This handout includes space for every question that requires a written response. Please feel free to use it to handwrite your solutions (legibly, please). If you choose to typeset your solutions, the README.md for this assignment includes instructions to regenerate this handout with your typeset LaTeX solutions.

1.

2.a

$$\ell(\theta^{(t+1)}) = \alpha \ell_{\mathsf{sup}}(\theta^{(t+1)}) + \ell_{\mathsf{unsup}}(\theta^{(t+1)}) \qquad \text{Definition}$$

$$\geq \alpha \ell_{\mathsf{sup}}(\theta^{(t+1)}) + \sum_{i=1}^{n} \sum_{z^{(i)}} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \theta^{(t+1)})}{Q_i^{(t)}(z^{(i)})} \qquad \text{Jensen's inequality}$$

$$\geq$$

2.b

2.c

List the parameters which need to be re-estimated in the M-step:

In order to simplify derivation, it is useful to denote

$$w_j^{(i)} = Q_i^{(t)}(z^{(i)} = j),$$

and

$$\tilde{w}_j^{(i)} = \begin{cases} \alpha & \tilde{z}^{(i)} = j \\ 0 & \text{otherwise.} \end{cases}$$

We further denote $S = \Sigma^{-1}$, and note that because of chain rule of calculus, $\nabla_S \ell = 0 \Rightarrow \nabla_\Sigma \ell = 0$. So we choose to rewrite the M-step in terms of S and maximize it w.r.t S, and re-express the resulting solution back in terms of Σ.

Based on this, the M-step becomes:

$$\phi^{(t+1)}, \mu^{(t+1)}, S^{(t+1)} = \arg\max_{\phi, \mu, S} \sum_{i=1}^{n} \sum_{j=1}^{k} Q_i^{(t)}(z^{(i)}) \log \frac{p(x^{(i)}, z^{(i)}; \phi, \mu, S)}{Q_i^{(t)}(z^{(i)})} + \alpha \sum_{i=1}^{\tilde{n}} \log p(x^{\tilde{(i)}}, z^{\tilde{(i)}}; \phi, \mu, S)$$

$$=$$

Now, calculate the update steps by maximizing the expression within the argmax for each parameter (We will do the first for you).

ϕ_j: We construct the Lagrangian including the constraint that $\sum_{j=1}^{k} \phi_j = 1$, and absorbing all irrelevant terms into constant C:

$$\mathcal{L}(\phi, \beta) = C + \sum_{i=1}^{n} \sum_{j=1}^{k} w_j^{(i)} \log \phi_j + \sum_{i=1}^{\tilde{n}} \sum_{j=1}^{k} \tilde{w}_j^{(i)} \log \phi_j + \beta \left(\sum_{j=1}^{k} \phi_j - 1 \right)$$

$$\nabla_{\phi_j} \mathcal{L}(\phi, \beta) = \sum_{i=1}^{n} w_j^{(i)} \frac{1}{\phi_j} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \frac{1}{\phi_j} + \beta = 0$$

$$\Rightarrow \phi_j = \frac{\sum_{i=1}^{n} w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{-\beta}$$

$$\nabla_{\beta} \mathcal{L}(\phi, \beta) = \sum_{j=1}^{k} \phi_j - 1 = 0$$

$$\Rightarrow \sum_{j=1}^{k} \frac{\sum_{i=1}^{n} w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{-\beta} = 1$$

$$\Rightarrow -\beta = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \right)$$

$$\Rightarrow \phi_j^{(t+1)} = \frac{\sum_{i=1}^{n} w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{\sum_{j=1}^{k} \left(\sum_{i=1}^{n} w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)} \right)}$$

$$= \frac{\sum_{i=1}^{n} w_j^{(i)} + \sum_{i=1}^{\tilde{n}} \tilde{w}_j^{(i)}}{n + \alpha\tilde{n}}$$

μ_j: Next, derive the update for μ_j. Do this by maximizing the expression with the argmax above with respect to μ_j.

First, calculate the gradient with respect to μ_j:

$$\nabla_{\mu_j} =$$

Next, set the gradient to zero and solve for μ_j:

$$0 =$$

Σ_j: Finally, derive the update for Σ_j via S_j. Again, Do this by maximizing the expression with the argmax above with respect to S_j.

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First, calculate the gradient with respect to S_j:

$$\nabla_{S_j} =$$

Next, set the gradient to zero and solve for S_j:

$$0 =$$

This results in the final set of update expressions:

$$\phi_j :=$$
$$\mu_j :=$$
$$\Sigma_j :=$$

2.f

3.a

3.b