Let σ be an arbitrary orientation of a graph X, and let D be the incidence matrix of X^σ. Then the *Laplacian* of X is the matrix $Q(X) = DD^T$. It is a consequence of Lemma 8.3.2 that the Laplacian does not depend on the orientation σ, and hence is well-defined.

Lemma 13.1.1 *Let X be a graph with n vertices and c connected components. If Q is the Laplacian of X, then* $\operatorname{rk} Q = n - c$.

Proof. Let D be the incidence matrix of an arbitrary orientation of X. We shall show that $\operatorname{rk} D = \operatorname{rk} D^T = \operatorname{rk} DD^T$, and the result then follows from Theorem 8.3.1. If $z \in \mathbb{R}^n$ is a vector such that $DD^T z = 0$, then

$z^T DD^T z = 0$. But this is the squared length of the vector $D^T z$, and hence we must have $D^T z = 0$. Thus any vector in the null space of DD^T is in the null space of D^T, which implies that $\operatorname{rk} DD^T = \operatorname{rk} D$. $\square$

Let X be a graph on n vertices with Laplacian Q. Since Q is symmetric, its eigenvalues are real, and by Theorem 8.4.5, $\mathbb{R}^n$ has an orthogonal basis consisting of eigenvectors of Q. Since $Q = DD^T$, it is positive semidefinite, and therefore its eigenvalues are all nonnegative. We denote them by $\lambda_1(Q)$, ..., $\lambda_n(Q)$ with the assumption that

$$\lambda_1(Q) \leq \lambda_2(Q) \leq \cdots \leq \lambda_n(Q).$$

We use $\lambda_i(X)$ as shorthand for $\lambda_i(Q(X))$, or simply λ_i when Q is clear from the context or unimportant. We will also use λ_∞ to denote λ_n. For any graph, $\lambda_1 = 0$, because $Q\mathbf{1} = 0$. By Lemma 13.1.1, the multiplicity of zero as an eigenvalue of Q is equal to the number of components of X, and so for connected graphs, λ_2 is the smallest nonzero eigenvalue. Much of what follows will concentrate on the information determined by this particular eigenvalue.

If X is a regular graph, then the eigenvalues of the Laplacian are determined by the eigenvalues of the adjacency matrix.

Lemma 13.1.2 *Let X be a regular graph with valency k. If the adjacency matrix A has eigenvalues $\theta_1, \ldots, \theta_n$, then the Laplacian Q has eigenvalues $k - \theta_1, \ldots, k - \theta_n$.*

Proof. If X is k-regular, then $Q = \Delta(X) - A = kI - A$. Thus every eigenvector of A with eigenvalue θ is an eigenvector of Q with eigenvalue $k - \theta$. $\square$

This shows that if two regular graphs are cospectral, then they also have the same Laplacian spectrum. However, this is not true in general; the two graphs of Figure 8.1 have different Laplacian spectra.

The next result describes the relation between the Laplacian spectrum of X and the Laplacian spectrum of its complement $\overline{X}$.

Lemma 13.1.3 *If X is a graph on n vertices and $2 \leq i \leq n$, then $\lambda_i(\overline{X}) = n - \lambda_{n-i+2}(X)$.*

Proof. We start by observing that

$$Q(X) + Q(\overline{X}) = nI - J. \tag{13.1}$$

The vector $\mathbf{1}$ is an eigenvector of $Q(X)$ and $Q(\overline{X})$ with eigenvalue 0. Let x be another eigenvector of $Q(X)$ with eigenvalue λ; we may assume that x is orthogonal to $\mathbf{1}$. Then $Jx = 0$, so

$$nx = (nI - J)x = Q(X)x + Q(\overline{X})x = \lambda x + Q(\overline{X})x.$$

Therefore, $Q(\overline{X})x = (n - \lambda)x$, and the lemma follows. $\square$

Note that $nI - J = Q(K_n)$; thus (13.1) can be rewritten as

$$Q(X) + Q(\overline{X}) = Q(K_n).$$

From the proof of Lemma 13.1.3 it follows that the eigenvalues of $Q(K_n)$ are n, with multiplicity $n-1$, and 0, with multiplicity 1. Since $K_{m,n}$ is the complement of $K_m \cup K_n$, we can use this fact, along with Lemma 13.1.3, to determine the eigenvalues of the complete bipartite graph. We leave the pleasure of this computation to the reader, noting only the result that the characteristic polynomial of $Q(K_{m,n})$ is

$$t(t-m)^{n-1}(t-n)^{m-1}(t-m-n).$$

We note another useful consequence of Lemma 13.1.3.

Corollary 13.1.4 *If X is a graph on n vertices, then $\lambda_n(X) \leq n$. If $\overline{X}$ has $\bar{c}$ connected components, then the multiplicity of n as an eigenvalue of $Q(X)$ is $\bar{c}-1$.* □

Our last result in this section is a property of the Laplacian that will provide us with a lot of information about its eigenvalues.

Lemma 13.1.5 *Let X be a graph on n vertices with Laplacian Q. Then for any vector x,*

$$x^T Q x = \sum_{uv \in E(X)} (x_u - x_v)^2.$$

Proof. This follows from the observations that

$$x^T Q x = x^T D D^T x = (D^T x)^T (D^T x)$$

and that if $uv \in E(X)$, then the entry of $D^T x$ corresponding to uv is $\pm(x_u - x_v)$. □