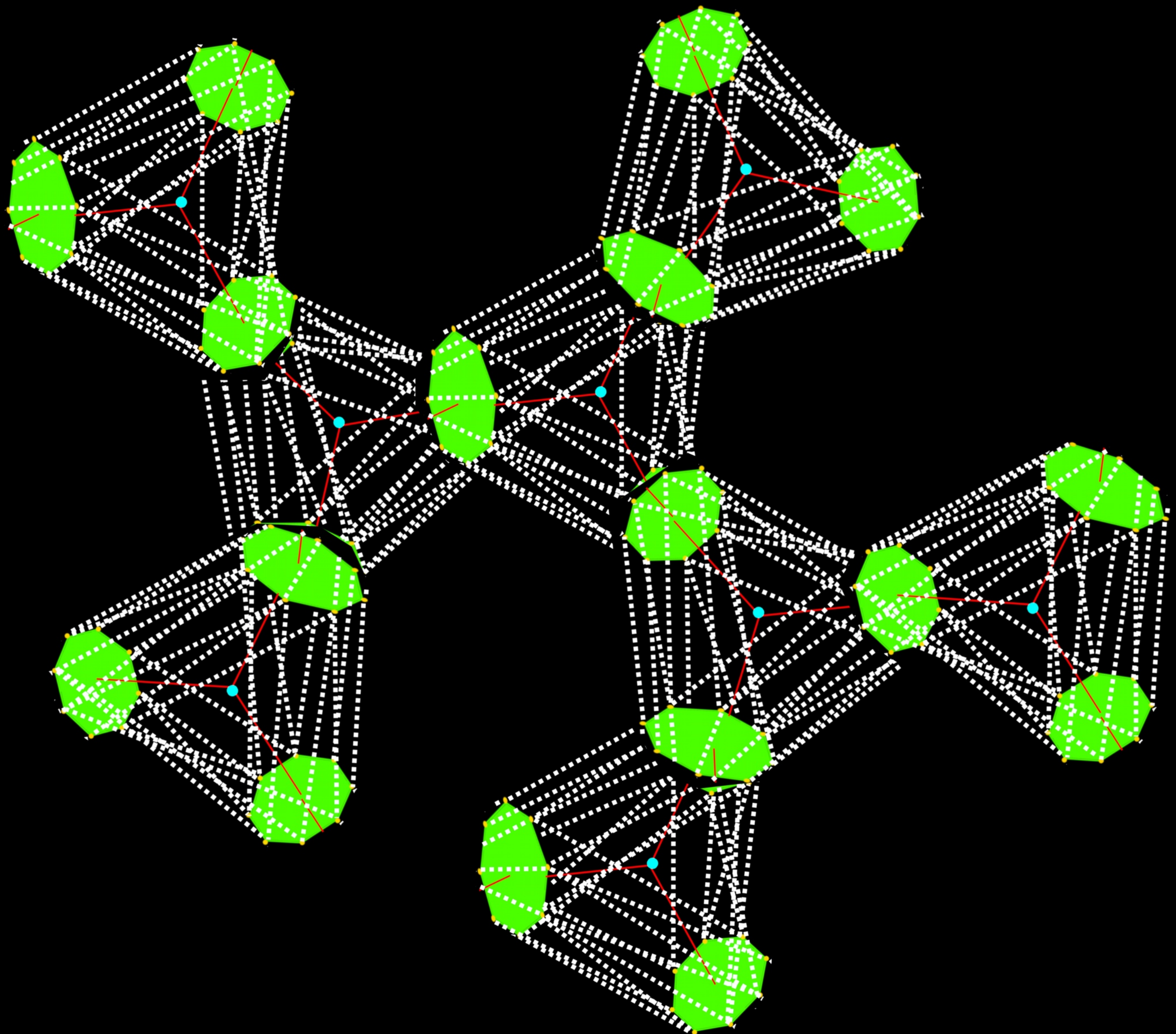
where F := conv[P, P.γ, P.γ', ...]

- *(CS) Hypotheses imply:*

the convex hull F is well-defined,

and

- *translates F.γ, γ ε Γ , have well-separated gates {G} coincident with Γ-orbit of Steinberg symbols G=P=FILL[B]*

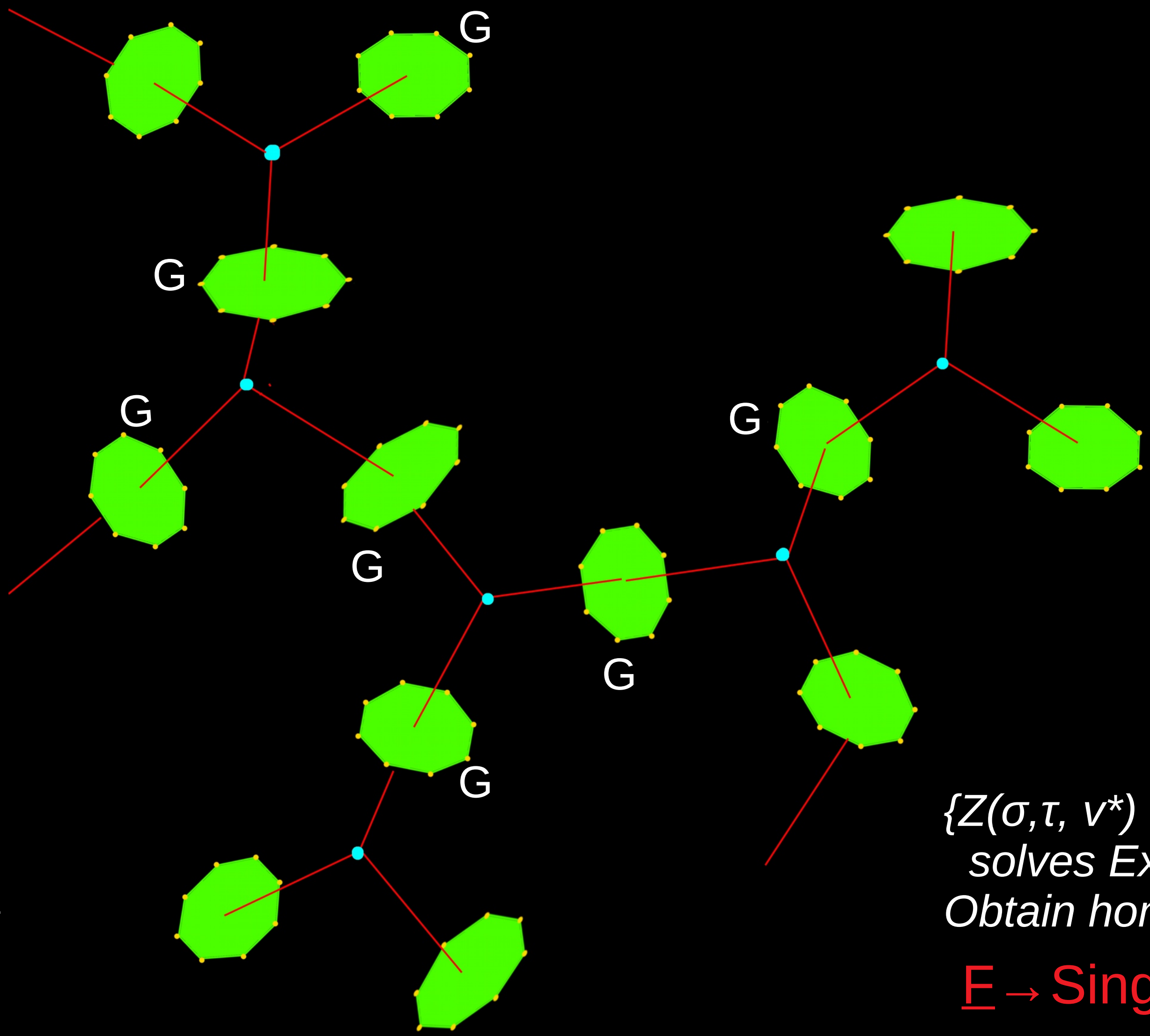


(CS) replaces X
with a chain sum

$$\underline{F} = \Sigma F.\gamma$$

where

$$F := \mathrm{conv}[P, P.\gamma, P.\gamma', \ldots]$$



v*-optimal
semicouplings
from source X=$\underline{F}$
to target Y=E[$\underline{F}$]

{Z(σ,τ, v) + Global Red.Thm}*
solves Extension Problem
Obtain homotopy-reduction:

$\underline{F}$ → Singularity Z{2}

The End.

T

Thank you.