WEBER ELECTRODYNAMICS AND SURPRISING (-2) MOLECULES

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Wilhelm Eduard Weber (1804-1891) was a German physicist, esteemed colleague of C.F. Gauss in Gottingen, and leading experimentalist of the 19th century. But Weber was largely forgotten throughout the 20th century. Professor A.K.T. Assis has recently translated and edited Weber's main works [1, 2] revealing Weber as perhaps the most significant physicist of the 21st century.

1. Two-Body Weber Hamilton

If $H(x, v)$ is a Hamiltonian on xv state space, then a one parameter flow $t \mapsto (x(t), v(t))$ satisfies Conservation of Energy iff

$$\frac{d}{dt}H(x(t), v(t)) = (x', v') \cdot \nabla H = 0. \tag{1}$$

Thus conservation of energy requires that (x', v') be everywhere orthogonal to ∇H. Classical mechanics achieves orthogonality by setting (x', v') parallel to $J\nabla H$, with the constant of proportionality equal to the inertial mass of the particles.

Weber's Hamiltonian for a two body electrical system between particles p_1, p_2 with radial distance $x = r_{12} > 0$ and relative radial velocity $v = \nu$ is

$$H(x, v) = \frac{1}{2}\mu v^2 + \frac{q_1 q_2}{x}(1 - \frac{v^2}{2c^2}),$$

where $\mu := \frac{m_1 m_2}{m_1 + m_2}$ is the reduced inertial mass of the system. It is convenient to set

$$\mu_{\text{eff}}(x) := \mu - \frac{q_1 q_2}{c^2 x} \tag{2}$$

allowing us to express H in the equivalent form

$$H(x, v) = \frac{1}{2}\mu_{\text{eff}} v^2 + \frac{q_1 q_2}{x}.$$

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The effective inertial mass μ_{eff} depends on position x unlike the reduced mass μ defined above.

Hamilton's equations become

$$\frac{d}{dt}(x,v) = (x',v') = \mu_{\text{eff}}^{-1} \cdot \ J\nabla H.$$

This expression is well defined when μ_{eff} is invertible, which is satisfied for radii $x = r$ satisfying $0 < r \neq r_c < +\infty$ where

$$r_c := \frac{q_1 q_2}{\mu c^2}.$$

The quantity r_c is called Weber's critical radius.

For two-body systems the effective inertial mass is a multiplicative scalar μ_{eff}. For N-body systems, the effective inertial mass is represented as a canonical L^2 minimizing symmetric operator A which satisfies the equations $x' = v = A^{-1}\nabla_v H$, and $v' = a = -A^{-1}\nabla_x H$. Thus the two-body effective mass is a multiplicative scalar, while the N-body effective mass is a symmetric operator.

2. Weber's Stable [−2] Molecule

Weber's electrodynamic model and the effective inertial mass μ_{eff} has the following prediction: that equal charges $[-1e]+[-1e]$ repel *except* at a sufficiently small critical distance $r < r_c$ where the charges acquire a *negative* effective inertial mass when $q_1 q_2 > 0$. The effective inertial mass becoming negative $\mu_{\text{eff}} < 0$ means that Weber's force between $[-1e]$ and $[-1e]$ becomes attractive, and the stable $[-2e] = [-1e] + [-1e]$ indivisible molecule is obtained.

The question arises if we have two identical electrically charged particles, then how much work is required to drive $[-1e]$ and $[-1e]$ from $r = +\infty$ to Weber's critical radius $r < r_c$? Surprisingly the energy predicted by Weber is $E = \mu c^2$.

Lemma 1. *In Weber's isolated two body system* $[-1e, -1e]$*, the total energy required to drive particles from* $r = +\infty$ *to Weber's critical radius* $r = r_c$ *is equal to* $E = \mu c^2$*.*

Proof. We compute using Weber's potential. The existence of a potential implies the energy is path independant. We find

$$\Delta E = \Delta U + \Delta T = \frac{1}{r_c}(1 - \frac{\dot{r}^2}{2c^2}) + \frac{1}{2}\mu\dot{r}^2$$

which is equal to μc^2, as claimed. □

Remark. The cancellation of the relative kinetic energy $T = \frac{1}{2}\mu\dot{r}^2$ by the Weberian term is key to providing the position and velocity independance of ΔE.

Remark. The above formula provides the total energy in the reduced system. The total energy of the system is actually $2\mu c^2$ which coincides with Einstein's formula since $c^2_{\text{Einstein}} = 2c^2 = 2c^2_{\text{Weber}}$.

In Weber's electrodynamics this $E = \mu c^2$ formula is a not interpreted as a "mass-energy equivalence", but represents the work energy required to drive similar electric particles into the critical radius $(r = +\infty) \to (r = r_c)$. By contrast, Einstein hypothesizes the "mass-energy" equivalence as a general property of *all* matter, and c^2 is assumed to represent what Einstein called "the intrinsic energy per unit mass". We view Weber's electrodynamic interpretation of $E = \mu c^2$ as physically explaining how and why certain molecules have large assembly energies, whereas Einstein's hypothesis was a supposition.

Remark. The observed 511 keV ($= 8.187 \times 10^{-14}$ Joules) annihalation of electron-positron pairs is an electrodynamic experiment confirming Weber's derivation. It suggests the possibility of distinct molecules which have equal inertial mass, but different assembly energy, i.e. $\mu_1 = \mu_2$ and yet the assembly energies being distinct. This would be contrary to the Einsteinian hypothesis.

Remark. What is scale of r_c in standard physical units? Recall the experimental conversion factor

$$\frac{1}{4\pi\epsilon_0} := (9.0 \times 10^9)\frac{[N][m]^2}{[C]^2}.$$

Recall also that electron has unit of electric charge equal to 1.60×10^{-19} Coulombs. Thus we find r_c is approximately computed in standard units to be

$$r_c = (9.0 \times 10^9)\frac{(1.60 \times 10^{-19})^2}{8.187 \times 10^{-14}}[m] \approx 1.758 \ldots \times 10^{-15} \text{ meters.}$$

This agrees with the scale of atomic nuclei in the standard atomic models.

3. N-Body Weber Hamilton

Consider an N-body electrodynamic system consisting of particles $p_1, \ldots, p_N$ with electrical charges $q_1, \ldots, q_N$, respectively. Let r_{ij}

denote the relative distance of p_i, p_j, and let ν_{ij} denote the relative radial velocity of p_i, p_j. Recall the definitions $\nu_{ij} := v_{ij} \cdot \hat{\mathbf{r}_{ij}}$. Weber's Hamiltonian for the N-body system is given by

$$H(r_{ij}, \nu_{ij}) = \sum_{i,j} \frac{1}{2}\mu_{ij}\nu_{ij}^2 + \frac{q_i q_j}{r_{ij}}(1 - \frac{\nu_{ij}^2}{2c^2}) \tag{3}$$

Let $x_1, \dots, x_N, v_1, \dots, v_N$ be Cartesian coordinates on xv state space. Here we introduce the assumption that Newton's Second Law holds for every two-body interaction, namely

$$-\frac{\partial U}{\partial r_{kj}} = \mu_{\text{eff},kj}\nu'_{kj} \tag{4}$$

Now we compute the gradients of the Hamiltonian using the chain rule:

$$\nabla_{v_k} H = \sum_j \mu_{\text{eff},kj}\nu_{kj}\hat{r}_{kj} = \sum_j \mu_{\text{eff},kj}\text{proj}_{\hat{r}_{kj}}(v_k - v_j).$$

Claim: For every $k = 1, \dots, N$, let A_k be a linear transformation satisfying

$$x'_k = v_k = A_k^{-1}(\nabla_{v_k} H). \tag{5}$$

We argue below 2 that there exists canonical symmetric operators A_k satisfying equation (5).

Next we compute

$$-\nabla_{x_k} H = \sum_j \frac{\partial U}{\partial r_{kj}}\hat{r}_{kj}.$$

Assumption (4) implies that

$$-\nabla_{x_k} H = \sum_j \mu_{\text{eff},kj}\text{proj}_{\hat{r}_{kj}}(v'_k - v'_j),$$

which implies that the operators A_k satisfy (5) and simultaneously satisfy

$$v'_k = a_k = A_k^{-1}(-\nabla_{x_k} H) \tag{6}$$

for every $k = 1, \dots, N$. Thus the operators A_k are linear maps which simultaneously solve two linear equations, namely (5) and (6). Moreover Conservation of Energy requires that the operators A_k also be symmetric, and this provides canonical choice of operators.

Lemma 2. *Let H be Weber's Hamiltonian* (3) *for an N-body system. There exists a unique L^2-energy minimizing symmetric operator $A = A_1 \otimes \cdots \otimes A_N$ satisfying Conservation of Energy* (1) *and* (4), (5), (6).

If $A = A_1 \otimes \cdots \otimes A_N$ is the unique minimal symmetric operator (${}^tA = A$) satisfying Lemma 2, then Weber's Hamilton equations are given by

$$(x', v') = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} J\nabla H. \tag{7}$$

The operator A^{-1} represents the effective inertial mass of the system, i.e. the coefficient of proportionality between force and acceleration in Newton's Second Law. We observe that A^{-1} is velocity independant.

4. Two-Body Gibbs Liouville

For classical Hamilton equations the vector field $J\nabla H$ is divergence free $div(J\nabla H) = 0$. This implies Poincare-Gibbs-Liouville's classical theorem that the canonical volume form $dxdv$ on state space is invariant with respect to Hamiltonian evolution. E.g. if $H = \omega(x^2 + v^2)$ is the classical Hamiltonian for a one-particle oscillator, then the energy levels are circles and the Hamilton evolves like rigid rotation. Therefore the area form $dxdv$ is invariant with respect to time evolution. We remark that existence of an invariant background measure $dxdv$ on state space is necessary a priori to define the entropy of probability measures ρ on statespace.

In Weber electrodynamics we find that the volume form $dxdv$ is *not* invariant under Weber-Hamilton flow. Here we derive the proper Weberian invariant measure on two-body state space. This means finding a function f such that the measure $\rho = f.vol$ is invariant with respect to Weber-Hamilton flow. Invariance means $\rho = f.vol$ satisfies $L_X(f.vol) = 0$ identically. Cartan's formula $L_X = \iota_X d + d\iota_X$ implies ρ is invariant if

$$div(fX) = 0. \tag{8}$$

In coordinates if $X = X_1\partial_x + X_2\partial_v$, then $\iota_X dxdv = X_1 dv - X_2 dx$. Therefore $L_X(f.dxdv) = d(fX_1 dv - fX_2 dx)$ and the key equation (8) becomes

$$f(X_{1,x} + X_{2,v}) + f_{,x}\, X_1 + f_{,v}\, X_2 = 0.$$

Everything is computed explicitly using Hamilton's equations. We have $X_1 = x' = v$ and therefore $f X_{1,x} = 0$. We also have

$$X_2 = \mu_{\text{eff}}^{-1} \cdot \frac{-\partial H}{\partial x} = \mu_{\text{eff}}^{-1} \, (\frac{q_1 q_2}{x^2}) \, (1 - \frac{v^2}{2c^2})$$

and therefore

$$X_{2,v} = \mu_{\text{eff}}^{-1} \frac{q_1 q_2}{x^2} \frac{(-v)}{c^2}.$$

We obtain

$$f(\mu_{\text{eff}}^{-1} \frac{q_1 q_2}{c^2} \frac{v}{x^2}) + f_{,x} v + f_{,v} \mu_{\text{eff}}^{-1} \frac{q_1 q_2}{x^2} (1 - \frac{v^2}{2c^2}) = 0. \tag{9}$$

Every function f satisfying the ODE (9) defines an invariant measure on state space. We make further assumptions on the density, for example whether the distribution f satisfies either $f_{,x} = 0$ or $f_{,v} = 0$ identically.

Proposition 3 (Spatially Homogeneous Invariant Densities)**.** *The density $\rho = (1 - \frac{v^2}{2c^2})^{-1} dxdv$ is the unique invariant measure on state space $\{(x,v) | x > 0\}$ which is position independant and satisfies $\rho(x,0) = 1$ identically.*

Proof. Suppose that $f_{,x} = 0$ identically. Then (9) can be rewritten simply

$$f X_{2,v} + f_{,v} X_2 = 0,$$

which is equivalent to

$$(\log f)_{,v} = -(\log X_2)_{,v}.$$

This implies

$$f = (1 - \frac{v^2}{2c^2})^{-1}$$

is the unique invariant density which is position independant and $\rho(x,0) = 1$. $\square$

The proof of 3 shows that f is multiplicative inverse of the distance dependant multiplicative factor in $X_2 = -\mu_{eff}^{-1} \nabla_x H$, and this factor is computed to be $(1 - \frac{v^2}{2c^2})$.

5. N-Body Gibbs Liouville

We continue with our previous notation. Let

$$X = (X_1, X_2) := (x', v') = (v, a) = (A^{-1}\nabla_v H, -A^{-1}\nabla_x H)$$

be the Weber Hamilton flow of an N-body electrodynamic system. In this section we study the question of X-invariant absolutely continuous volume measures $\rho = f dx dv$ on state space. We demonstrate the unique existence of a spatially homogeneous X-invariant measure $\rho = f dx dv$ on the N-body state space.

Recall the defining relations

$$Av = \nabla_v H, \quad Av' = -\nabla_x H.$$

We differentiate these relations to obtain

$$\nabla^2_{xv} H = grad_x \nabla_v H = grad_x (Av) = D_x A.v \tag{10}$$

Recall the key equation (8)

$$div(fX) = f.divX + \nabla f \cdot X = 0. \tag{11}$$

The divergence term satisfies

$$divX = div_v X_2 = div_v(-A^{-1}\nabla_x H)$$

and the key equation (11) thus becomes

$$\nabla_v \log f \cdot A^{-1}\nabla_x H = -div_v(A^{-1}\nabla_x H). \tag{12}$$

The definition $X_2 = -A^{-1}\nabla_x H$ implies equation (12) can be written

$$\nabla_v \log f \cdot X_2 = -div_v X_2.$$

The dot in the LHS represents a vector dot product, and is not readily inverted. Therefore equation (12) does not uniquely prescribe f.

Proposition 4. *The transformed key equation* (12) *has unique solution* $f = f(x, v)$ *satisfying* $\nabla_x f = 0$ *and* $f(x, 0) = 1$ *for all* x.

Proof. The assumption $\nabla_x f = 0$ implies $\nabla^2_{xv} f = 0$ identically. Differentiating (12) with respect to x we obtain

$$\nabla_v \log f \cdot grad_x X_2 = -\nabla_x div_v X_2 \tag{13}$$

Thus the dot product in (12) is replaced with the matrix product in (13) between the vector $\nabla_v \log f$ and the matrix derivative $grad_x X_2$.

It follows that $\nabla_v \log f$ satisfying the assumptions of Proposition 4 is uniquely defined by

$$\nabla_v \log f = -\nabla_x div_v X_2 \cdot [grad_x X_2]^{-1} \tag{14}$$

wherever $grad_x X_2$ is uniquely invertible.

Claim: The operator $grad_x X_2 = grad_x(A^{-1}\nabla_x H)$ is invertible for all x, v satisfying $H(x,v) < +\infty$.

The proposition follows from observing that the RHS expression in (14) is explicit, and uniquely prescribes $\nabla_v \log f$. Therefore f is uniquely defined by the initial conditions. $\square$

The above proposition is an existence result. We observe that (14) is equivalent to $\nabla_v \log f$ being equal to $-\nabla_v tr \log grad_x X_2$. This implies the invariant density f satisfying 4 is the x-independant factor of $e^{-tr \log grad_x X_2}$, which by Jacobi's identity is equal to the x-independant factor of

$$(\det e^{\log grad_x X_2})^{-1} = (\det grad_x X_2)^{-1},$$

and this factor is unique.

References

[1] André Koch Torres Assis. *Weber's Electrodynamics*. Springer, 1994.
[2] Wilhelm Eduard Weber. *Electrodynamic Measurements, Eighth Memoir, relating specially to the Connection of the Fundamental Law of Electricity with the Law of Gravitation*. 1894.